The Green's Function

1 Laplace Equation

Consider the equation

$$\nabla^2 G = -\delta(\vec{x} - \vec{y}), \tag{1}$$

where $\vec{x}$ is the observation point and $\vec{y}$ is the source point. Let us integrate (1) over a sphere Σ centered on $\vec{y}$ and of radius $r = |\vec{x} - \vec{y}|$

$$\int \nabla^2 G \, d\vec{x} = -1.$$

Using the divergence theorem,

$$\int \nabla^2 G \, d\vec{x} = \int_\Sigma \nabla G \cdot \vec{n} \, d\Sigma = \frac{\partial G}{\partial n} 4\pi r^2 = -1$$

This gives the *free-space Green's function* as

$$G = \frac{1}{4\pi r} = \frac{1}{4\pi|\vec{x} - \vec{y}|}. \tag{2}$$

2 The Wave Equation

We look for a spherically symmetric solution to the equation

$$\left(\nabla^2 - \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\right) G = -\delta(\vec{x} - \vec{y})\,\delta(t - \tau). \tag{3}$$

Such a solution is of the form

$$G = \frac{f(t - \tau \pm \frac{|\vec{x}-\vec{y}|}{c_o})}{4\pi|\vec{x} - \vec{y}|}, \tag{4}$$

where f is an arbitrary function. For $r \neq 0$, G satisfies

$$\left(\nabla^2 - \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\right) G = 0.$$

As $r \to 0$

$$\begin{aligned} \left(\nabla^2 - \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\right) G \to f(t-\tau)\nabla^2\left(\frac{1}{4\pi r}\right) &= f(t-\tau)(-\delta(\vec{x} - \vec{y})) \\ &= -\delta(\vec{x} - \vec{y})\,\delta(t - \tau). \end{aligned}$$

Hence $f(t) = \delta(t)$. The Green's function then becomes

$$\boxed{G = \frac{\delta\left(t-\tau\pm\frac{|\vec{x}-\vec{y}\,|}{c_o}\right)}{4\pi|\vec{x}-\vec{y}\,|}} \tag{5}$$

3 The Helmholtz Equation

For harmonic waves of angular frequency ω, we seek solutions of the form $g(r)exp(-i\omega t)$. The Green's function $g(r)$ satisfies the constant frequency wave equation known as the Helmholtz equation,

$$\left(\nabla^2 + \frac{\omega^2}{c_o^2}\right) g = -\delta(\vec{x} - \vec{y}). \tag{6}$$

For $r \neq 0$, $g = Kexp(\pm ikr)/r$, where $k = \omega/c_0$ and K is a constant, satisfies

$$\left(\nabla^2 + \frac{\omega^2}{c_o^2}\right) g = 0.$$

As $r \to 0$

$$\begin{aligned} \left(\nabla^2 + \frac{\omega^2}{c_o^2}\right) g \to K\nabla^2 \left(\frac{1}{r}\right) &= K(-4\pi\delta(\vec{x} - \vec{y})) \\ &= -\delta(\vec{x} - \vec{y}). \end{aligned}$$

Hence $K = 1/4\pi$ and

$$g(r) = \frac{e^{\pm ikr}}{4\pi r}. \tag{7}$$

Note this result can be obtained directly using the general expression for the Green's function in (5)

4 Application to Acoustics

Begin by assuming isentropic flow, no viscosity. The governing equations can be written

$$\rho_o \frac{\partial}{\partial t}\vec{u} = \vec{F} - \nabla p \tag{8}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho_o \vec{u} = Q \tag{9}$$

where $\vec{F}$ is a force per unit volume and Q is a mass flow rate per unit volume.
Combining the two equations gives

$$\left(\nabla^2 - \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\right) p = -\frac{\partial Q}{\partial t} + \nabla \cdot \vec{F} \tag{10}$$

For harmonic oscillations, the equation can be written as

$$\left(\nabla^2 + k^2\right) p = -\frac{\partial Q}{\partial t} + \nabla \cdot \vec{F} \tag{11}$$

Now, use Green's function to define waves propagating away from a source:

$$\text{1D}: \quad \left(\tfrac{d^2}{dx^2}+k^2\right) g = -\delta(x-y) \quad \Rightarrow g = -\frac{i}{2k}e^{-ik|x-y|} \tag{12}$$

$$\text{2D}: \quad (\nabla^2+k^2)\, g = -\delta(\vec{x}-\vec{y}\,) \quad \Rightarrow g = -\frac{i}{4}H_o^{(2)}(kr) \tag{13}$$

$$\text{3D}: \quad (\nabla^2+k^2)\, g = -\delta(\vec{x}-\vec{y}\,) \quad \Rightarrow g = \frac{1}{4\pi}\frac{e^{-ikr}}{r} \tag{14}$$

Note that

$$\begin{aligned} H_n^{(2)}(kr) &\sim \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{kr}}e^{-i\left(kr-n\frac{\pi}{2}-\frac{\pi}{4}\right)} \\ H_1^{(2)}(kr) &\sim \frac{2i}{\pi}\frac{1}{kr}+O(r\ln r) \end{aligned}$$

4.1 Single Source at $\vec{y}$: $Q = q\delta(\vec{x}-\vec{y}\,)$

$$\begin{aligned} \text{3D}: \quad p &= \frac{1}{4\pi}\frac{dq}{dt}\frac{e^{-ikr}}{r} \\ \vec{u} &= \frac{1}{4\pi}\frac{q'(t)}{\rho_o}\left(\frac{ik}{r}+\frac{1}{r^2}\right)e^{-ikr}\,\frac{\vec{x}-\vec{y}}{r} \\ \text{2D}: \quad p &= -\frac{i}{4}\frac{dq}{dt}H_o^{(2)}(kr) \\ \vec{u} &= -\frac{i}{4}q'(t)kH_1^{(2)}(kr)\,\frac{\vec{x}-\vec{y}}{r} \\ \text{1D}: \quad p &= -\frac{i}{2k}\frac{dq}{dt}e^{-ik|x-y|} \\ u &= \pm\frac{1}{2}q(t)e^{-ik|x-y|} \end{aligned}$$

4.2 Dipole at $\vec{y}$ in the τ-Direction

Equation of a dipole can be written:

$$p\left(\vec{x},\vec{y}+\frac{l}{2}\vec{\tau}\right)-p\left(\vec{x},\vec{y}-\frac{l}{2}\vec{\tau}\right) = l\vec{\tau}\cdot\nabla_{\vec{y}}\,p(\vec{x},\vec{y})$$

.

The solution (in terms of pressure) can be written:

$$\begin{aligned} \text{3D}: \quad p(\vec{x},\vec{y}) &= \frac{1}{4\pi}\left(ik+\frac{1}{r}\right)\frac{e^{-ikr}}{r}\left(\vec{\tau}\cdot\frac{\vec{x}-\vec{y}}{r}\right)\left(l\frac{dq}{dt}\right) \\ \text{2D}: \quad p(\vec{x},\vec{y}) &= -\frac{i}{4}kH_1^{(2)}(kr)\left(l\frac{dq}{dt}\right)\left(\vec{\tau}\cdot\frac{\vec{x}-\vec{y}}{r}\right) \end{aligned}$$

$$r \to \infty: \quad p(\vec{x}, \vec{y}) = \frac{1}{\sqrt{8\pi}} k^{\frac{1}{2}} \frac{e^{-i\left(kr - \frac{\pi}{4}\right)}}{\sqrt{r}} \left(l \frac{dq}{dt}\right) \left(\vec{\tau} \cdot \frac{\vec{x} - \vec{y}}{r}\right)$$

$$\text{1D}: \quad p(x, y) = \pm \frac{1}{2} e^{-ik|x-y|} \left(l \frac{dq}{dt}\right)$$

4.3 Single Force at $\vec{y}$: $\vec{F} = \vec{f}\delta(\vec{x} - \vec{y})$

$$\left(\nabla^2 + k^2\right) p = \nabla \cdot \vec{f}' \delta(\vec{x} - \vec{y}) = \vec{f}' \cdot \nabla \delta(\vec{x} - \vec{y})$$

$$p = -\vec{f} \cdot \nabla_{\vec{x}} g$$

The solutions for pressure are of the form:

$$\text{3D}: \quad p = \frac{1}{4\pi} \left(ik + \frac{1}{r}\right) \frac{e^{-ikr}}{r} \frac{(\vec{x} - \vec{y}) \cdot \vec{f}}{r}$$

$$\text{2D}: \quad p = -\frac{i}{4} k H_1^{(2)}(kr) \frac{\vec{x} - \vec{y}}{r} \cdot \vec{f}$$

$$r \to \infty: \quad p = \frac{1}{\sqrt{8\pi}} k^{\frac{1}{2}} \frac{e^{-i\left(kr - \frac{\pi}{4}\right)}}{\sqrt{r}} \frac{\vec{x} - \vec{y}}{r} \cdot \vec{f}$$

$$\text{1D}: \quad p = \pm \frac{1}{2} e^{-ik|x-y|} f$$

Note that a force $\vec{f}$ is equivalent to a dipole of strength (lq) and whose direction is the same as $\vec{f}'$:

$$f = l \frac{dq}{dt}$$

$$\vec{\tau} = + \frac{\vec{f}}{\left|\vec{f}\right|}$$