UNIVERSITY OF NOTRE DAME
DEPARTMENT OF AEROSPACE AND MECHANICAL ENGINEERING

Professor Atassi *AE-342*

THE CAUCHY PRINCIPAL VALUE

The fundamental equation of *thin airfoil theory*

$$\frac{1}{2\pi}\oint_0^c \frac{\gamma(\xi)d\xi}{x-\xi} = V_\infty(\alpha - \frac{dz}{dx}) \tag{1}$$

was established by representing the airfoil by a vortex sheet of strength $\gamma(\xi)$ along the airfoil chord stretching from 0 to c. Around ξ, an infinitesimal vortex sheet of strength $\gamma(\xi)d\xi$ induces a vortical velocity at x equal to

$$dw = -\frac{\gamma(\xi)d\xi}{2\pi(x-\xi)} \tag{2}$$

Equation(1) is then derived by summing up the contribution of all the vortex sheet extending from 0 to c. However, unthoughtful generalization of the mathematics may lead to erroneous results. At $x = \xi$, dw becomes infinite which contadicts our model of a single vortex where the velocity at the center is zero. As a result, the integration along the vortex sheet must stop just before reaching its center, x, and resume just after the center to avoid the singular behavior which, otherwise, would render our model useless. The circular symmetry implied by the vortex analogy suggests that the vortex sheet integration must cease at equal distance from x. The integral in (1) must then be calculated as

$$\oint_0^c \frac{\gamma(\xi)d\xi}{x-\xi} = \lim_{\epsilon\to 0}\{\int_0^{x-\epsilon} \frac{\gamma(\xi)d\xi}{x-\xi} + \int_{x+\epsilon}^{c} \frac{\gamma(\xi)d\xi}{x-\xi}\} \tag{3}$$

This is known as the *Cauchy principal value.* Note that when the integrand is not singular, the Cauchy principal value is identical to the usual definition of a *definite integral.* Equation (1) is called a *singular integral equation* because the integrand is infinite at $x = \xi$, and the integral must be calculated according to (3).

It may be interesting to examine the Cauchy principal value with an example. The simplest example is to consider a vortex sheet of constant strength equal to -2π along the segment [0,1], then, using(3), we obtain

$$\oint_0^1 \frac{d\xi}{\xi - x} = ln\frac{1-x}{x} \tag{4}$$

Let us ignore the difficulty associated with the singular behavior and integrate (4) as if it were an ordinary integral. We integrated (4) using Simpson rule for 11 values of x from 0.500 to 0.510. [0,1] was divided into 100 intervals for good accuracy. The attached Table shows the calculated values compared with the exact solution (4). The results are also plotted in the attached Figures. Note how the results depend on the integration scheme. Note also that the result obtained for $x = 0.505$, corresponding to the mid inteval of integration [0.5, 0.51], is very accurate because the integration scheme satisfies (3).