Pseudofinite model theory and combinatorics

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Anand Pillay

University of Notre Dame

Introduction I

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- In general we are concerned with finite graphs, which we will normally take to be bipartite, for technical reasons. Namely (V, W, R) where V, W are finite sets and $R \subseteq V \times W$.
- One class of problems is what we call Erdös-Hajnal-type problems.
- This means trying to find "large" $V_0 \subseteq V$ and $W_0 \subseteq W$ such that $V_0 \times W_0$ is homogeneous for R, namely $V_0 \times W_0 \subseteq R$, or $V_0 \times W_0 \subseteq R^c$ (the complement of R). (So Ramsey-type theorems.)

Introduction II

- The actual Erdös-Hajnal conjecture, restricts attention to the class of finite graphs (V, W, R) omitting a given induced finite subgraph H, and asks there to be $\delta > 0$ (depending on H), such that for all (V, W, R), there is homogeneous $V_0 \times W_0$ with $|V_0| \geq |V|^\delta$, and $|W_0| \geq |W|^\delta$.

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- In this most general formulation, H is an arbitrary finite graph. But we could restrict attention to specific H and aim for better resuts (which we do later).
- The second class of problems concerns trying to decompose, or partition, V and W into a "small" number of sets $V = V_1 \cup ..... \cup V_n$, $W = W_1 \cup ... \cup W_m$, such that each induced subgraph $(V_i, W_j, R|(V_i \times W_j))$ is "regular". Namely sufficiently large induced subgraphs of $(V_i, W_j, R|(V_i \times W_j))$ have approximately the same density.

Introduction III

- In this general context we have Szemeredi's regularity theorem, which says that given $\epsilon > 0$, there is N_ϵ such that for all (V, W, R), we can partition V, W as above, with $n, m \leq N_\epsilon$, and such that outside an "ϵ-small" exceptional set Σ of (i, j), each $(V_i, W_j, R|(V_i \times W_j))$ is ϵ-regular. "ϵ-small" means that $|\cup_{i,j \in \Sigma} V_i \times W_j| \leq \epsilon |V \times W|$.

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- And ϵ-regularity of $(V_i, W_j, R|(V_i \times W_j))$ means that for any induced subgraph $(V', W', R|(V' \times W'))$ of $(V_i, W_j, R|(V_i \times W_j))$, with $|V'| \geq \epsilon|V_i|$ and $|W'| \geq \epsilon|W_j|$, the densities of $(V_i, W_j, R|(V_i \times W_j))$ and $(V', W', R|(V' \times W'))$ differ by $< \epsilon$.

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- (The regularity lemma also includes a statement that the V_i's are roughly the same size. Also the W_j's.) Under additional assumptions on the relation R we would like to obtain stronger conclusions, with for example homogeneity replacing regularity, and maybe with no exceptional set.

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- This problematic falls under the description of "arithmetic regularity theorems". An important paper of Ben Green deals with the case where G is commutative, and X arbitrary.
- We will give some results where G is arbitrary (not necessarily commutative), but under some restrictions on X (or on the associated relation R).

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- From the logic point of view, we have to pass from classical first order logic to so-called continuous first order logic, where formulas have values in $\mathbb{R}$, or $[0, 1]$, rather than just $\{0, 1\}$.
- This last topic is really "work in progress", so I will not say so much about it in these lectures.

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- What is kind of new in the recent applications of model theory is that the nonstandard methods are combined with applying nontrivial structural theorems in the nonstandard (pseudofinite) model.
- This point of view was in a sense initiated when model theorists found another proof (valid in all characteristics) of Tao's algebraic regularity theorem (Tao) for graphs defined in finite fields (Pillay-Starchenko, Hrushovski).

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- Actually among the themes of our recent work with Conant and Terry (CPT1, CPT2) and my expository paper "Domination and regularity", is that certain "domination statements" yield fairly directly, the relevant graph regularity statements, in the infinite setting. Hopefully I will try to explain some of this in these talks.

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- However I should also mentioin that our methods do not, as a rule, give optimal bounds, although the problem of good bounds *is* an important aspect of the combinatorial conjectures and results.

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- For simplicity I will assume we are in a 1-sorted situation (namely just one sort), so the relation and function symbols come with a finite "arity". We also assume a distinguished binary relation symbol $=$ (for equality). The many-sorted context is an easy generalization, and I may freely work in such a context.

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- From these symbols, together with the logical connectives ($\neg, \vee, \wedge, \exists, \forall$ and parentheses) as well as a supply of variables v_i or x_i or y_i, we build L-formulas.

Model theory II

- L-formulas are typically denoted ϕ, ψ, or $\phi(\bar{x})$, $\psi(\bar{y})$ to witness the free variables. L-sentences, namely L-formulas with no free variables, are typically denoted $\sigma, \tau, \ldots$.

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- For M an L-structure, $\phi(\bar{x})$ an L-formula, and $\bar{a}$ a tuple of the appropriate length from M, "$M \models \phi(\bar{a})$" means that the formula is true in the structure M when $\bar{x}$ is interpreted as $\bar{a}$. If ϕ is a sentence we also say M is a model of ϕ.

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- If $\phi(\bar{x}, \bar{y})$ is an L-formula, and $\bar{b}$ a tuple from M then $X = \{\bar{a} \in M : M \models \phi(\bar{a}, \bar{b})\}$ is called a set definable in M over $\bar{b}$, or a $\bar{b}$-definable set in M. If B is a subset of M containing the tuple $\bar{b}$ we may also say "B-definable in M".

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- A collection $\Sigma(\bar{x})$ of L_M-formulas (with free variables among $\bar{x}$ is said to be *consistent* (with M) if it is *finitely satisfiable* in M, namely for each finite subset Σ' of Σ there is $\bar{a}$ in M such that $M \models \wedge\Sigma'(\bar{a})$.

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- The compactness theorem says that a collection Σ of L-sentences has a model if every finite subset of Σ has a model. It implies that any L-structure M has an elementary extension N with the property that for every consistent (with M) collection $\Sigma(\bar{x})$ of L_M-formulas, there is a tuple $\bar{a}$ from N such that $N \models \Sigma(\bar{a})$ (where the latter notation means that $N \models \phi(\bar{a})$ for all $\phi(\bar{x}) \in \Sigma$, and we also say that $\bar{a}$ *realizes* $\Sigma(\bar{x})$ in N.)

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- We mention a couple of consequences. First modulo some set theory, for any L-structure M and sufficiently large cardinal κ, M has an elementary extension N which is κ-saturated and is of cardinality κ.

Model theory V

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- Such a κ-saturated model N of cardinality κ is unique up to isomorphism, in the sense that its isomorphism type is determined by its *first order theory* $T = Th(N)$, the set of L-sentences σ such that $N \models \sigma$.
- Secondly, fixing M, a subset B of M, an $n < \omega$, the Stone space (space of ultrafilters) of the Boolean algebra of formulas $\phi(\bar{x})$ in L_B up to equivalence in M, coincides with $\{tp_N(\bar{a}/B) : \bar{a} \in N\}$ where N is some sufficiently saturated elementary extension of M. We call the space $S_n(B)$ (although it depends on the L_B-theory of M).

Model theory VI

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- T identifies, up to bi-interpretability with the category $Def(T^{eq})$.

Definable families I

- We fix a complete L-theory T and typically work in a κ-saturated model $\bar{M}$ of cardinality κ for some large κ. M, N etc denote small elementary substructures, and A, B,.. small subsets. It is also convenient to let $x, y, ..$ range over finite tuples of variables, rather than individual variables.

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- Such a formula $\phi(x, y)$ can be seen in at least three different ways: (i) as defining a bipartite graph (in some/any) model of T, (ii) as giving rise to a family of definable sets, namely the sets defined in $\bar{M}$ (or in a model M) by the formulas $\phi(x, b)$, as b varies over tuples of the right length in $\bar{M}$ (or M), (iii) as a collection of continuous $\{0, 1\}$-valued functions on a suitable compact space.

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- Maybe the third point needs some comments.

Definable families II

- Fix a model M. Then $S_x(M)$ is a compact space. For each $b \in M$, we obtain a continuous function f_b on $S_x(M)$ where $f(p) = 1$ if $\phi(x,b) \in p$ and $= 0$ otherwise. So we get a (definable) family of functions f_b, $b \in M$.

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- This makes a connection of model theory with functional analysis, and in fact some of the basic theorems of stability theory were proved by Grothendieck in his thesis (1951) in this context. (First noticed by Ben-Yaacov).

Definition 0.1

The formula $\phi(x, y)$ is k-stable (for T) if there do not exist $a_1, .., a_k, b_1, .., b_k$ in some/any model M of T such that $M \models \phi(a_i, b_j)$ iff $i \leq j$.

Definable families III

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The formula is k-NIP (for T), if there do not exist $a_1, .., a_k$ and b_s for $s \subseteq \{1, .., k\}$ in some/any model M of T such that $M \models \phi(a_i, b_s)$ iff $i \in s$.

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- By compactness $\phi(x, y)$ is k-stable (in T) for some k iff there do not exist $a_i, b_i \in \bar{M}$ for $i = 1, 2, 3, ...$ such that $\bar{M} \models \phi(a_i, b_j)$ iff $i \leq j$. We just say that $\phi(x, y)$ is stable (for T).

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- There is a similar statement for k-NIP. It is left to the reader. We just say $\phi(x, y)$ is NIP for T.
- T is said to be stable if every formula $\phi(x, y)$ is stable (for T). Likewise T is said to be NIP if every formula $\phi(x, y)$ is NIP for T. In both cases it is enough to consider formulas where x is a single variable, rather than a tuple.

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- So dealing with the class of k-NIP graphs is relevant to studying graphs omitting a fixed finite subgraph H.

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- So for a tuple a in $\bar{M}$, $tp(a/M)$ for example means $tp_{\bar{M}}(a/M)$ in previous notation. Likewise for small sets A of parameters in place of M.
- For an L-formula $\phi(x,y)$, we also have the notion of a complete ϕ-type over a set A or model M.
- This is precisely the restriction of a complete type $p(x)$ over M to the collection of Boolean combinations of formulas $\phi(x,b)$ for $b \in M$. It is "determined" (when M is a model) by the collection of $\phi(x,b)$, $\neg\phi(x,b)$ for $b \in M$, true of a given $a \in \bar{M}$ (realizing p).

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- Secondly, if p' is a complete ϕ-type over $\bar{M}$ extending (or containing p) and p' is finitely satisfiable in M, then p is precisely the ϕ-type over $\bar{M}$ obtained from applying the definition mentioned above; namely for $b \in \bar{M}$, $\phi(x,b) \in p'$ iff $\bar{M} \models \psi(b)$.

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- We call this the local (i.e. formula by formula) theory in stability. (References: GST for example, but also done in Groithendieck's thesis.)

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- Where $(b_i : i < \omega)$ is A-indiscernible means that $tp(b_{i_1}, ...b_{i_n}/A) = tp(b_{j_1}, .., b_{j_n}/A)$ for all $i_1 < .. < i_n$ and $j_1 < ... < j_n$.

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- And $\psi(x, b)$ forks over A if it implies a finite disjunction of formulas each of which divides over A.
- In any case, with the previous assumptions (stability of $\phi(x, y)$ etc.) p' can be characterzed also by: $p \subset p'$ and no formula in p' divides (forks) over A.

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- When T is stable (namely every L-formula $\phi(x, y)$ is stable for T), then the local theories cohere to give a nice theory of "independence", the characteristic feature of which is uniqueness of free extensions.

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- (iii) no formula in p' forks (divides) over M.
- Moreover we have essentially the same conclusions when M is replaced by an algebraically closed set A (finite equivalence relation theorem).

Types V

- Even though the theory T may be unstable, there still may be types $p(x)$ over some model M which satisfy conditions (i), (ii), (iii) above, and we call these *generically stable types*. (We sometimes assumed the theory T to be NIP.)

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- Consider the theory of pairs $(F < K)$ of algebraically closed fields (with a predicate P for the bottom model F) such that there is moreover some additional structure on F, such as adding an additional predicate for a real closed subfield F_0 of F such that $F = F_0(i)$).

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- Then the theory T of K equipped with all this structure is NIP and unstable.

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- $p(x)$ is generically stable, and can be considered to be the "generic type" over M. It is also "regular" in the sense of Pillay-Tanovic.
- Working out the details of all of this is left to the reader. i mainly introduced generically stable types as a motivation for the notion of generically stable measure that will come later.

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- Fix again T and $\bar{M}$ a monster model and $M \prec \bar{M}$.
- Let $p(x)$, $q(y)$ be complete types over M (in variables x, y respectively). $p(x)$ and $q(y)$ are said to be *weakly orthogonal* if $p(x) \cup q(y)$ extends to a unique complete type $r(x, y)$ over M.

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- Now let (V, W, R) be a (bipartitite graph) definable in $\bar{M}$ with parameters from M. So $(V(M), W(M), R(M))$ is a bi-partite graph definabke in M (with parameters).

Types and graphs II

- Now let $p(x) \in S_V(M)$ (i.e. $p(x)$ is a complete type over M containing the formula "$x \in V$"). Likewise let $q(y) \in S_W(M)$.

Types and graphs II

- Now let $p(x) \in S_V(M)$ (i.e. $p(x)$ is a complete type over M containing the formula "$x \in V$"). Likewise let $q(y) \in S_W(M)$.
- We can think of p as defining a $\{0,1\}$ valued measure on the Boolean algebra of definable subsets of $V(M)$. (Namely a definable set has measure 1 or is "large" if the formula defining it is in p). Similarly for $q(y)$ and $W(M)$.

Theorem 0.3

In this context, suppose $p(x)$ and $q(y)$ are weakly orthogonal. Then there are large definable subsets V_0 of $V(M)$ and W_0 of $W(M)$ such that (V_0, W_0) is homogeneous *for $R(M)$. Namely either $(V_0, W_0, R|(V_0 \times W_0))$ is a complete graph or an empty graph.*

Types and graphs III

Proof.

Let $r(x, y)$ be the unique complete type over M extending $p(x) \cup q(y)$.

Case (i) $R(x, y) \in r(x, y)$.

So working in $\bar{M}$, $p(x) \cup q(y) \models R(x, y)$. By compactness (i.e. saturation of $\bar{M}$), there are formulas $\phi(x) \in p(x)$, $\psi(y) \in q(y)$ such that $\bar{M} \models (\forall x)(\forall y)(\phi(x) \wedge \psi(y) \to R(x, y))$. So the sentence $(\forall x)(\forall y)(\phi(x) \wedge \psi(y) \to R(x, y))$ is also true in M. Let V_0 be the subset of V defined by $\phi(x)$ in M, Likewise for W_0, and we see that $(V_0, W_0, R|(V_0 \times W_0)$ is a complete graph. Both V_0, W_0 are large.

Case (ii), $\neg R(x, y) \in r(x, y)$.

Similarly we obtain large V_0, W_0 such that $(V_0, W_0, R(V_0 \times W_0))$ is the empty graph. □

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Theorem 0.4

Suppose that $p(x)$ and $q(y)$ are weakly orthogonal for all $p(x) \in S_V(M)$ and $q(y) \in S_W(M)$. Then we can partition $V(M)$ into definable sets $V_0, .., V_n$, and partition $W(M)$ into definable sets $W_0, .., W_m$ such that each (V_i, W_j) is homogeneous for R.

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Definition 0.5

Let M be an L-structure, and A a subset of some sort X of M (e.g. if M is 1-sorted then X could be the sort consisting on n-tuples from M). We will say that "A is pseudofinite in M" if whenever σ is a sentence in the language L together with an additional predicate symbol for A, and $(M, A) \models \sigma$, then there is an L-structure M' and subset A' of $X(M')$ such that $(M', A') \models \sigma$.

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- Let's make some remarks: Pseudofiniteness of A in M is a property of $Th(M, A)$ (in the language $L(P) = L \cup \{P\}$).
- If M is 1-sorted and A is M itself, we say that M is pseudofinite.

Pseudofiniteness II

- Suppose that A is definable in the L-structure M by a formula $\phi(x, b)$. Then peudofiniteness of A in M is equivalent to : for every L-formula $\psi(y)$ in $tp_M(b)$, there is an L-structure M' and $b' \in M'$ such that $M \models \psi(b')$ and $\phi(x, b')(M')$ is finite.

Pseudofiniteness II

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- We now give some routine equivalences to pseudofiniteness.

Pseudofiniteness III

Lemma 0.6

For M an L-structure and A a subset of a sort X in M, the following are equivalent:

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Proof.

Let Σ be as in (ii). Then obviously $(M, A) \models \Sigma$ iff (M, A) is pseudofinite. On the other hand, assuming (M, A) to be pseudofinite, let I be the collection of finite subsets of $Th(M, A)$, for each $i \in I$, Let $(M_i, A_i) \models i$ with A_i finite. Then any nonprincipal ultraproduct of the A_i is a model of $Th(M, A)$. □

Nonstandard analysis I

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- An object in $\mathbb{V}^*$ is said to be *internal* if it is definable (with parameters) in $(\mathbb{V}^*, \epsilon^*)$.

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- In $\mathbb{V}^*$ we have the nonstandard versions $\mathbb{N}^*$, $\mathbb{R}^*$ of $\mathbb{N}$ and $\mathbb{R}$, (as well as of cardinals). Moreover any internal object which is a $*$-set, has a (nonstandard) cardinality.

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- First the easy direction: Suppose that (M, A) is an $L(P)$-structure in V^* and A is finite in the sense of $\mathbb{V}^*$, and let σ be an $L(P)$-sentence true in (M, A).
- This is expressed by the satisfaction of some formula $\chi(x, y, z)$ of set theory by (M, A, σ) in $\mathbb{V}^*$. So as $\mathbb{V} \prec \mathbb{V}^*$ we can find (M', A') in $\mathbb{V}$ such that A' is finite and $(M', A') \models \sigma$.

Nonstandard analysis III

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Lemma 0.7

Suppose M is an L-structure, A a subset of a sort of M and A is pseudofinite in M (in the sense of Definition 0.5). Then there is some appropriate (M^*, A^*) in $\mathbb{V}^*$ such that

- *(i) (M^*, A^*) is an $L(P)$-structure elementarily equivalent to (M, A),*

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- *(iii) whenever $\chi(y, z)$ is a formula of set theory true of (M^*, A^*) in $\mathbb{V}^*$ then there is $(M, A) \in \mathbb{V}$ such that A is finite and $\chi(y, z)$ is true of (M, A) (in $\mathbb{V}$).*

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- This collection of formulas is finitely satisfiable in $\mathbb{V}$, so realized in a saturated elementary extension $\mathbb{V}^*$, as required.

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(iii)' Whenever $\chi(y, z)$ is a formula of set theory true of (M^*, A^*) in $\mathbb{V}^*$, then $\chi(y, z)$ is true of infinitely many (M_n, A_n) (in $\mathbb{V}$).

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- So if (M, A) was already κ-saturated of cardinality κ, then it will be isomorphic to (M^*, A^*), so can be assumed to live in the nonstandard model with A^* finite in the sense of the model.
- So in this sense the 2 notions of pseudofinite cohere, when (M, A) is "saturated".

Nonstandard analysis VI

- Suppose A is an internal object in $\mathbb{V}^*$ which is finite in the sense of $\mathbb{V}^*$. (In particular A is a *set* in V^*). So for each internal $Z \subseteq A$ we have $|Z| \in \mathbb{N}^*$, and we define $\mu^*(Z)$ to be $|Z|/|A| \in [0,1]^*$.

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- Our rather roundabout way of constructing this "pseudofinite Keisler measure" is partly to avoid an appeal to ultraproducts, which I am allergic to.

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- This means that μ has values in $[0,1]$, $\mu(x = x) = 1$, $\mu(x \neq x) = 0$ and for disjoint M-definable Y, Z, $\mu(Y \cup Z) = \mu(Y) + \mu(Z)$.

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- A Keisler measure on X over M is the same thing as a regular Borel probability measure on the Stone space $S_X(M)$. (To be explained.)

Keisler measures III

Example 0.10

A complete type $p(x) \in S_x(M)$ is a $\{0,1\}$-valued Keisler measure on X over M. From the point of view of the last bullet point, it is a “Dirac”.

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- Let M be the unique countable model of T. Let μ be the Keisler measure on the universe, over M say, determined by assigning 0 to each formula $x = a$ and assigning $1/2$ to each equivalence class.
- The μ is the average of the two nonrealized 1-types $p(x)$, $q(x)$ over M.

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- Note that μ is $Aut(M)$-invariant.

Keisler measures IV

Example 0.12

Let A be a finite subset of $X(M)$, and for Z an M-definable subset of X let $\mu_A(Z) = |Z \cap A|/|A|$. μ_A is a "counting" Keisler measure (on X over M).

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Example 0.13

Likewise let M be an L-structure living in a nonstandard model $\mathbb{V}^*$ of set theory, and let A be a finite, in the sense of $\mathbb{V}^*$, subset of a sort $X(M)$. For Z a definable subset of M, let $\mu_A(Z)$ be as defined earlier ($st(|Z \cap A|/|A|)$. μ_A is a "pseudofinite counting" Keisler measure on X over M.

Keisler measures V

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- Let $\bar{M}$ be the saturated elementary extension of $\mathbb{R}$, another real closed ordered field.

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- The property of unique extension to a larger model is called *smoothness* and shows the difference with types where the only smooth types over a model are realized ones ($tp(a/M)$ for $a \in M$).
- Such measures as well as *generically stable measures* (generalizing generically stable types) will appear later.

Stable regularity lemma I

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Theorem 0.15

For every $\epsilon > 0$ there is N_ϵ such that for every k-stable finite graph (V, W, R), there are partitions $V = V_1 \cup .. \cup V_n$, $W = W_1 \cup .. \cup W_m$ with $m, n \leq N_\epsilon$, and such that for every i, j, $(V_i, W_j, R|(V_i \times W_j))$ is ϵ-homogeneous, namely either almost complete ($|(V_i \times W_j) \setminus R| \leq \epsilon|V_i \times W_j|$) or almost empty ($|(V_i \times W_j) \cap R| \leq \epsilon|V_i \times W_j|$)

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- The original proof by Malliaris-Shelah of (a version of) Theorem 0.15, was not a pseudofinite proof, and gave good bounds (on N_ϵ). I will follow my treatment in "Domination and regularity" which is close to the Malliaris-Pillay account. (See subsequent references.)

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- The original proof by Malliaris-Shelah of (a version of) Theorem 0.15, was not a pseudofinite proof, and gave good bounds (on N_ϵ). I will follow my treatment in "Domination and regularity" which is close to the Malliaris-Pillay account. (See subsequent references.)
- The general idea is simply to study graphs (V, W, R) definable in a an arbitrary structure such that the relation R is defined by a stable formula $\phi(x, y)$, and where the V-sort is equipped with a Keisler ϕ-measure μ, and then apply Lemmas 0.7 and 0.8 where μ is taken to be the pseudofinite counting measure.

Stable measures I

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Lemma 0.16

Suppose that $\phi(x, y)$ is a stable formula, and μ is a Keisler ϕ-measure over M. Then there are $p_i(x) \in S_\phi(M)$, and $\alpha_i \in (0, 1]$ for $i = 1, 2, \ldots$ (maybe finite) such that $\sum_i \alpha_i = 1$ and $\mu = \sum_i \alpha_i p_i$.

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- We have seen in the section on types that from stability of $\phi(x, y)$ every $p(x) \in S_\phi(M)$ is definable. In particular for any countable $M_0 \prec M$, $S_\phi(M_0)$ is countable.spsace

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- It follows that the space $S_\phi(M)$ is scattered, in the sense that it is exhausted by the Cantor-Bendixon analysis.
- Where recall that for given a topological space S, the CB analysis is as follows: the points $p \in S$ of CB-rank 0 are the isolated points.

Stable measures III

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- Now let $p_1, .., p_k$ be the elements in $S_\phi(M)$ of CB-rank γ.
- Without loss of generality $p_1, .., p_r$ have positive μ-measure (say $\alpha_1, .., \alpha_r$) and $p_{r+1}, .., p_k$ have μ-measure 0. (Maybe $r = 0$).

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- Now we can find clopens $U_1 \subset U_2 \subset ... \subset U$, and positive reals $\beta_1 < \beta_2 < .....$ such that $lim_{i \to \infty} \beta_i = \beta$ and $\mu(U_i) = \beta_i$ (using regularity of μ).

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- So we can apply induction to write each of $\mu | U_1$ and $\mu | (U_{i+1} \setminus U_i)$ as a suitable $\sum_j \delta_j q_j$, and put them together with $\alpha_1 p_1 + .. + \alpha_r p_r$ to find the required expression for μ.

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- End of proof sketch.

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Moreover, each V_i can be defined by a ϕ-formula (over M), and each W_j by a ϕ^*-formula (over M).

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- Let $B = S_\phi(M) \setminus \{p_i : i \in I\}$. So B is Borel and $\mu(B) = 0$.

Stable graphs IV

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- Moreover by the choice of U we have that $\mu(V_0') < \alpha_0/(1-\epsilon)$.
- Let us now replace V_0 by V_0' (i.e. V_0' is the new V_0).

Stable graphs V

- The aim and end result of the manipulations so far is to obtain clopen sets $V_0, .., V_n$ partitioning $S_\phi(M)$ (equivalently ϕ-formulas $V_0, .., V_n$ which partition V) such that $V_i \in p_i$ and $\mu(V_i \setminus p_i) < \epsilon\mu(V_i)$, for $i = 0, .., n$

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- $V = V_0 \cup .. \cup V_n$ and $W = \cup W_J$ will be the desired partitions.

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- End of proof of Lemma 0.17 (which one sees is almost tautological).

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- The sizes of the V_N can be assumed to be growing (by thinning the sequence).
- So we can find a saturated model (V, W, R) of the common theory of the (V_N, W_N, R) such that V is infinite, and clearly pseudofinite in the structure (V, W, R).

Proof of stable regularity lemma II

- By Lemmas 0.7 and 0.8, and the Remarks following it, we may assume (V, W, R) to be in $\mathbb{V}^*$, with V finite in the sense of $\mathbb{V}^*$, equipping V with the nonstandard Keisler counting measure μ.

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- So Lemma 0.17 can be applied, for $\epsilon/2$, yielding some partitions $V = V_1 \cup .. \cup V_n$, and $W = W_1 \cup ... \cup W_m$ (into definable sets, so sets internal in $\mathbb{V}^*$) such that for each i, j, either for all $b \in W_j$, $\mu(V_i \setminus R(x, b)) \leq (\epsilon/2)\mu(V_i)$, or $\mu(V_i \cap R(x, b)) \leq (\epsilon/2)\mu(V_i)$.

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- By clause (iii)' of Lemma 0.8, mentioned above, the formula is true of infinitely many of the (V_N, W_N, R_N) in $\mathbb{V}$.
- Now choose $N \geq n, m$. So there are partitions $V_{N,1} \cup .. \cup V_{N,n}$ of V_N and $W_{N,1} \cup .. \cup W_{N,m}$ of W_N with the property (*).

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- This contradiction ends the proof of Theorem 0.15. The proof can also modified slightly to yield that in Theorem 0.15 the V_i can be defined by ϕ-formulas and the W_j by ϕ^*-formulas.

Distal regularity I

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- We still focus on the bipartitite case although a lot of work goes on in the unipartitite case. The context studied by combinatorics people was *semialgebraic graphs*, namely graphs $G = (V, W, R)$ definable in the structure $(\mathbb{R}, +, \times)$.
- For such a fixed such semialgebraic graph G, one can consider the family of finite graphs $(V', W', R|(V' \times W'))$ as V', W' range over finite subsets of V, W respectively.

Distal regularity II

- Strong Erdös-Hajnal (which is a theorem in this situation) says that there is δ depending on G such that for each such finite V', W' there are $V_0 \subseteq V'$ and $W_0 \subseteq W'$, with $|V_0| \geq \delta|V'|$ and $|W_0| \geq \delta|W'|$, such that V_0, W_0 is homogeneous for R.

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- The closely related strong regularity theorem, provides, given $\epsilon > 0$ some N_ϵ such that for every finite V', W' there is a decomposition $V' = V_1 \cup .. \cup V_n$, $W' = W_1 \cup .. \cup W_m$ with $m, n < N$ such that outside a small exceptional set Σ of pairs (i, j), each V_i, W_j is outright homogeneous for R.

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- The distal theorems give the same results but replacing the structure $(\mathbb{R}, +, \times)$ by any first order structure M such that $Th(M)$ is *distal* (and our pseudofiniite formalism adapts well to this set-up).
- Distality was introduced by Simon in his thesis and is supposed to capture the idea of a "purely unstable" NIP theory.

Distal regularity III

- Examples of distal first order theories are RCF (more generally o-minimal theories), $Th(\mathbb{Q}_p, +, \times)$, $Th(\mathbb{Z}, +, <)$, $RCVF$ (real closed valued fields).
- The theory of algebraically closed valued fields is an important unstable NIP theory, but is not distal because the residue field is stable (in the correct sense of a sort or definable set in an ambient theory being stable).

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- A characterization of distality which is convenient for our purposes is:

Definition 0.18

A (complete) theory is distal if T is NIP and every generically stable Keisler measure is smooth.

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- Before defining generically stable measures, let us remark on how established notions for types generalize to measures.
- For some of these definitions a global assumption that T has NIP may be useful.

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- We say that $\mu(x)$ is finitely satisfiable in M_0, if whenever $\phi(x,b)$ is over M and $\mu(\phi(x,b)) > 0$, then $\phi(x,b)$ is realized by an element (tuple) of M_0.

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- Assume M is $|A|^+$-saturated. We say that μ is definable over A if for every L-formula $\phi(x,y)$, and closed set $C \subseteq [0,1]$, $\{b \in M : \mu(\phi(x,b)) \in C\}$ is "type-definable" over A. (explain..).

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- Assume M is $|A|^+$-saturated. We say that μ is definable over A if for every L-formula $\phi(x,y)$, and closed set $C \subseteq [0,1]$, $\{b \in M : \mu(\phi(x,b)) \in C\}$ is "type-definable" over A. (explain..).
- Note that these definitions agree with the usual ones when $\mu(x)$ is a complete type.

Smooth and generically stable measures III

- Let us remark for interested members of the audience that measures behave similarly to types with respect to forking if T is NIP.
- Namely, assume T is NIP, and μ is a Keisler measure over $\bar{M}$. Then μ does not fork over M_0 iff μ is $Aut(\bar{M}/M_0)$-invariant.

Definition 0.20

(Assume T is NIP). Let $\mu(x)$ be a Keisler measure over a model M. We say that μ is *generically stable* if μ has an extension $\mu'(x)$ over $\bar{M}$ which is both definable over M and finitely satisfiable in M (and in fact μ' turns out to be the unique global nonforking extension of μ).

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Lemma 0.21

(Assume T NIP.) Let $\mu(x)$ be a Keisler measure over M. The following are equivalent:
(i) μ is generically stable,
(ii) For any L-formula $\phi(x,y)$, and $\epsilon > 0$, there are $a_1, ..., a_n$ in M such that for any $b \in M$, $\mu(\phi(x,b))$ is within ϵ of the proportion of a_i which satisfy $\phi(x,b)$.

Smooth and generically stable measures IV

- One source of generically stable measures (in an NIP theory) is so-called average measures: let $I = (a_i : i \in [0,1])$ be an indiscernible "segment" in a model M and for $\phi(x)$ over M, define $\mu_I(\phi(x))$ to be the Lebesgue measure of $\{i : M \models \phi(a_i)\}$. This makes sense, because $\phi(x,y)$ being NIP, the set of $\{i \in [0,1] : M \models \phi(a_i)\}$ is a finite union of points and convex sets, hence finite unions of points and intervals, so measurable.

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- For an NIP formula $\phi(x,y)$, there should be (and maybe already is) a good theory of generically stable ϕ-types (as well as a notion of ϕ-distality), which would help place subsequent results and proofs in a formula-by-formula context.

Distality theorems I

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- We first give our version of distal regularity.

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Given $\mathcal{G}$, suppose that one of the following happens:

- *(i) The graphs in $\mathcal{G}$ are uniformly definable in some model M of a distal theory,*

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THEN for any $\epsilon > 0$ there is N_ϵ such that that for every $(V, W, R) \in \mathcal{G}$ there are partitions $V_1, .., V_n$ of V and $W_1, .., W_m$ of W with $n, m \leq N_\epsilon$ such that outside a small exceptional set of pairs (i, j), each pair V_i, W_j is homogeneous for R.

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- Note that with assumption (ii), 0.22 recovers the Fox et al results.
- Our strong Erdos-Hajnal theorem has the same assumptions as in Theorem 0.22, but the conclusion is that there is $\delta > 0$ such that for each (V, W, R) in $\mathcal{G}$ there are $V_0 \subseteq V$, $W_0 \subseteq W$ with $|V_0| \geq \delta|V|$ and $|W_0| \geq \delta|W|$ such that V_0, W_0 is homogeneous for R. This clearly follows from Theorem 0.22.

Regularity theorem for smooth measures I

- Our proof of Theorem 0.22 will use a couple of results, first a regularity theorem for arbitrary definable graphs (V, W, R) equipped with Keisler measures on V, W, at least one of which is smooth, which we do in this section. The other, discussed later is the fact that in the NIP environment the pseudofinite counting measure is generically stable (whoich follows from the Vapnik-Chervonenkis theorem).

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- We start with a basically immediate "domination" statement for smooth measures in arbitrary theories.

Regularity theorem for smooth measures II

Lemma 0.23

- *(T an arbitrary theory.) Let $\mu(x)$ be a Keisler measure over a model M_0 on the sort X. Suppose μ to be smooth. Let μ also denote the induced (Borel probability) measure on $S_X(M_0)$. And let $\pi : X = X(\bar{M}) \to S_X(M_0)$ be the tautological map $\pi(a) = tp(a/M_0)$.*

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- *Then for every definable (with parameters from $\bar{M}$) subset Y of X, there is a closed subset E of $S_X(M_0)$ of μ-measure 0, such that for all $p \in S_X(M_0)$ such that $p \notin E$, either $\pi^{-1}(p) \subset Y$ or $\pi^{-1}(p) \cap Y = \emptyset$.*

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- Let E be the (closed) subset of $S_X(M_0)$ consisting of those p which are consistent with both $x \in Y$ and $x \notin Y$.
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- Then $(\mu)_E$ has two different extensions to a Keisler measure over $\bar{M}$, one giving Y measure 1 and one giving Y measure 0.

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- From which it follows that μ itself has two different extensions to $\bar{M}$, contradicting smoothness.

$\square$

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- E_b clearly only depends on $tp(b/M)$, so we write E_b as E_q where $q = tp(b/M)$. Let Z_q be an M-definable set containing E_q with $\mu(Z_q)$ with μ-measure $< \epsilon$.

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- By compactness we can partition $V \setminus Z_q$ into M-definable sets $V_{q,1}, ..., V_{q,n_q}$ such that for each i, $\pi^{-1}(V_{q,i})$ is either contained in $R(x, b)$ for some/all b realizing q, or is disjoint from $R(x, b)$ for some/all b realizing q.

Regularity theorem for smooth measures VI

- By compactness we can replace q by a formula (or M-definable set) W_q in q such that for all $i = 1, .., n_q$, either $V_{q,i}$ is contained in $R(x, b)$ for all $b \in W_q$, or $V_{q,i}$ is disjoint from $R(x, b)$ for all $b \in W_q$.

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- And for $(i, q_j) \notin E$, V_i must be contained in $V_{q_j,s}$ for some s, so by (*) $V_i \times W_{q_j}$ is contained in or disjoint from R.

Regularity theorem for smooth measures VII

- Let $V_1, .., V_t$ be a common refinement of this finite collection of partitions of V (into M-definable sets).
- We claim that this partition of V together with the partition $W = W_{q_1} \cup ... \cup W_{q_m}$, is as required.
- It is just a matter of identifying the exceptional set and checking the measure constraint.
- Let $E = \{(i, q_j) : V_i \subseteq Z_{q_j}\}$.
- For each q_j, $\cup_{(i,q_j)\in E} V_i \times W_{q_j} = Z_{q_j} \times W_{q_j}$ which has $\mu \times \nu$ measure $< \epsilon\nu(W_{q_j})$.
- Summing over the q_j, gives $(\mu \times \nu)(\cup_{(i,q_j)\in E}(V_i \times W_{q_j})) < \epsilon$.
- And for $(i, q_j) \notin E$, V_i must be contained in $V_{q_j,s}$ for some s, so by (*) $V_i \times W_{q_j}$ is contained in or disjoint from R.
- End of proof of Lemma 0.24.

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- The Vapnik-Chervonenkis theorem is a uniform law of large numbers for "families of events" with finite VC dimension.
- It has the following consequence for Keisler measures:
- Suppose $\mu(x)$ is a Keisler measure over M. Let $\phi(x,y)$ be an L-formula which has k-NIP.
- Then for any ϵ, there is $N = N_{k,\epsilon}$ depending only on k and ϵ, such that there are $p_1(x), .., p_N(x) \in S_x(M)$, such that for all $b \in M$, $\mu(\phi(x,b))$ is within ϵ of the proportion of the $p_1, .., p_N$ which contain $\phi(x,b)$.

NIP and pseudofinite measures II

- In the special case when A is a finite set of tuples from M of the appropriate length, and $\mu = \mu_A$ is the counting measure with respect to A (which we could recall), then this says that there are $a_1, .., a_N \in A$ such that for all $b \in M$, $\mu(\phi(x, b))$ is within ϵ of the proportion of the $a_1, .., a_N$ which satisfy $\phi(x, b)$.

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- We conclude the following:

Lemma 0.25

Suppose M is a model of an NIP theory, A is a subset of $X(M)$ for some sort X, A is pseudofinite in M, (M, A) is saturated (?), and $\mu(x)$ is a pseudofinite counting measure on $X(M)$ (over M) given after Lemma 0.8. Then μ is generically stable.

NIP and pseudofinite measures III

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- So we know (from Lemma 0.7 and the construction) that $\mu(Z)$ is the standard part of $|Z \cap A|/|A|$ for Z a definable subset of $X(M)$, and where $|.|$ denotes cardinality in $\mathbb{V}^*$ (which is finite in the sense of $\mathbb{V}^*$ for A and its internal subsets).

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- Fix a formula $\phi(x, y)$ of L which we know has k-NIP in M, for some k, so we may assume that in every relevant (M', A') with A' finite, $\phi(x, y)$ has k-NIP in M'.

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- Fix a formula $\phi(x, y)$ of L which we know has k-NIP in M, for some k, so we may assume that in every relevant (M', A') with A' finite, $\phi(x, y)$ has k-NIP in M'.
- So fixing $\epsilon > 0$ and letting $N = N_{k,\epsilon/2}$ be as above, it follows that there are $a_1, .., a_N$ in A such that for any $b \in M$, $|\phi(x, b)(M) \cap A|/|A|$ is within $\epsilon/2$ of the proportion of the a_i which satisfy $\phi(x, b)$ in M.

NIP and pseudofinite measures IV

- So for each $b \in M$, $\mu(\phi(x,b))$ is within ϵ of the proportion of the a_i which satisfy $\phi(x,b)$ in M.

NIP and pseudofinite measures IV

- So for each $b \in M$, $\mu(\phi(x,b))$ is within ϵ of the proportion of the a_i which satisfy $\phi(x,b)$ in M.
- By Lemma 0.21, μ is generically stable, completing the proof of Lemma 0.25.
- Assuming that we have a good notion of generically stable ϕ-measure where $\phi(x,y)$ is a NIP-formula, then the proof above will show that a pseudofinite counting measure, restricted to a NIP-formula $\phi(x,y)$, will be generically stable.

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- We prove Theorem 0.22. We will give the proof under assumption (ii) which is the context of the combinatoricists results on semialgebraic graphs, as well as the Chernikov-Starchenko theorem.

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- As in the proof of stable regularity, assume the conclusion fails. So there is $\epsilon > 0$ such that for every N there is a finite induced subgraph (V'_N, W'_N, R_N) for which there is no suitable partition (into at most N sets).
- We may assume that at least the cardinalities of the V_N are strictly increasing.

Proof of distal regularity II

- Add new predicates P and Q for the distinguished finite subsets of V, W, to get a family of $L(P, Q)$ structures, and as usual take a saturated model of the common $L(P, Q)$- theory of the (M, V_N, W_N).

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- Call this model (M^*, V^*, W^*) (where V^*, W^* are pseudofinite subsets of $V(M^*), W(M^*)$.
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- Both V^*, and W^* induce the pseudofinite counting measures μ, ν, on $V(M^*)$, $W(M^*)$ respectively.
- By Lemma 0.25, μ is generically stable (as is ν). By distality μ is also smooth.
- Fix ϵ and apply Lemma 0.24 with $\epsilon/2$ to $(V(M^*), W(M^*), R(M^*))$ equipped with μ and ν, to get a partitions of size n, m of the vertex sets with the appropriate properties.

Proof of distal regularity III

- Apply Lemma 0.8 to obtain (M, V_N, W_N) satisfying the appropriate formulas of set theory in $\mathbb{V}$, to get a contradiction, as in the proof of the stable regularity lemma.

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- Note that there is a difference with the stable proof, as the V_N, W_N etc are not in the language L.

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- And the "compact domination" statement for smooth measures (Lemma 0.23) on which 0.24 depends is replaced by a "generic compact domination" statement for generically stable measures.

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Lemma 0.26

Suppose T is NIP and $\mu(x)$ is a Keisler measure on a sort X over a model M_0, such that $\mu|M_0$ is generically stable. Let $\pi : X = X(\bar{M}) \to S_X(M_0)$ be as before. Let $Y \subseteq X$ be definable over $\bar{M}$. Then there is closed set $E \subseteq S_X(M_0)$ of μ-measure 0 such that all $p(x) \in S_X(M_0) \setminus E$, exactly one of $p(x) \cup$ "$x \in Y$" and $p(x) \cup$ "$x \notin Y$" is μ-random.

Remarks on the NIP case III

- Finally there is a regularity lemma just for finite bipartitite graphs (V, W, R) for which the edge relation R is k-NIP, or equivalently, as we have mentioned earlier, which omit a fixed induced subgraph. This is again proved by the combinatoricists, and in fact is a celebrated theorem of Lovasz-Szegedy, if I am not mistaken, and implies the results above.

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- Also proved later by Chernikov and Starchenko with model-theoretic methods.
- This could be obtained by our methods, given a generic compact domination theorem for generically stable ϕ measures where $\phi(x, y)$ is NIP.
- In any case the regularity lemma alluded to above, still has the exceptional pairs, but has ϵ-homogeneity rather than ϵ-regularity.

Arithmetic or group regularity lemmas; introduction I

- In this last part of the course we will discuss Szemeredi type theorems for the class of finite groups G equipped with a distinguished subset X.

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- First what can be said in general?
- In all the work by combinatoricists on this problem, there is a blanket assumption that G is commutative, probably so as to be able to use Fourier analytic methods.

Arithmetic or group regularity lemmas; introduction II

- As mentioned in the introduction from (G, X) we obtain a bipartitite graph (G, G, R) where $R(x, y)$ iff $xy \in X$, so one would expect some improved statement of Szemeredi regularity in which the group structure is respected in some sense.

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- Green's paper, A Szemeredi-type regularity lemma in abelian groups, GAFA, 2005, (possibly) initiated the topic, and has a rather complicted Fourier-analytic statement, which is difficult to parse.
- However when restricted to the class of finite-dimensional vector spaces over $\mathbb{F}^2$ (equipped with a distinguished subset X), it yields the following:

Arithmetic or group regularity lemmas III; introduction

Theorem 0.27

For every ϵ there is N such that for all (G, X) (where $G = \mathbb{F}_2^n$ some n), there is a partition of G into cosets $H + 0, H + g_1, .., H + g_k$ with respect to a subgroup (vector subspace) H of G of index at most N, such that outside a smal exceptional set of pairs, each graph $(H + g_i, H + g_j, R|((H + g_i) \times (H + g_j)))$ is ϵ-regular. (where remember the graph relation $R(x, y)$ is $x + y \in X$).

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- Terry and Wolf (TW) in 2017 considered the case where $G = \mathbb{F}_p^n$ for p fixed, AND where the relation $x + y \in X$ is k-stable, obtaining stronger structural results; X is almost a union of cosets of a subspace of small index.

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- Alon, Fox, and Zhao, subsequently considered the case where G is (finite) abelian and $x + y \in X$ is k-NIP.

Arithmetic or group regularity lemmas IV; introduction

- With Conant and Terry, we considered first arbitrary (G, X) where G is arbitrary (not necessarily abelian) and $xy \in X$ is k-stable, and then the more general case where $xy \in X$ is k-NIP.

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- One cannot expect such kinds of results in general, even when $G = \mathbb{F}_2^n$.
- However we do have a general rather soft "coset regularity" statement (for arbitrary (G, X)), which we may give later.
- In the next section we will state the "new" results (mainly from 2017-2018) and then discuss ingredients of the proof.

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Fix k. For any $\epsilon > 0$ there is N depending on ϵ (and k) such that for any pair (G, A) where G is a finite group and A is a k-stable subset, there is a normal subgroup H of G of index at most N, such that

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- *(i) for each coset C of H in G, either $|C \setminus A| \leq |H| (= |C|)$, or $|C \cap A| \leq |H|$. Moreover*

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- *(i) for each coset C of H in G, either $|C \setminus A| \leq |H| (= |C|)$, or $|C \cap A| \leq |H|$. Moreover*
- *(ii) There is a union Y of cosets of H such that $A = Y$ up to a set of cardinality $\leq \epsilon |H|$.*

Stable and NIP arithmetic regularity lemmas II

- When A is k-NIP, and G is of bounded exponent, we obtain the same conclusion, but now with an exceptional set of cosets of H.

Theorem 0.29

Fix k and r. Then for any $\epsilon > 0$ there is N such that for any pair (G, A) where G is a finite group of exponent $\leq r$ and A is a k-NIP subset of A, there is a normal subgroup H of G of index at most N, and a union Z of cosets of H (the exceptional set) with $|Z| \leq \epsilon|G|$ such that

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- In fact the $\mathbb{T}^n$'s are precisely the compact connected commutative Lie groups.
- So we define an (ϵ, n)-Bohr neighbourhood of a (possibly finite) group H to be the preimage of the open ball of radius ϵ around the identity under a homomorphism $\pi : H \to \mathbb{T}^n$.

Stable and NIP arithmetic regularity lemmas IV

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Fix k. Then for any $\epsilon > 0$, there is N (depending on ϵ and k) such that for any pair (G, A) where G is a finite group and A is a k-NIP subset of G, there are

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- *(ii) there is a union Y of translates of B such that A is equal to Y up to a set of cardinality $\leq \epsilon|B|$, after throwing away Z.*

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- Sufficiently large means that $|K| \geq \epsilon |H|$.
- This is a natural notion of regularity of a coset C of a subgroup H of G with respect to A, but where we only consider the densities with respect to large subsets of C which are themselves cosets of subgroups.

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Theorem 0.31

For any ϵ there is N, such that if (G, A) is any pair consisting of a finite group G and a subset A, then there is a normal subgroup H of index at most N, and a union Z of cosets of H (the exceptional set) with $|Z| \leq \epsilon|G|$, such that for any coset C of H in G such that C is not contained in Z, then C is ϵ-coset-regular with respect to A.

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- Note that when G is simple (noncommutative), Theorem 0.31 says that G is itself ϵ-coset regular. But anyway Theorem 0.31 is only meaningful when G has a reasonable supply of subgroups.

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- As before a δ-formula (over M) is a Boolean combination of formulas $\delta(x, b)$ for $b \in M$, and the subset of G it defines is called a δ-definable set. (We treat $x = x$, $x \neq x$ as degenerate δ-formulas, and sometimes we may want to include Boolean combinations of $x = g$ etc. too....)

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- With this notation and assumptions, here is the fundamental theorem of local stable group theory.

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Theorem 0.32

- *G has a smallest δ-definable subgroup of finite index which we call G^0_δ.*

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- *(iii) There is a unique left-invariant (Keisler) δ-measure on G, μ say, and moreover*
- *(iv) for any δ-definable set X, $\mu(X) > 0$ iff X is generic.*

Corollary 0.33

(In the same context as that of Theorem 0.32, and the same notation.)
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- But the crucial point is that the pseudofinite Keisler measure on G (coming from $\mathbb{V}^*$) will be left invariant, hence by the uniqueness aspect of Theorem 0.32(iii), must coincide on δ-definable sets with the δ-measure μ from Theorem 0.32.

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- This allows us to pull Theorem 0.32 and Corollary 0.33 to the finite (of course using some approximations) and obtain Theorem 0.28.
- Note that Theorem 0.28 also implies that k-stable sets in finite simple groups better be (asymptotically) either almost everything or almost nothing.

G/G^{00} I

- We saw in the discussion at the end of the last section that up to small cardinality, suitable subsets of the finite groups G are controlled by bounded index subgroups, i.e. all the action is going on in G/H for some bounded index subgroup H.

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- A slight variant of this will actually be the case in general. That is, roughly speaking, for an infinite pseudofinite group G with its pseudofinite Keisler measure μ, internal sets of positive measure will be controlled in a sense by a compact (rather than finite) quotient G/G^{00}.

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- And passing to approximations, this will be reflected in various ways in the finite.
- It is a rather surprisingly important role for thos compact group, although variants are also behind the classification of approximate subgroups.

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- Bounded index means of index at most $\leq 2^{|M|+|L|}$, which can be shown to be equivalent to $< \kappa$ where κ is the degree of saturation of $\bar{M}$.

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- In fact, because of the bounded index assumption, the coset of g modulo $(G^*)^{00}_M$ depends only on $tp(g/M)$, whereby the canonical homomorphism from G^* to $(G^*)^{00}_M$ factors through the tautological map to the type space $S_G(M)$, and this equips $G^*/(G^*)^{00}_M$ with its compact Hausdorff topology. It is a definable groups analog of the so-called KP Galois group.

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- Likewise we could consider a collection Δ of L-formulas $\delta(x,y)$ (or even a single such formula), and consider $(G^*)^{00}_{M,\Delta}$, the smallest subgroup of G^* of "bounded index" defined by a collection of Δ-formulas over M. (Not necessarily normal any more.)

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- In particular we have an exact sequence $1 \to H^0 \to H \to H/H^0 \to 1$, where H^0 denotes the connected component of the identity of H as a topological group;

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Lemma 0.34

Suppose G is a pseudofinite group, considered as definable in the structure $M = \mathbb{V}^*$. Then the definable Bohr compactification of G is profinite-by-commutative, that is the connected component of $G^*/(G^*)^{00}_M$ (as a topological group) is an inverse limit of connected commutative compact Lie groups.

Local generic compact domination I

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- We work in a saturated model $\bar{M}$, and one of the main results is "generic compact domination" of G by G/G^{00}.
- What this means (at least one form), is that given a definable (with parameters from $\bar{M}$) subset Y of G, there is a closed $E_Y \subset G/G^{00}$ of (normalized) Haar measure 0, such that for all cosets C of G^{00} outside E_Y, not both "$x \in C \wedge x \in Y$" and "$x \in C \wedge x \notin Y$" are μ-random.

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- This implies in particular that μ is the unique translation invariant measure on G.

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- In work with Conant, we developed such a theory, but assuming also pseudofiniteness. It is an analogue of the fundamental theorem of local stable group theory.
- Together with Lemma 0.34, which explains where the Bohr neigbourhoods come from, this will suffices to prove Theorem 0.30.

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Theorem 0.35

- *(i) There is a unique left invariant Keisler δ-measure μ on G.*
- *(ii) The δ-definable sets of positive μ-measure are precisely the (left) generic δ-definable sets.*
- *(iii) Given a δ-definable (over $\bar{M}$) set $Y \subseteq G$, there is a closed subset $E_Y \subset G/G^{00}_{\delta}$, of μ-measure 0 such that for $C \in G/G^{00}_{\delta}$, $C \notin E_Y$, exactly one of $x \in C \cup x \in Y$, $x \in C \cup x \notin Y$ is μ-random (equivalently by (ii) extends to a global generic type).*

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