Model theory, stability theory, and the free group

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1 Overview

Overview

- I will take as a definition of model theory the study (classification?) of first order theories T.
- A "characteristic" invariant of a first order theory T is its category $Def(T)$ of definable sets. Another invariant is the category $Mod(T)$ of models of T.
- Often $Def(T)$ is a familiar category in mathematics. For example when $T = ACF_0$, $Def(T)$ is essentially the category of complex algebraic varieties defined over Q.
- Among first order theories the "perfect" ones are the *stable theories*.
- *Stability theory* provides a number of tools, notions, concepts, for understanding the category $Def(T)$ for a stable theory T.
- Among stable theories are the theory of algebraically closed fields, the theory of differentially closed fields, as well as the theory of abelian groups (in the group language).
- An ingenious method, "Hrushovski constructions", originally developed to yield counterexamples to a conjecture of Zilber, produces new stable theories with surprising properties.
- However, modulo the work of Sela, nature has provided us with another complex and fascinating stable first order theory, the theory T_{fg} of the noncommutative free group.
- One can argue that the true "algebraic geometry over the free group" should be the study of $Def(T_{fg})$.

- Some references are given at the end of the notes. The references (1), (2) include all the material covered in the first section of these notes (and much more).
- I have given several references (3)-(6) for stability and stable groups. But let me mention here that the first chapter of reference (5) (my Geometric Stability Theory) gives an exposition of the basics of stability and stable groups, more or less in the style of these lecture notes.

2 Model Theory

Beginning model theory

- We fix a "language" L consisting of some relation symbols R_i, function symbols f_j and function symbols c_k (countable if you wish), where each R_i and f_j comes equipped with an "arity" (positive integer). We impose that the relation symbols include a privileged binary relation symbol $=$.
- From this, together with the logical connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\exists$, $\forall$, an infinite supply of "variables" $x_0, x_1, x_2, ...$ and maybe some parentheses (,), we build up the collection of *first order L-formulas*.
- Bear in mind that when L is L_g the language of groups, we have only function symbols for "multiplication" and "inversion" and a constant symbol for the identity, and no relation symbols other than equality.

- In fact we first define L-terms: A constant symbol or variable is a term, and if $t_1, .., t_n$ are terms and f is an n-ary function symbol then $f(t_1, .., t_n)$ is a term.
- Now for L-formulas: If $t_1, .., t_n$ are L-terms and R and n-ary relation symbol then $R(t_1, ., t_n)$ is an L-formula (called an *atomic* L-formula). E.G. $t_1 = t_2$
- If ϕ, ψ are L-formulas, so are $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $(\neg\phi)$, $(\exists x \phi)$ and $(\forall x \phi)$ (where x is any variable).
- There is an obvious notion of an occurrence x of a variable in a formula ϕ being "in the scope of a quantifier $\exists$ or $\forall$", or as we say *bound*. The free variables of ϕ are by definition the variables which have some non bound occurrence in ϕ. EXAMPLE!!

- If a formula ϕ has free variables among $x_1, .., x_n$ we sometimes write ϕ as $\phi(x_1, .., x_n)$. A formula with no free variables is called an L-sentence, sometimes denoted by $\sigma, \tau, ...$
- Now we bring in structures. By an L-structure M we mean a set (or universe), sometimes notationally identified with M, together with for each relation R, function f, or constant symbol c of L a corresponding relation $R^M \subseteq M \times .. \times M$, function $f^M : M \times .. \times M \to M$, of appropriate arities, and element $c^M \in M$. E.G. for $L = L_g$ any group $G = (G, \cdot, ^{-1}, e)$ is an L_g structure.
- Finally truth: for an L-structure M, and L-formula $\phi(x_1, .., x_n)$ and $a_1, .., a_n$ in (the universe of) M, we can define (inductively) "$M \models \phi(a_1, .., a_n)$", in words "$\phi(x_1, .., x_n)$ is true in M when x_i is assigned a_i for $i = 1, .., n$" or even "$\phi(x_1, ., x_n)$ is true of $(a_1, .., a_n)$ in M".
- Again there are several steps involved. The first is for a term $t(x_1, .., x_n)$ (i.e. where the variables in t are among $x_1, .., x_n$) and $a_1, .., a_n \in M$, to define (inductively) $t^M(a_1, .., a_n)$ (the interpretation of t in M when a_i is assigned to x_i for $i = 1, .., n$.).
- The next step, is for an atomic formula ϕ of the form $R(t_1, .., t_k)$ with (free) variables among $x_1, .., x_n$, and $a_1, .., a_n \in M$, $M \models \phi(a_1, .., a_n)$ if $(t_1^M(a_1, .., a_n), ..., t_k^M(a_1, .., a_n)) \in R^M$.
- The key inductive step involves the quantifiers. So suppose ϕ has free variables among $x_1, .., x_n$ and has the form $\exists x_{n+1}\psi$ (so ψ has free variables among $x_1, .., x_n, x_{n+1}$).
- Then $M \models \phi(a_1, .., a_n)$ if for some $a_{n+1} \in M$, $M \models \psi(a_1, .., a_n, a_{n+1})$.
- Note there is nothing special in this definition about the choice of variables $x_1, .., x_n$.
- What we have essentially defined is: for a formula ϕ with free variables among $y_1, .., y_k$ say, and $a_1, .., a_k \in M$, when ϕ is true in M under the assignment of a_i to y_i for $i = 1, .., k$.
- When σ is an L-*sentence*, then we read $M \models \sigma$ as "σ is true in M", or M is a model of σ. And for Σ a set of L-sentences, $M \models \Sigma$ means $M \models \sigma$ for all $\sigma \in \Sigma$ and is read M is a model of Σ.

- There are obvious notions of substructure, extension, isomorphism, for L-structures, but the key model-theoretic notion is *elementary substructure/extension*: Suppose $M \subseteq N$ are L-structures. We say that M is an elementary substructure of N (N is an elementary extension of M), notationally $M \prec N$, if for every L-formula $\phi(x_1.., x_n)$ and $a_1, .., a_n \in M$, $M \models \phi(a_1, .., a_n)$ iff $N \models \phi(a_1, .., a_n)$. ($(N, <)$ is a substructure of $(Z, <)$ but not an elementary substructure.)

- Finally we have *elementary equivalence*: L-structures M, N are elementarily equivalent if for every L-sentence σ, $M \models \sigma$ iff $N \models \sigma$. (So elementary substructure implies elementary equivalence, BUT $(2Z, <)$ is a substructure of $(Z, <)$, they are isomorphic so elementarily equivalent, and NOT $(2Z, <) \prec (Z, <)$.)

$Mod(T)$ **and** $Def(T)$

- A first order L-theory T is simply a collection of L-sentences which has a model (usually T is also assumed to be closed under "logical consequence").

- $Mod(T)$ is the category whose objects are models of T and whose morphisms are elementary embeddings (where an elementary embedding of M into N is an isomorphism of M with an elementary substructure of N).

- Fix an L-theory T. L-formulas $\phi(x_1, .., x_n)$, $\psi(x_1, .., x_n)$ are said to be *equivalent modulo* T if the L-sentence $\forall \bar{x}(\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$ is true in all models of T (we sometimes write $T \models \forall \bar{x}(\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$ for this).

- $B_n(T)$ denotes the *Boolean algebra* of equivalence classes.

- For $\phi(x_1, .., x_n)$ an L-formula and $M \models T$ (or any L-structure for that matter) ϕ^M denotes $\{\bar{a} \in M^n : M \models \phi(\bar{a})\}$, the *set defined by* $\phi(x_1, .., x_n)$ *in* M. It clearly depends only on the equivalence class of ϕ.

- T is said to be *complete* if any two models of T are elementarily equivalent (equivalently for any L-sentence σ, $\sigma \in T$ or $\neg\sigma \in T$).

- If T is complete, and M is ANY model of T, then $B_n(T)$ identifies with the Boolean algebra of subsets of M^n defined by L-formulas.

- A morphism between $[\phi(x_1, .., x_n)] \in B_n(T)$ and $[\psi(y_1, .., y_k)] \in B_k(T)$ is given by (the equivalence class modulo T) of some formula $\chi(x_1, .., x_n, y_1, .., y_k)$ which in any model M of T defines the graph of a function from ϕ^M to ψ^M.

- We can now define $Def(T)$ as the category whose set of objects is the disjoint union of the $B_n(T)$ (as n varies) and with morphisms as defined above.
- Again when T is complete, and M any model of T, $Def(T)$ identifies with the category of sets and functions defined by L-formulas in M.
- We may and should consider also sets definable *with parameters*. There are different formalisms for this.
- For example, fix some L-structure M and some subset A of the universe of M. Let L_A be the language obtained from L by adding new constant symbols c_a for the elements $a \in A$.
- Then if $\phi(x_1, .., x_n)$ is an L_A formula, it will define naturally a subset of M^n which we call a set definable in M *over* A and again write as ϕ^M.
- Let us make the connection with $Def(T)$. M can be canonically viewed as an L_A-structure which we write as $(M, a)_{a \in A}$. Let T_A be the "complete theory" of this L_A-structure, namely the set of L_A-sentences true in $(M, a)_{a \in A}$.
- Then we obtain, as before, the category $Def(T_A)$ which identifies with the category of sets and functions definable in M over A.
- A final remark is that that, for T a theory and $\phi(\bar{x})$ a formula, it is useful to regard ϕ (or $[\phi]$) as a functor from $Mod(T)$ to Set, namely the functor just takes M to ϕ^M.

Types

- There are some fundamental compact (and totally disconnected) spaces attached to a first order theory T, the *type spaces*.
- $S_n(T)$ is by definition the Stone space of the Boolean algebra $B_n(T)$, that is the collection of ultrafilters on $B_n(T)$.
- The topology on $S_n(T)$ is given by: a basic open (in fact clopen) is $\{p \in S_n(T) : \phi(\bar{x}) \in p\}$ for $\phi(\bar{x})$ an L-formula.
- A type $p(\bar{x}) \in S_n(T)$ gives, potentially, a complete description of the first order properties of some n-tuple $\bar{a}$ in some model of T.
- To actualize this potential description depends on the *compactness theorem* for first order logic.

- In its basic form the compactness theorem says that if Σ is a collection of L-sentences such that every finite subset Σ_0 of Σ has a model, then Σ has a model.
- An application of the compactness theorem (via adjoining constant symbols to the language) yields the following: Let $p(\bar{x}) \in S_n(T)$. Then there is a model M of T and n-tuple $\bar{a}$ from M such that for all L-formulas $\phi(\bar{x})$, $M \models \phi(\bar{a})$ if and only if $\phi(\bar{x}) \in p$.
- In this situation we say that $\bar{a}$ *realizes* $p(\bar{x})$ in M, and that $p(\bar{x})$ is the *type* of $\bar{a}$ in M, $p(\bar{x}) = tp_M(\bar{a})$.
- All this can again be done "over parameters": Namely fix a model M of T and a subset A of M. Then any $p(\bar{x}) \in S_n(T_A)$ can be *realized* in some elementary extension N of M by some n-tuple $\bar{a}$, and we say that $p(\bar{x})$ is the type of $\bar{a}$ over A in N, $p(\bar{x}) = tp_N(\bar{a}/A)$. We sometimes write $S_n(A)$ for $S_n(T_A)$
- Looking ahead, a basic problem is to describe in some fashion $S_n(T_{fg})$.
- It makes perfect sense to speak of the space of complete "quantifier-free" n-types of a theory T, $S_{n,qf}(T)$, consisting of complete descriptions of the quantifier-free formulas true of a potential n-tuple in a model of T.
- And in the case of T_{fg} these complete quantifier-free n-types are in natural $1-1$ correspondence with the collection of isomorphism types of pairs $(G, \bar{a})$ where G is a limit group and $\bar{a}$ an n-tuple which generates G.
- This makes use of the characterization of limit groups as ω-residually free, and the compactness theorem.

Saturation

- I will now assume, for convenience, that T is a *complete* first order L-theory.
- Does T have "canonical models" from the point of view of model theory?
- This is different from asking whether T has "natural" models. For example (looking ahead) when $T = T_{fg}$ the natural, or standard, models are precisely the free groups F_n ($n \geq 2$, maybe infinite).
- In any case the answer to the canonical models question is *yes*, modulo some set theory (such as assuming GCH).
- Fix an infinite cardinal κ and a model M of T of cardinality κ.
- We will say that M is saturated if for any subset A of M of cardinality $< \kappa$, any $p(\bar{x}) \in S_n(T_A)$ (i.e. complete type over A) is realized in M.

- An equivalent, and possibly more accessible characterization is: whenever A is a subset of M of cardinality $< \kappa$ and $\Sigma(x)$ is a collection of formulas of L_A with free variable x, such that for every finite subset Σ_0 of Σ, $M \models \exists x(\wedge\Sigma_0(x))$, then there is $a \in M$ such that $M \models \phi(a)$ for all $\phi(x) \in \Sigma(x)$.

Theorem 2.1. *(i) Any two saturated models of T of the same cardinality are isomorphic.*
(ii) If M is a saturated model of T of cardinality κ then any model of T of cardinality $\leq \kappa$ elementarily embeds in M.
(iii) Under GCH for example, for any regular cardinal κ, T has a saturated model of cardinality κ.

- So saturated models are our canonical models.
- They are all "essentially" the same apart from the accidental fact of having different cardinalities.
- Saturated models are analogous to Grothendieck's universes or Weil's universal domains.
- If T is stable we don't need any set-theoretic hypothesis to get saturated models.
- It has become customary to "work inside a saturated model of large cardinality" of T in order to either study models of T or definability in T.
- There are several important consequences/properties of saturated models and I will discuss two, homogeneity, and definable families.

- The first (homogeneity) concerns the ubiquity of automorphisms and/or an automorphism-theoretic accouunt of types.

Lemma 2.2. *Suppose that $M \models T$ has cardinality κ and is saturated. Let $A \subset M$ have cardinality $< \kappa$ and let $\bar{b}, \bar{c}$ be two n-tuples from M (n finite). Then $tp_M(\bar{b}/A) = tp_M(\bar{c}/A)$ if and only if there is an automorphism g of M which fixes A pointwise and takes $\bar{b}$ to $\bar{c}$.*

- A very important notion in model theory is that of a definable family of definable sets (sometimes called a uniformly definable family of definable sets).
- This is simply given by an L-formula ϕ together with a division of the free variables in ϕ into two tuples of variables x, y, and so we may write ϕ as $\phi(x, y)$. For any choice of a tuple b in a model M (where the length of b is the length of the tuple y of variables), we obtain the definable set $\phi(x, b)^M$, a subset of M^n if x is an n-tuple of variables).

- The family $\{\phi(x,b)^M : b \in M\}$ is then a definable family of definable sets.
- For example, working with the structure $(C, +, \times, 0)$, the collection of all lines in $C \times C$ is the definable family of definable sets defined by the formulas $y = ax + b$, as (a, b) varies in C^2.

- A *special case* of a definable family of definable sets is a "definable fibration".
- This is the case when the formula $\phi(x, y)$ defines a partial function from x-space to y-space: namely the sentence $\forall x \exists^{\leq 1} y(\phi(x, y)$ is in the theory T (so true in all models).
- Fixing a model M, and writing X for $\exists y(\phi(x,y))^M$, and Y for $\exists x(\phi(x,y))^M$, we see that the formula $\phi(x, y)$ defines a surjective function $f : X \to Y$, and the family $(X_b)_{b \in Y}$ of fibres is the definable family of definable sets given by $\phi(x, y)$.
- The question is whether we really see all possible properties or behaviour of the fibres X_b in the model M.

- Of course passing to an elementary extension N of M gives rise to the family which I will write $f(N) : X(N) \to Y(N)$, and if $b \in Y(N) \setminus Y$ then $X(N)_b$ is a "new fibre". But if for some $c \in Y$, $tp_N(b) = tp_M(c)(= tp(_N(c))$, then the new fibre $X(N)_b$ will have similar properties to X_c (e.g. one is infinite iff the other is infinite).
- So if M is already saturated then all possible types $p(y)$ containing the formula $\psi(y)$ will be *realized* in M and so all possible types of fibres X_b will be already present.
- Everything I said about definable families makes sense for definable with parameters families.
- In general it is properties of definable families (essentially of L-formulas $\delta(x, y)$) which we use to define and distinguish classes of first order theories, as we see in the next section.

Example

- A basic "nice" example for model theory in general and stability in particular is the theory ACF_0 of algebraically closed fields of characteristic 0 (characteristic p too), and there is no harm in repeating it.
- The language here is the language $L_r = \{+, -, \cdot, 0, 1\}$ of rings. It is not hard to write down an (infinite) set of L_r-sentences whose models are precisely the algebraically closed fields of characteristic 0.

- This set of sentences (or rather its logical closure) is ACF_0.
- ACF_0 is complete, which means that all algebraically closed fields of characteristic 0 are elementarily equivalent (a special case of the Lefschetz principle).

- ACF_0 has *quantifier elimination* meaning that every L_r-formula $\phi(x_1, .., x_n)$ is equivalent modulo ACF_0 to a formula $\psi(x_1, .., x_n)$ without quantifiers.
- Any model of ACF_0 is determined up to isomorphism by its transcendence degree (over Q).
- Any model of infinite transcendence degree is saturated, in particular every uncountable model is saturated.
- Another basic example is where T is the theory of $(Z, +, 0)$ and where the saturated models are of the form $\hat{Z} \oplus Q^{(\kappa)}$.

3 Stability

Beginning stability

- Let us fix a (first order) L-theory T, which we assume to be complete.
- $x, y, z, ..$ now range over finite tuples of variables. When we write an L-formula δ as $\delta(x, y)$ we mean as above that x, y include all free variables in δ and we do not insist that there is no overlap.
- So a formula $\delta(x, y)$ gives rise to a definable family (as above, in fact in two ways), but could also be viewed as defining in any model M a "bipartitite" graph, namely $((\exists y\delta(x, y))^M, (\exists x\delta(x, y))^M, \delta(x, y)^M)$.

Definition 3.1. (i) The formula $\delta(x, y)$ is *stable* (for T) if it NOT the case that there is some model M of T and tuples a_i, b_i in M for $i \in N$ such that for all $i, j \in N$, $M \models \delta(a_i, b_j)$ iff $i < j$.
Equivalently (by the compactness theorem) there is some $k \in N$ such that for some (any) model M of T, it is not the case that there are a_i, b_i in M for $i < k$ such that $M \models \delta(a_i, b_j)$ iff $i < j$.
(ii) T is stable if every L-formula $\delta(x, y)$ is stable for T.

Lemma 3.2. *If $\phi(x, y)$, and $\psi(x, y)$ are stable, so is $\phi^*(y, x)$ $(= \phi(x, y)))$ as well as $\neg\phi(x, y)$, $\phi(x, y) \wedge \psi(x, y)$, $\phi(x, y) \vee \psi(x, y)$.*

- A interesting strengthening of the notion of a stable formula is an "equation".

Definition 3.3. (i) A formula $\delta(x, y)$ is an *equation in the sense of Srour*, if it is NOT the case that there exists a model M of T and a_i, b_i in M for $i \in N$ such that $M \models \delta(a_i, b_j)$ for all $i > j$ but $M \models \neg\delta(a_j, b_j)$ for all j.
(ii) T is said to be *equational in the sense of Srour* if every L-formula $\phi(x, y)$ is equivalent modulo T to some "finite Boolean combination" of equations $\delta(x, y)$.

- For $\delta(x, y)$ to be an equation is a *Noetherian* condition on the uniformly definable family $(\delta(x, b))_b$ of definable sets: In any model, any intersection is a finite subintersection.

Lemma 3.4. *(i) Equations are preserved under positive Boolean combinations (but not negations).*
(ii) If $\delta(x, y)$ is an equation, then $\delta(x, y)$ is stable.
(ii) So by (ii) and Lemma 3.2, if T is equational then it is stable.

- The basic examples of stable theories are equational and this moreover "explains" stability.

- For example we mentioned that ACF_0 has quantifier elimination implying that any L_r-formula $\phi(x, y)$ is equivalent modulo ACF_0 to a Boolean combination of formulas of the form $P(x, y) = 0$ where P is a polynomial over Z.

- Clearly $P(x, y) = 0$ is an equation in the sense of Srour. In fact the Noetherian condition holds for the solution sets of all $P(x, b) = 0$ even as P varies.

On the other hand:

Theorem 3.5. *(Sela,...) Let T_{fg} be the first order theory of the noncommutative free groups F_n ($n \geq 2$) in the group language L_g (i.e. the collection of L_g-sentences true in ALL the F_n). Then*
(i) T_{fg} is complete (Tarski's problem).
(ii) Any formula $\phi(\bar{x})$ of L_g is equivalent modulo T_{fg} to a Boolean combination of $\forall\exists$ formulas. (EXPLAIN.)
(iii) T_{fg} is stable.
(iv) T_{fg} is NOT equational.
(v) T_{fg} is decidable (?)
(vi) The natural embeddings of F_n in F_m ($n < m$) are elementary *embeddings.*

Example

- I want to describe a or the characteristic feature of stability of a theory T.

- A motivating example again comes from ACF_0 (or ACF_p).

- Fix an algebraically closed field K, an algebraically closed subfield $k < K$ and an irreducible (affine say) variety X over k, which we identify with its set $X(K) \subseteq K^n$ of K-rational points, a definable (with parameters from k) set in K.

- Then (i) the collection of proper subvarieties of X defined over K is a (proper) ideal in the Boolean algebra of definable (over K) subsets of X (by virtue of irreducibility of X).

- But more importantly, by virtue of QE, there is a *unique* type $p(\bar{x}) \in S_n(K)$ which contains (the formula defining) X and the negation (complement) of any (formula defining a) proper subvariety of X, i.e. p avoids the ideal mentioned above.

- $p(\bar{x})$ is none other than the *generic point* of $X \otimes_k K$, viewed as a scheme over K.

- If we realize $p(\bar{x})$ by a point $\bar{a}$ in an (elementary) extension K' of K then we call $\bar{a}$ a generic point of X over K (following Weil) and any two such generic points will be conjugate via $Aut(K'/K)$.

- So suitable definable sets (irreducible varieties) have "unique" generic points over any set of parameters, where the notion of "generic" appears to depend on facts about ACF_0.

- A similar kind of analysis works for equational theories.

- For an arbitrary stable theory where there is no a priori distinction between "positive" (closed) and "negative" (open) formulas or definable sets, we must work with (the set of realizations of) a *complete type* $p(\bar{x})$ rather than an irreducible closed set, and with the ideal of "small" formulas given by Shelah's combinatorial notions of forking (or dividing).

- Another point I will want to make (relevant to the free group) is that theory is "local", working for so-called $\phi(x, y)$-types, where $\phi(x, y)$ is a stable formula in a possibly unstable theory.

Forking

- We fix a saturated model $\bar{M}$ of a complete first order theory T which we will later assume to be stable. As mentioned before all models of T of cardinality at most that of $\bar{M}$ can be assumed to be elementary substructures of $\bar{M}$.

- A pervasive notion in model theory is that of an *indiscernible sequence*: Let $I = (b_i : i \in N)$ be a sequence of n-tuples from $\bar{M}$, and A a subset of $\bar{M}$. We say that I is indiscernible over A, if for any $i_1 < i_2 < .. < i_k$, and $j_1 < j_2 < .. < j_k$ in N, $tp(b_{i_1}, ..., b_{i_k}/A) = tp(b_{j_1}, .., b_{j_k}/A)$.

- Now fix a "small" set A of parameters, and a formula $\phi(x, b)$ over $\bar{M}$ where we make explicit the parameters b in the formula.

- We say that $\phi(x, b)$ *divides over* A if there is some indiscernible sequence $(b_i : i \in N)$ over A, with $b_0 = b$ such that the set $\{\phi(x, b_i) : i \in N\}$ is *inconsistent* i.e. has no common solution in $\bar{M}$ (which by compactness amounts to saying that for some k, $\bar{M} \models \neg\exists x(\bigwedge_{i<k} \phi(x, b_i)))$.

- Example: if $tp(b/A)$ has infinitely many solutions in $\bar{M}$ (we say $b \notin acl(A)$), then the formula $x = b$ divides over A (with in fact $k = 2$ above).

Lemma 3.6. *Assuming T stable, then the collection of formulas $\phi(x, b)$ which divide over A ($\phi(x, y)$ and b varying) is a proper ideal in the Boolean algebra $B_n(\bar{M})$ of formulas over $\bar{M}$.*

- In fact more is true. Let us fix a complete type $p(x) \in S(A)$, and let $X = p^M$ be the set of realizations of $p(x)$ in $\bar{M}$ (a so-called type-definable over A subset of $\bar{M}^n$). For a (small) set $B \supseteq A$ of parameters we will say that $c \in X$ is a *generic point* of X over B if $\bar{M} \models \neg\phi(c, b)$ whenever $\phi(x, b)$ is a formula over B which divides over A; namely if $tp(c/B)$ avoids the ideal of formulas which divide over A.

With this notation, assuming stability of T, we have the following "fundamental theorem of stability theory":

Theorem 3.7. *(i) There is a generic point c of X over B, and we call $tp(c/B)$ a "nonforking extension" of $p(x)$ over B.*
(ii) There are at most continuum many such distinct types $tp(c/B)$ for $c \in X$ generic over B.
(iii) Moreover if A is an elementary substructure M_0 of $\bar{M}$ (or more generally "algebraically closed") then there is a unique such type, i.e. X has a unique generic point over B (up to $Aut(\bar{M}/B)$-conjugacy), equivalently $p(x)$ has a unique nonforking extension over B.

In general we will say that a type $p(x) \in S(A)$ is *stationary* if it has a unique nonforking extension over any $B \supseteq A$.

- A brief comment on nomenclature.
- The reader might have expected us to talk about "nondividing extensions" rather than "nonforking extensions".
- There is a definition of a formula $\phi(x,b)$ forking over A which is a slight weakening of the notion dividing, but in stable theories is equivalent, and this equivalence is the content of Lemma 3.6.
- So this explains the use of "nonforking".

- So we have given a generalization of the "uniqueness of generic points of irreducible algebraic varieties" to arbitrary stable theories.
- Moreover the above machinery gives rise to a notion of independence or freeness in stable theories with symmetry, transitivity properties.
- With previous notation, we see that $c \in X$ is a generic point of $X = p^{\bar{M}}$ over B precisely if "$\bar{M} \models \neg\phi(c,b)$ whenever $\phi(x,b)$ is a formula over B which divides over A" and we will also say that *c is independent from B over A.*
- Then we have for example symmetry: c is independent from $A \cup \{b\}$ over A if and only b is independent from $A \cup \{c\}$ over A, and
- transitivity: Assuming $A \subseteq B \subseteq C$, c is independent from C over A iff c is independent from C over B and from B over A.

- The existence, symmetry and transitivity properties of dividing as discussed above are valid for a larger class of first order theories, the so-called "simple theories" (including theories of the random graph, pseudofinite fields,..) but not the uniqueness part which is characteristic of stability.
- In any case these "algebraic" properties of dividing/forking provide a useful "calculus" in stable theories.
- In a smaller case of stable theories, the *superstable theories*, types come equipped with an ordinal valued dimension, the U-rank, with the feature that for c and $A \subseteq B$, c is independent from B over A iff $U(tp(c/B)) = U(tp(c/A))$. Such ranks are computationally useful (but are not present in T_{fg} which is unsuperstable).
- And in the special case of ACF this U-rank of a type is just the algebraic geometric dimension of the associated algebraic variety.

- I will briefly discuss definability and finite satisfiability of types. We assume T to be stable.
- For any $M \models T$ and $p(x) \in S_n(T)$, $p(x)$ is *definable* meaning that for any L-formula $\phi(x, y)$, $\{b \in M : \phi(x, b) \in p(x)\}$ is a definable set in the structure M (and this property in turn implies stability of T).
- Moreover, if M_0 is an elementary substructure of M, the following are equivalent:
  (i) $p(x)$ is definable over M_0, (ii) $p(x)$ is finitely satisfiable in M_0, (iii) $p(x)$ is the nonforking extension of $p|M_0$.

The local theory

- Let T be an arbitrary complete theory, and let $\Delta(x)$ be a collection of stable formulas $\delta(x, y)$ (y varying with δ), for example $\Delta(x) = \{\delta(x, y)\}$ a single stable formula, or when T is stable, it might be all formulas $\delta(x, y)$. There is no harm to assume Δ is Booleanly closed.
- For $A \subseteq M \models T$, by a complete Δ type over A we mean an ultrafilter on the Boolean algebra $B_\Delta(A)$ generated by $\phi(x, b)$ for $\phi(x, y) \in \Delta$ and $b \in A$.
- Then the previous theory goes through, for example if $p(x) \in S_\Delta(M)$, and $M \prec N$ then $p(x)$ has a unique "nonforking" extension to some $p'(x) \in S_\Delta(N)$.
- And there are suitable versions of symmetry and transitivity.
- This formula-by-formula theory explains even the fully stable case, where the nonforking extension of a complete type $p(x) \in S(M)$ to N is just the union of the nonforking extensions of the various $\delta(x, y)$ types.
- Looking ahead a bit, for T_{fg}, and taking for Δ the quantifier-free formulas one obtains traditional "algebraic geometry over the free group", but looking at the case of diophantine sets could be interesting.

Stable groups

- The above theory of forking in stable theories has an "equivariant version" namely in the presence of a definable transitive group operation, due to Zilber in the $\aleph_1$-categorical case, Cherlin-Shelah in the superstable case, and Poizat in full generality.
- I will restrict myself to just groups (i.e. acting on themselves by left or right translation) rather than the more general homogeneous space case.
- By a stable group we mean in full generality a group definable in (a model of) a stable theory.

- For simplicity I will consider the case where T is a stable theory and the language includes suitable functions (for multiplication and inversion) and a constant (for the identity), suchg that with respect to the interpretations of these symbols any (some) model of T is a group.

- We allow of course other relations, functions etc in addition to the group structure, and definability will mean with respect to all this structure.

- I will use G in place of M (an arbitrary model) and $\bar{G}$ for a saturated model. x denotes a single variable (ranging over the group) rather than a tuple. Stability of T is assumed.

- Let X be a definable (with parameters) subset of $\bar{G}$ defined by a formula $\phi(x,b)$ say. We will say that X is left generic if for every $g \in \bar{G}$, the left translate gX (defined by $\phi(g^{-1}x,b)$) does not divide over $\emptyset$.

- For an arbitrary model $G \prec \bar{G}$ and X a definable (with parameters) subset of G we call X left generic if $\bar{X} \subset \bar{G}$ is generic where $\bar{X}$ is the set defined in $\bar{G}$ by the formula which defines X in G. (Maybe $X(\bar{G})$ for $\bar{X}$.)

With this notation:

Lemma 3.8. *Let X be a definable (with parameters) subset of G. Then the following are equivalent.*
(i) X is left generic,
(ii) Finitely many left translates of X (by elements of G) cover G.
Moreover we also have that X is left generic iff right generic (iff finitely many right translates of X cover G) and we just say "generic".

In particular we see that genericity of a definable subset X of G can be understood without passing to the saturated model $\bar{G}$.

- We will call G *connected* if G has no definable (with parameters from G) proper subgroups of finite index.

- Clearly this is a property of T, namely one model G is connected if all are.

- It is a fact coming from stability that if H is a definable subgroup of G of finite index, then H contains another subgroup of G of finite index which is definable without parameters.

- Hence we can define $\bar{G}^0$ (connected component of $\bar{G}$) to be the intersection of all ($\emptyset$-) definable subgroups of finite index. It will be a normal subgroup of $\bar{G}$ defined by at most countably many formulas, and the quotient $\bar{G}/\bar{G}^0$ is naturally a profinite group (finite iff $\bar{G}^0$ is definable). In general we need saturation of $\bar{G}$ to "see" $\bar{G}/\bar{G}^0$. In any case it is a fundamental invariant of T.

- EXAMPLE: When T is the theory of $(Z, +, 0)$ then $\bar{G}/\bar{G}^0$ is $\hat{Z}$, the profinite completion of the group Z.

- Now we discuss generic types. A complete type $p(x) \in S(A)$ (A any set of parameters) is said to be generic if it contains only generic formulas. It is a fact that a nonforking extension of a generic type is generic (and vice versa).

- Let G be a model, and let $S_{gen}(G)$ be the space of generic types over G. It is a closed subspace of the Stone space $S_x(G)$ and is invariant under the natural left or right action of G. (Note that G acts on the Boolean algebra of definable subsets of G so also on the space of ultrafilters.)

- Moreover there is a natural (continuous) group operation on $S_{gen}(G)$: given $p(x)$, $q(x)$ in $S_{gen}(G)$, let $p * q = tp(ab/G)$ where a, b realize p, q respectively in $\bar{G}$ in such a way that a is independent from b over G. (This of course has to be proved and is in fact the main content of the following theorem.)

The fundamental theorem of stable groups is the following:

Theorem 3.9. *Fix any $G \models T$. Then there is a natural isomorphism of topological groups between $\bar{G}/\bar{G}^0$ and $(S_{gen}(G), *)$.*

- Any type $p(x) \in S(G)$ picks out a coset of each finite index definable subgroup hence an element of $\bar{G}/\bar{G}^0$. The point of the theorem is that restricted to $S_{gen}(G)$ this induces a bijection, and moreover an isomorphism of topological groups.

- In particular if G is connected then there is exactly one generic type over any model and in fact over any set.

- In this context we call the unique generic type over $\emptyset$, p_0, and denote by $p_0(A)$ or $p_0|A$ the unique generic type over a set A of parameters.

- p_0 is stationary and for any A, $p_0|A$ is the unique nonforking extension of p_0.

4 The free group

Introduction

- I will first use some elementary methods to obtain information regarding "the" generic type p_0 of T_{fg}, and then discuss further results and problems.

- I will assume the statements (vi) and (iii) of Theorem 3.5, but it would be interesting to see to what extent things go through without assuming in advance stability of T_{fg}.

- Note that by 3.5 (vi) for any cardinals $2 \leq \kappa < \lambda$, the natural embedding of F_κ in F_λ is elementary, in particular every $F_\kappa \models T_{fg}$.
- By a "standard model" of T_{fg} I mean some finite rank free group F_n ($n \geq 2$).

Connectedness

Lemma 4.1. *(In (10) but essentially due to Poizat) Let $e_1, e_2, ...$ be free generators of F_ω. Let $X \subseteq F_\omega$ be definable (with parameters). Then X is generic if and only if it contains all but finitely many e_i.*

- Proof:
- Suppose first X to be generic and let $g_1 X \cup ... g_n X = F_\omega$.
- Let r be such that the parameters in the formula defining X as well as $g_1, .., g_n$ are all words in $e_1, .., e_r$.
- Let $i > r$. Then $e_i \in g_t X$ for some t. So $g_t^{-1} e_i \in X$.
- But there is an automorphism of F_ω which fixes $e_1, .., e_r$ and takes $g_t^{-1} e_i$ to e_i (as $\{e_1, .., e_r, g_t^{-1} e_i\} \cup \{e_j : j > r, j \neq i\}$ is also a basis of F_ω).
- So $e_i \in X$. We have shown left to right.
- Conversely if X is non generic then its complement X^c is generic (by stability), so by the first part X contains only finitely many of the e_i.

Corollary 4.2. *(In (10).) The free group is connected (no proper definable subgroup of finite index, and a unique generic type over any set of parameters).*

- In fact we can either use Lemma 4.1 to see that F_ω (so all models of T_{fg}) has no proper definable subgroup of finite index (and then use 3.9), OR directly, using stability of T_{fg} (definability of types) and Lemma 4.1, conclude that for any model G of T_{fg} and definable $X \subseteq G$ exactly one of X, $G \setminus X$ is generic, giving unique generic types over any model.
- As mentioned earlier we let $p_0(x)$ be the (unique) generic type over $\emptyset$, i.e. the collection of formulas $\phi(x)$ without parameters which are generic (or define generic sets in some/any model).
- We obtain a bit more from Lemma 4.1:

Corollary 4.3. *Any basis $\{e_1, .., e_n\}$ of a standard model F_n is an independent (in the stability-theoretic sense) set of realizations of p_0 in F_n (in other words $(e_1, .., e_n)$ realizes $p_0^{(n)}$). Likewise in any F_κ. In particular a primitive element in any F_κ realizes p_0 (where by definition a primitive element is a member of a basis).*

Weak homogeneity

I will prove a converse to Corollary 4.3:

Lemma 4.4. *(In (11).) (i) Any realization of p_0 in a standard model F_n (and in fact in any F_κ) is a primitive.*
(ii) Any maximal independent set of realizations of p_0 in a standard model F_n is a basis.
(iii) In particular we have "homogeneity" with respect to realizations of $p_0^{(k)}$ in standard models. (Explain.)

The proof uses a nice result of Chloé Perin:

Theorem 4.5. *(8) Any elementary substructure of a standard model F_n is a free factor of F_n (i.e. a free group F_k for some $2 \leq k \leq n$ with a basis which extends to a basis of F_n)*

- We sketch the proof of Lemma 4.4.
- Let $a_1, .., a_n$ be a basis of $F_n = F$. Let b realize p_0 in F. Let $F \prec F_{n+1}$ where the latter has basis $a_1, .., a_n, a_{n+1}$.
- Then b and a_{n+1} are independent realizations of p_0, hence by Corollary 4.3 (b, a_{n+1}) has the same type in F_{n+1} as a basis in F_2, whereby the subgroup G of F_{n+1} generated by $\{b, a_{n+1}\}$ is an elementary substructure (isomorphic to F_2).
- By Theorem 4.5, G is a free factor of F_{n+1}, so $\{b, a_{n+1}\}$ extends to a basis $\{b, a_{n+1}, c_1, .., c_{n-1}\}$ of F_{n+1}.
- Let ϕ be the retraction from F_{n+1} to F taking a_{n+1} to 1. Then $\{b, \phi(c_1), .., \phi(c_{n-1})\}$ generates F_n hence must be a basis.
- This proves (i), and (ii) is similar (and easier).
- Let us remark that natural extensions of Lemma 4.4 (ii), (iii) and Perin's Theorem 4.5 fail for free for free groups on infinitely many generators, and this is related to material in the next subsection.

Weight

- I want here to describe a result which may seem a bit esoteric to non stability-theorists, although the proof is at the classical combinatorial group theory level.
- As mentioned earlier, for a stable theory to be superstable means that an ordinal valued dimension (or rank) can be attached to any type, which moreover "reflects forking". Gibone and Poizat gave proofs of the non-superstability of T_{fg}.

- A consequence of superstability of a theory T is "strong stability", namely any type has *finite weight*, namely (working over some set of parameters in a saturated model), for any finite tuple a there is a finite bound on the cardinality of any *independent* set $\{b_i : i \in I\}$ such that a depends on b_i for each i.
- On the other hand a theory may be nonsuperstable but at the same time strongly stable. EXAMPLE?

So we prove the following strengthening of nonsuperstability of T_{fg}.

Theorem 4.6. *((11), (12)). T_{fg} is not strongly stable. In fact p_0 has "infinite weight". In fact in F_ω there is a realization a of p_0 and an independent set $(b_i : i \in N)$ of realizations of p_0 such that a depends on (forks with) b_i for all $i \in N$.*

We will use the following elementary consequence of the theory of Whitehead automorphisms of a free group.

Lemma 4.7. *If $\{a_1, .., a_n\}$ is a basis of F_n, $m \leq n$ and $k_1, .., k_m$ are all > 1, then $a_1^{k_1}...a_m^{k_m}$ is not a primitive.*

Corollary 4.8. *Let G be any model of T_{fg} and $a_1, .., a_m$ an independent set of realizations of p_0 in G. Suppose $k_1, .., k_m$ are all > 1. Then $a_1^{k_1}....a_m^{k_m}$ does not realise p_0 in G.*

- There is a "forking calculus" for "generics" in stable groups, remarkably similar to the theory of Whitehead automorphisms.
- In particular $(*)$: (if in a stable connected group G), a realizes p_0 and a is independent from $b \in G$, then each of ab, ba realizes p_0.

- Let us prove Theorem 4.6.
- This is an improvement, due to Sklinos (12), over the original proof of "infinite weight" in (11).
- Let $\{a_1, a_2, ...\}$ be a basis of F_ω.
- Let $b_1 = a_1a_2^2$, $b_2 = a_1a_2^3a_3^2$, $b_3 = a_1a_2^3a_3^3a_4^2$, $b_n = a_1a_2^3....a_n^3a_{n+1}^2$,.....
- We claim first that $\{b_i : i = 1, 2, ..\}$ is an independent set of realizations of p_0 in F_ω. The fact that each b_i realizes p_0 is due to $(*)$ above.
- But independence is because for each n, $\{b_1, .., b_n, a_{n+1}\}$ generates $F_{n+1} \prec F_\omega$, hence is a basis of F_{n+1}.

- Hence by 4.3, $\{b_1, .., b_n\}$ is an independent set of realizations of p_0 in F_{n+1} so also in F_ω.
- Secondly we claim that a_1 depends on each b_n:
- If not then also b_n is independent of a_1^{-1} so by (*), $a_1^{-1}b_n$ realizes p_0, namely $a_2^3 a_3^3 ... a_{n+1}^2$ realizes p_0 which contradicts Corollary 4.8. End of proof of 4.6.
- The same construction yields (12) non homogeneity of the free groups F_κ for κ uncountable, as well as that in F_ω not every maximal independent set of realizations of p_0 is a basis:
- For example, work in $F = F_\kappa$ generated freely by $\{a_1, a_2, ..., a_\alpha, ...\}$. Then as above $(a_1, a_2, ..., a_n, ..)$ and $(b_1, b_2, .., b_n, ...)$ have the same type in F.
- But the map taking a_i to b_i for $i = 1, 2, ..$ does not extend to an automorphism f of F. If it did, let c be such that $f(c) = a_1$. Now c (being a word in finitely many generators) will be independent of some a_n.
- But then by automorphism a_1 will be independent of b_n, giving a contradiction as above, and proving non homogeneity of F_κ.
- Finally, it is not difficult to see that $\{b_i : i \in N\}$ is a *maximal independent set of realizations of* p_0 in F_ω which is *not* a basis.
- Hence if G is the subgroup of F_ω generated by the b_i then $G \prec F_\omega$ but is not a free factor. So 4.4(ii) and Perin's 4.5 fail for F_ω.

Diophantine sets

- A positive primitive L_g-formula $\phi(x)$ (x a finite tuple of variables) is something of the form $\exists y(\wedge_{i=1,..,k} w_i(x, y) = 1)$ (where again y is some finite tuple of variables, and the w_i are just words (or terms).
- By a (strict) definable (with parameters) diophantine set in a model G of T_{fg} I mean a subset of G^n defined by a formula of the form $\phi(x, b)$ where b is a tuple from G and $\phi(x, z)$ is a positive primitive L_g-formula.
- Passing to finite disjunctions we get "diophantine formulas" and definable (with parameters) diophantine sets.

- Let me first recall that if G is a *commutative* group (still written in multiplicative notation and with x, z tuples of variables) and $\phi(x, z)$ is a positive primitive formula of L_g, then $\phi(x, 1)$ is an L^g-formula which defines a subgroup H say of G^n.

- And for any tuple b from G, $\phi(x, b)$ defines a coset of H.

- Hence, with respect to the first order theory of the commutative group G, a *p.p*- formula $\phi(x, z)$ is an *equation in the sense of Srour* in a very strong sense: any two "instances" $\phi(x, b)$, $\phi(x, c)$ of $\phi(x, z)$ are either equivalent or disjoint (in the obvious sense).

- I understand that Sela has proved "diophantine" equationality of T_{fg} namely that with respect to T_{fg} any positive primitive formula $\phi(x, z)$ is an equation in the sense of Srour.

- We recall that this means that in any model G of T_{fg}, if $b_1, b_2, ...$ are tuples from G (with length that of z) and if $X_i \subset G^n$ is that the set defined by $\phi(x, b_i)$ in G, THEN the sequence:

- $X_1 \supseteq X_1 \cap X_2 \supseteq X_1 \cap X_2 \cap X_3 \supseteq ....$ stabilizes, in fact (by compactness) stabilizes at some k which depends only on the formula $\phi(x, z)$, not on the model G or the b_i's.

- On the other hand we do not have the DCC (or Noetherianity) on the class of all (strict) diophantine sets:

Lemma 4.9. *There are infinite descending chains of (strict) diophantine definable sets. In fact T_{fg} is not "diophantine-superstable".*

- Proof outline (Sklinos-thesis).

- Work in F_ω with basis $\{a_1, a_2, ...\}$.

- Let $\phi_1(x)$ be $\exists y(x = a_1 a_2^2 y^{-2})$, $\phi_2(x)$ be $\exists y(x = a_1 a_2^2 (a_2 a_3^2 y^{-2})^{-2})$, $\phi_3(x)$ be $\exists y(x = a_1 a_2^2 (a_2 a_3^2 (a_3 a_4^2 y^{-2})^{-2})^{-2})$, etc.

- Then the diophantine sets defined by the $\phi_n(x)$ form a descending chain, and moreover

- letting $A_n = \{a_1 a_2^2, a_1 a_2^3 a_3^2, .., a_1 a_2^3, , a_n^3 a_{n+1}^2\}$, each ϕ_n is defined over A_n and $\phi_{n+1}(x)$ divides over A_n.

- END OF PROOF.

- Can we also witness "infinite weight" in a diophantine way?
- Sela has defined the "diophantine envelope" of a definable set (in a standard model), a kind of local diophantine Zariski closure, and a stability-theoretic interpretation would be useful.

Negligibility

- Bestvina and Feighn have introduced, in as yet unpublished work, what is in effect a *combinatorial* but still conjectural account of genericity for definable sets in a free group.
- Rather roughly speaking a subset X of $F_n = \langle a_1, .., a_n \rangle$ is *negligible* if for some $k > 0$ and $0 \leq \epsilon < 100$, for all $w \in X$ except finitely many, all but ϵ percent of w can be covered by at most k "pieces".
- Where a "piece" of w is a (proper) subword of w appearing in w in two "different ways".
- The "BF-conjecture or hypothesis" is that
- (*) if $X \subseteq F_n$ is *definable*, then either X or the complement of X is negligible.

- It follows quickly (using connectedness of the free group) that a definable (with parameters) subset of F_n is generic (finitely many translates cover F_n) iff it is non-negligible.
- The BF-conjecture provides a computational substitute for ranks (Morley rank, U-rank) in free groups, and I used a version of it (which was stated by Sela as a theorem) in (10) to show that that the free group is rather complicated from the point of view of "geometric stability theory" (non CM-trivial) , and hence "morally" can not be obtained via a Hrushovski-type construction.
- There are several appealing model-theoretic properties of the free group, the only proofs of which I know go through the BF-conjecture.
- These include:

- (i) If $G \models T_{fg}$ then the proper definable subgroups of G are precisely the centralizers $C_G(a)$ of elements of G. They coincide with the maximal abelian subgroups of G, and in "standard models" are cyclic.
- (ii) G is *definably simple* (no proper nontrivial definable normal subgroups), although G is not simple as an abstract group. This would be the only known example of a definably simple, non simple, stable group.

Axiomatizing p_0

- The issue here is to give an explicit axiomatization of the generic type p_0.
- Namely to describe explicitly a collection $\Sigma(x)$ of formulas (without parameters) $\phi(x)$, such that for any model G of T_{fg}, and $a \in G$, a realizes p_0 if and only if a satisfies all the formulas in $\Sigma(x)$.
- It is not hard to see that a single formula does not suffice (i.e. p_0 is not "isolated" in the space $S(T_{fg})$).
- On the other hand, if we take for granted the decidability of T_{fg} then by uniqueness of the generic type p_0, $p_0(x)$ (as a collection of formulas) is already decidable. (EXPLAIN.)
- Maybe this is an appropriate place to mention a formula (in p_0) which is thought to be not (equivalent to) a Boolean combination of universal (existential) formulas (Razborov).
- $\phi(x) : \forall y, z, w(x = y^2 z^2 w^3 \to (\exists r, s, t)(rw = wr \wedge x = t^{-1} r^{-1} ts^2))$.

Homogeneity and forking

- Perin and Sklinos (9) have substantially generalised the results in Lemma 4.4 (and (i) below was independently proved by Ould-Houcine (7)).

Theorem 4.10. *(i) Any F_κ ($\kappa \geq 2$) is $\aleph_0$-homogeneous (in the model-theoretic sense): if $\bar{b}$, $\bar{c}$ are n-tuples from F_κ with the same type in F_κ then there is an automorphism f of F_κ such that $f(\bar{b}) = \bar{c}$.*
(ii) Let $\bar{b}$ and $\bar{c}$ be finite tuples in a standard model F_n and let G be a free factor of F_n (maybe trivial). Then $\bar{a}$ and $\bar{b}$ are independent over G (in the sense of forking) if and only if F_n admits a free decomposition $H * G * K$ such that $\bar{b} \in H * G$ and $\bar{c} \in G * K$.

- Concerning (i): when $\kappa = 2$ the result is relatively easy (and was first observed by Nies).
- Namely one reduces to the case when neither $\bar{b}$ nor $\bar{c}$ is contained in a free factor of F_2. Let $\bar{b} = \bar{w}(a_1, a_2)$ where (a_1, a_2) is a basis of F_2 and $\bar{w}$ an n-tuple of words.
- The formula (without parameters) $\phi(\bar{x})$: $\exists y_1, y_2(\bar{x} = \bar{w}(y_1, y_2) \wedge [y_1, y_2] \neq 1)$ is true of $\bar{b}$ so also of $\bar{c}$.
- One obtains an injective endomorphism f_1 of F_2 taking $\bar{b}$ to $\bar{c}$ and an injective endomorphism f_2 taking $\bar{c}$ to $\bar{b}$. The composition f takes $\bar{b}$ to $\bar{b}$ and is injective, hence by the indecomposability assumption, f is an automorphism.
- The key point here is that injectivity of an endomorphism h of F_2 is equivalent to $[h(a_1), h(a_2)] \neq 1$.

- Generalizing the proof to $n > 2$ requires dealing with this "expressibility" of injectivity, and the proof (at least that by Perin-Sklinos) depends on a relative version of the "K-M,S" finiteness theorem.
- I won't say much about (ii) (forking) other than that it uses (i) heavily.

DOP

- Here we are concerned with questions from "classification of stable theories" in the sense of Shelah, where the number of models of a theory T of this or that cardinal is a key problem.
- The property DOP is a so-called "nonstructure" property which if it holds for a stable theory T gives rise to the maximum possible number of $\aleph_1$-saturated models of cardinality κ for suitable κ (all $\kappa > \aleph_1$?)
- The DOP property (when it holds) is typically witnessed by a definable family of definable sets (as described earlier) $f : X \to Y$ (in a saturated model $\bar{M}$ say) where the base has a definable group structure, and mutually generic fibres X_g, X_h are "orthogonal", roughly meaning that there are no nontrivial definable (even with additional parameters) relations between these fibres.
- In the case of T_{fg} a natural candidate for such a witness to DOP is the definable family of centralizers, where the base Y is the universe of a model of T_{fg} with its given group structure.
- This more or less boils down to proving that in a standard model F_n, if a_1, a_2 are distinct members of a basis then there is no definable isomorphism between finite index subgroups of $C(a_1)$ $(= \langle a_1 \rangle)$ and $C(a_2)$ $(= \langle a_2 \rangle)$.

Description of saturated models and types

- Finally a few unstructured questions/comments.
- A very basic problem is to say something meaningful about a (or the) saturated model $\bar{G}$ of T_{fg}.
- The asymptotic cone is part of the picture in some sense (as a quotient of some set in $\bar{G}$).
- But even in the case of $(Z, +)$ where a saturated model is $\hat{Z} \oplus Q^{\kappa}$ and the asymptotic cone is R (as a group), neither tells us much about the other.

- A related question is to give a description of the space of types over arbitrary models. This is the "normal" way by which one proves stability of specific theories, as stability of T is equivalent to there being "few types" over models.

- Already, a description of $S(T_{fg})$ $(= \cup_n (S_n(T_{fg}))$, i.e. complete types of the theory T_{fg} would be useful. In this case I now take as given or understood the types p_0, $p_0^{(n)}$, and thus also any type of a tuple in a standard model (as a word in $p_0^{(n)}$).

- Of course any $p(x) \in S(T_{fg})$ is an ultraproduct of types realized in any given standard model, but this is not very informative.

- It is somewhat interesting that we can take the types realized in finitely generated models of T_{fg} as "known" or "classified".

- This is because of what I understand from Perin to be a classification (Sela?) up to isomorphism of pairs of the form $(G, \bar{a})$ where G is a model of T_{fg} and $\bar{a}$ is a finite tuple of generators of G, using "hyperbolic towers".

- And the set of complete types of T_{fg} whose realization in some (any) model of T_{fg} generates an elementary substructure, is in 1-1 correspondence with the collection of isomorphism types above.

Some references

- BASIC MODEL THEORY:

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- STABILITY THEORY AND STABLE GROUPS:

- (3) J. Baldwin, Fundamentals of stability theory, Springer.

- (4) A. Pillay, Stability Theory, lecture notes, web-page, http://www.maths.leeds.ac.uk/ pillay/

- (5) A. Pillay, Geometric Stability Theory, OUP.

- (6) B. Poizat, Stable groups, AMS. (English translation of the better French original.)

- PAPERS WITH WORK REFERRED TO IN THE TALKS (in addition to papers by Bestvina-Feign, Kharlampovich-Myasnikov, Sela):
- (7) A. Ould Houcine, Homogeneity and prime models in torsion-free hyperbolic groups, preprint (see his web page).
- (8) C. Perin, Elementary embeddings and torsion-free hyperbolic groups, to appear in Annales de l'ENS.
- (9) C. Perin and R. Sklinos, Homogeneity in the free group, to appear in Duke Math. Journal.
- (10) A. Pillay, Forking in the free group, J. Inst. Math. Jussieu, vol. 7 (2008), 375-389.
- (11) A. Pillay, On genericity and weight in the free group, Proceedings AMS, 137 (2009), 3911-3917.
- (12) R. Sklinos, On the generic type of the free group, J. Symbolic Logic, 76 (2011), 227-234.