Remarks and questions on infinitesimal stabilizers

Anand Pillay

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Suppose first T o-minimal (expansion of RCF), $M \models T$ (maybe saturated), G definable group in M with its o-minimal topology. Let $\bar{M}$ be a bigger saturated model. Let $\mu(x)$ be the partial type over M saying that $x \in G$ is infinitesimally close to the identity e with respect to M. Note that the collection of open M-definable neigbourhoods of the identity is uniformly definable, so we have a formula $\mu(x,y)$ such that $\mu(x)$ is $\{\mu(x,b) : b \in G(M)\}$. Let $p(x) \in S_G(M)$ be a definable type.

Let $\mu(x).p(x)$ be the partial type over M axiomatized by the set of formulas $\chi(x).\phi(x)$, $\chi \in \mu$ and $\phi \in p$. So the set of realizations of $\mu.p$ in $\bar{M}$ is $\mu(\bar{M}) \cdot p(\bar{M})$.

Remark 0.1. *Let $\phi(x)$ be over M. Then $\phi(x) \in \mu.p$ iff $\mu(\bar{M}).a \subset \phi(\bar{M})$ for some (any) realization a of p.*

Lemma 0.2. *Let $\phi(x)$ be over M. Then $\phi(x) \in \mu.p$ iff there is $b \in M$ such that $\mu(x,b)(\bar{M}).a \subseteq \phi(\bar{M})$ for some (any) a realizing p.*

Proof. By Remark 0.1 and compactness. □

Proposition 0.3. *$\mu.p$ is definable: for any L (or L_M)-formula $\phi(x,w)$, $\{d \in M : \phi(x,d) \in \mu.p\}$ is definable.*

Proof. As p is definable, $\{(b,d) \in M : \forall z(\mu(z,b) \to \phi(zx,d) \in p(x)\}$ is definable, by $\psi_\phi(y,w)$ a formula over M. So by Lemma 1.1, for any $d \in M$, $\phi(x,d) \in \mu.p$ iff $M \models \exists y(\psi_\phi(y,d))$.

□

Corollary 0.4. *$Stab(\mu.p) = \{g \in G(M) : g.\mu.p = \mu.p\}$ is a definable subgroup of $G(M)$.*

Proof. Let $\phi(x, y)$ be a formula such that for each c and g, the left translate $g.\phi(x, b)$ is equivalent to a formula $\phi(x, d)$. We define $Stab_\phi(\mu.p)$ to be the set of $g \in G(M)$ such that for each $c \in M$, $\phi(x, c) \in \mu.p$ iff $g.\phi(x, c) \in \mu.p$. By Proposition 0.3, $Stab_\phi(\mu.p)$ is definable and is clearly a subgroup of $G(M)$. Now $Stab(\mu.p)$ is the intersection of all such $Stab_\phi(\mu.p)$ as $\phi(x, y)$ varies. By the DCC on definable subgroups in o-minimal theories, we obtain the result. □

Remark 0.5. *Let T be a theory with a definable topology (so a first order topological theory in the sense of [1]). Namely there is a Hausdorff topology on models of T (and also on definable groups) with a uniformly definable basis. So given a definable group G the there is a uniformly definable neighbourhood basis of the identity. Then everything above makes sense. But one needs to choose M to be a reasonably saturated model and then we see that for p a definable type of G, $Stab(\mu.p)$ is an intersection of definable subgroups of $G(M)$.*

Here are several question/problems in the general context.

Problem 0.6. *Suppose that T is NIP, M big model, G definable group, and $p(x) \in S_G(M)$ a definable type. So $Stab(p) = H$ is an intersection of definable subgroups. Then*
(i) Is $H^{000} = H$ (probably easily yes)?
(ii) Does H have a definable H-invariant type?

Problem 0.7. *Same as Problem 0.6 but in the context of Remark 0.5 and with $Stab(\mu.p)$ instead of $Stab(p)$.*

Problem 0.8. *Again work in the context of Remark 0.5 Assume also that T has NIP and that for any model M and formula $\phi(x, b)$ over the monster, $\phi(x, b)$ does not fork over M iff $\phi(x, b)$ is contained in a global type definable over M (in particular the definable types in $S(M)$ are dense). Let G be a definable group over M, and maybe assume M saturated. Is it the case that G has fsg if and only if G is definably compact, in the sense that any definable type $p(x) \in S_G(M)$ has a limit point in $G(M)$ (namely there is $g \in G(M)$ such that every open neighbourhood of g is in p)?*

References

[1] A. Pillay, First order topological structures and theories, JSL, vol. 52, 1987.