SPLITTING THEOREM

Consider a uniform flow with a velocity $\vec{U}_0$, pressure p_0, and density ρ_0. Upon this flow a small unsteady disturbance is superimposed with velocity $\vec{u}(\vec{x},t)$, pressure $p'(\vec{x},t)$ and density $\rho'(\vec{x},t)$. The total velocity, pressure, density and entropy can then be written as

$$\vec{V}(\vec{x},t) = \vec{U}_0 + \vec{u}(\vec{x},t) \tag{1}$$
$$p(\vec{x},t) = p_0 + p'(\vec{x},t) \tag{2}$$
$$\rho(\vec{x},t) = \rho_0 + \rho'(\vec{x},t) \tag{3}$$
$$s(\vec{x},t) = s'(\vec{x},t) \tag{4}$$

Substituting (1–4) into the continuity, Euler, and energy equations and neglecting the quadratic terms, one obtains

$$\frac{D_0\rho'}{Dt} + \rho_0\nabla\cdot\vec{u} = 0, \tag{5}$$
$$\frac{D_0}{Dt}\vec{u} = -\frac{1}{\rho_0}\nabla p', \tag{6}$$
$$\frac{D_0}{Dt}s' = 0, \tag{7}$$

where

$$\frac{D_0}{Dt} \equiv \frac{\partial}{\partial t} + \vec{U}_0\cdot\nabla \tag{8}$$

We now split the unsteady velocity into solenoidal and potential fields,

$$\vec{u} = \vec{u}_s + \vec{u}_p, \tag{9}$$

where

$$\nabla\cdot\vec{u}_s = 0, \tag{10}$$
$$\nabla\times\vec{u}_p = 0. \tag{11}$$

The solenoidal field satisfies

$$\frac{D_0}{Dt}\vec{u}_s = 0, \tag{12}$$

and the potential field

$$\frac{D_0\rho'}{Dt} + \rho_0\nabla\cdot\vec{u}_p = 0, \tag{13}$$
$$\frac{D_0}{Dt}\vec{u}_p = -\frac{1}{\rho_0}\nabla p', \tag{14}$$
$$\tag{15}$$

Note that the splitting is not unique. Let

$$\vec{u} = \vec{u}'_s + \vec{u}'_p. \tag{16}$$

Then

$$\vec{u}_0 = \vec{u}_s - \vec{u'}_s = \vec{u'}_p - \vec{u}_p \tag{17}$$

is both solenoidal and potential, i.e.,

$$\nabla \cdot \vec{u}_0 = \nabla \times \vec{u}_0 = 0 \tag{18}$$

If $\vec{u}_0 = \nabla \Phi_0$, then Φ_0 is a harmonic function,i.e., solution to Laplace's equation. Moreover, it must satisfy

$$\frac{D_0}{Dt}\Phi_0 = 0. \tag{19}$$

The entropy s' and the solenoidal velocity $\vec{u}_s$ are governed by the first order partial diffrential equations (7 and 12) whose solutions are

$$s'(\vec{x}, t) = s'(\vec{x} - t\vec{U}_0) \tag{20}$$

$$\vec{u}_s(\vec{x}, t) = \vec{u}_s(\vec{x} - t\vec{U}_0) \tag{21}$$

If the values of s' and $\vec{u}_s$ are given upstream at $t = o$ as $s'_\infty(\vec{x})$ and $\vec{u}_{s\infty}(\vec{x})$, then

$$s'(\vec{x}, t) = s'_\infty(\vec{x} - t\vec{U}_0) \tag{22}$$

$$\vec{u}_s(\vec{x}, t) = \vec{u}_{s\infty}(\vec{x} - t\vec{U}_0) \tag{23}$$

By taking $\frac{D_0}{Dt}$ of (14) and the divergence of (13), eliminating the velocity, and using $p = c_0^2 \rho$, one gets

$$\frac{D_0^2}{Dt^2}\rho' - c_0^2 \nabla^2 \rho' = 0 \tag{24}$$

This is the convected wave equation. Note that the pressure depends only on the potential velocity $\vec{u}_p$(11) implies that there exists a potential function Φ such that $\vec{u}_p = \nabla \Phi$.

Thus the unsteady velocity is split into two parts obeying distinct and independent equations. The pressure is only related to the potential velocity,

$$p' = -\rho_0 \frac{D_0}{Dt}\Phi \tag{25}$$