Lecture notes - Stability Theory (Math 414) Spring 2003.

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1 Introduction and preliminaries

The aim of this course and these notes is to present an exposition of the basics of stability theory, stable group theory, and geometric stability theory. I will assume knowledge of my Autumn 2002 model theory lecture notes [1]. In fact the current notes are a natural continuation of the earlier ones.

The model theory lecture notes ended with a proof of Morley's Theorem. Stability theory developed historically (in the work of Shelah) as a chunk of machinery intended to help generalize Morley's theorem to a computation of the possible "spectra" of complete first order theories. Here the spectrum of T is the function $I(T, -)$, where $I(T, \kappa)$ is the number of models of T of cardinality κ. This project (at least for countable theories) was essentially completed by Shelah around 1980. In the meantime other perspectives developed, in which stability theory is seen rather as a way of classifying definable sets in a structure and describing the interaction between definable sets. Eventually this theory was seen as having a "geometric meaning". This means, on the one hand, that various structural results have a geometric flavour. On the other hand, it refers to the empirically discovered fact, that in numerous examples, the model-theoretic notions have "actually existing" mathematical meaning. This will be the perspective of these notes.

We have a choice of the level of abstraction at which to operate. The theory of forking can be developed smoothly for the class of simple theories. There have been recent developments, such as rosy theories which subsume stable theories, most simple theories, and 0-minimal theories. ω-stable (or totally transcendental) theories, on the other hand, have a somewhat more concrete flavour and many natural examples fit in here. An important aspect of general stability is that it is essentially a formula-by-formula theory. So we plan to find a kind of balance between the general stable context and the ω-stable context

For the remainder of this introductory section we will discuss definability and imaginaries. Apart from these being important issues in their own right, this will give us a chance to give examples of the kinds of arguments (including compactness) that we will be using.

Our conventions will be as follows: T will be a complete theory in a language L. T could be many-sorted, but as in the model theory notes we will usually work in the 1-sorted context. Note that there is no harm in asssuming T to have quantifier elimination. Fix some "big" cardinal $\bar{\kappa}$. By 4.11 of [1], T has a $\bar{\kappa}$-saturated, strongly $\bar{\kappa}$-homogeneous model. Let us fix such a model which we call $\bar{M}$. We know that any model of T of cardinality $< \bar{\kappa}$ is isomorphic to an elementary substructure of $\bar{M}$. So by a *model* we mean (unless we say otherwise) an elementary substructure of $\bar{M}$ of cardinality $< \bar{\kappa}$. $M, N, ...$ will denote models. $A, B, ...$ will usualy denote *small* (that is, of cardinality $< \bar{\kappa}$) subsets of $\bar{M}$. $a, b, ...$ will usually denote finite tuples of elements of $\bar{M}$, but sometimes also *small* tuples. (Sometimes we will need to distinguish between elements of $\bar{M}$ and tuples of elements. In that case we write $a, b, ..$ for elements, and $\bar{a}, \bar{b}, ..$ for tuples.) $\kappa, \lambda, ..$ will usually denote cardinals $< \bar{\kappa}$. By a *definable set* we mean some subset X of $\bar{M}^n$ which is definable in $\bar{M}$. $Aut(\bar{M})$ denotes the group of automorphisms of $\bar{M}$ and $Aut_A(\bar{M})$ those automorphisms which fix A pointwise. Note that $Aut(\bar{M})$ acts on everything associated with $\bar{M}$: tuples, definable sets,...
For an $L_{\bar{M}}$-sentence σ, we just write $\models \sigma$ in place of $\bar{M} \models \sigma$.

We will be using compactness *inside* $\bar{M}$ in a rather specific kind of manner. Let us suppose that $\Sigma(y)$ is a set of L_A-formulas, and y a finite (or even small) tuple of variables, and that $\phi(y)$ is also an L_A-formula. We will write $\Sigma(y) \models \phi(y)$ to mean that the implication is valid in $\bar{M}$, that is, for any b from $\bar{M}$ such that $\models \Sigma(b, a)$, also $\models \phi(b)$. Then we have:

Fact 1.1 *(with above notation.) If $\Sigma(y) \models \phi(y)$ then there is some some finite $\Sigma'(y) \subseteq \Sigma(y)$ such that $\wedge\Sigma'(y) \models \phi(y)$.*

Here is an application to a useful "Galois-theoretic" interpretation of "definability over A".

Lemma 1.2 *Let X be a definable set, and $A \subset \bar{M}$. Then X is A-definable (or defined over A) if and only if $f(X) = X$ (as a set) for all $f \in Aut_A(\bar{M})$.*

Proof. Left to right is immediate. (Suppose $\phi(x, a)$ defines X where a is from A. Then for any $b \in \bar{M}^n$, $\models \phi(b, a)$ iff $\models \phi(f(b), f(a))$ iff $\models \phi(f(b), a))$ (as $f(a) = a$).)
Right-to-left: Let $\phi(x, b)$ define X (where b is some finite tuple from $\bar{M}$). Let $p(y) = tp(b/A)$.
Claim I. $p(y) \models \forall x(\phi(x, y) \leftrightarrow \phi(x, b))$.
Proof. Let b' realize $p(y)$ in $\bar{M}$. So $tp(b'/A) = tp(b/A)$ so there is $f \in Aut_A(\bar{M}$ such that $f(b) = b'$. By our assumptions $f(X) = X$. On the other hand clearly $f(X) = \phi(x, b')(\bar{M})$.

By Fact 1.1, there is some $\psi(y) \in p(y)$ such that
(*) $\models \forall y(\psi(y) \rightarrow \forall x(\phi(x, y) \leftrightarrow \phi(x, b)))$.
Let $\sigma(x)$ be the formula $\exists y(\psi(y) \wedge \phi(x, y))$. Note that $\sigma(x) \in L_A$.
Claim II. $\sigma(x)$ defines X.
Proof. If $c \in X$, then $\models \phi(c, b)$, so as $\psi(y) \in tp(b/A)$, $\models \sigma(c)$. On the other hand if $\models \sigma(c)$, let b' be such that $\models \psi(b') \wedge \phi(c, b')$. Then by (*) $\models \phi(c, b)$, so $c \in X$.

By Claim II, X is A-definable.

Remark 1.3 *Consider the algebraic-geometric context, where $T = ACF_p$. Write $\bar{M} = \bar{K}$. Let $V \subseteq \bar{K}^n$ be an algebraic variety. Let k be a subfield of $\bar{K}$. V is said to be defined over k in the Weil-algebraic-geometric sense if the ideal $I \subset \bar{K}[x_1, .., x_n]$ of polynomials vanishing on V is generated by polynomials with coefficients from k. THEN V is defined over k in the model-theoretic sense if and only if V is defined over $k^{p^{-\infty}}$ in the Weil-algebraic-geometric sense. In particular when $p = 0$ the two notions coincide.*

Here is a slight generalization of Lemma 1.2. First:

Definition 1.4 *Let $X \subseteq \bar{M}^n$ be definable. We say that X is definable almost over A, if there is an A-definable equivalence relation E on $\bar{M}^n$, such that E has only finitely many classes, and X is a union of E-classes.*

Lemma 1.5 *X is definable almost over A if and only if $\{f(X) : f \in Aut_A(\bar{M})\}$ is finite.*

Proof. Again left to right is immediate. (Why?)
Right-to-left: Again let $\phi(x, b)$ define X, and $p(y) = tp(b/A)$. By our assumption there are $b = b_0, .., b_k$ realizing $p(y)$ such that for any $f \in Aut_A(\bar{M})$, $f(X) = \phi(x, b_i)(\bar{M})$ for some $i = 0, .., k$. Again by saturation of $\bar{M}$ we have: $p(y) \models \vee_{i=0,..,k} \forall x(\phi(x, y) \leftrightarrow \phi(x, b_i))$. By compactness (i.e. Fact 1.1) there is $\psi(y) \in p(y)$ such that
$\models \forall y(\psi(y) \rightarrow (\vee_{i=0,..,k} \forall x(\phi(x, y) \leftrightarrow \phi(x, b_i)))$.
Now define $E(x_1, x_2)$ by the formula:
(*) $\forall y(\psi(y) \rightarrow (\phi(x_1, y) \leftrightarrow \phi(x_2, y)))$.
Clearly E is an A-definable equivalence relation. By (*), E has finitely many classes. We leave it to you to check that X is a union of E-classes.

Exercise 1.6 *Show that X is definable almost over A iff $\{f(X) : f \in Aut_A(\bar{M})\}$ has cardinality $< \bar{\kappa}$.*

We will often identify formulas of $L_{\bar{M}}$ with the sets they define, so we may say $\phi(x)$ is over A, almost over A,..

Note that if the definable set X is simply $\{a\}$ for some $a \in \bar{M}$ then 1.2 becomes: $a \in dcl(A)$ iff a is fixed by all A-automorphisms of $\bar{M}$, and 1.6 becomes $a \in acl(A)$ iff a has $< \bar{\kappa}$ many images under A-automorphisms of $\bar{M}$. (In the latter case, if $\{a\}$ is a finite union of E-classes, where E is A-definable and has finitely many classes, then let $\chi(x)$ say that $\{x\}$ is an E-class, so $\models \chi(a)$ and $\chi(x)$ has only finitely many solutions.)

The machinery of T^{eq} will actually allow us to consider definable sets as elements in their own right, whereby the general and special cases in the paragraph above will be identical. T^{eq} will be important for many other reasons. In any case in the remainder of this section we will develop T^{eq}.

The nature and status of "quotient" objects is rather important in many parts of mathematics, especially those with a geometric flavour. For example, it is often important to know, given a manifold M and closed equivalence relation E on M, whether M/E has again (naturally) the structure of a manifold.

From our point of view we have $\bar{M}$, elements and tuples from $\bar{M}$ and definable sets of tuples, and complete types,... Suppose that E is a $\emptyset$-definable equivalence relation on $\bar{M}$. To what extent can we also talk about definable subsets of $\bar{M}/E$, types of elements or tuples of $\bar{M}/E$ over other sets, etc. Namely to what extent can we include E-classes, as *elements* (rather than definable sets) in our whole theory. There are two approaches. The first is somewhat informal: Define a *definable subset* of $\bar{M}/E$ to be a set X of E-classes such that $\cup X \subset \bar{M}$ is a definable subset of $\bar{M}$. Namely, a definable set of E-classes is something coming from an E-invariant definable subset of $\bar{M}$. Define $tp((b/E)/A)$ to be the set of A-definable subsets of $\bar{M}/E$ which contain b/E. Then show that all our tools, such as compactness hold for the generalized definable sets, types, etc... In fact this informal approach is how we deal with *hyperimaginaries*.

The more formal approach is to contruct a new complete many-sorted structure $\bar{M}^{eq}$ (whose theory is a first order many sorted theory T^{eq}) in which all these quotient objects are by definition elements.

Let us first define a many-sorted language L^{eq} (which actually depends on T as well as L, in spite of the notation). For each L-formula $\phi(x_1,..,x_n,y_1,..,y_n)$ such that T says that ϕ defines an equivalence relation on n-tuples, let S_ϕ be a sort symbol. (Thinking semantically we sometimes write S_E in place of S_ϕ). So among the sorts is $S_=$. Let us also introduce, for each such $\phi(x_1,..,x_n,y_1,..,y_n)$ a new function symbol f_ϕ whose domain sort is n-tuples of sort $S_=$ and whose range sort is S_ϕ. Finally for every m-place relation symbol R of L, R will still exist in L^{eq} but as a relation on m-tuples of sort $S_=$. Likewise for old function symbols of L. Note that any L-sentence σ can be identified with an L^{eq}-sentence: just take all variables to be of sort $S_=$. (Note that by definition of the many-sorted logic, if x is a variable of a given sort S then $\forall x(...)$ is interpreted as "for all x of sort S".)
We define T^{eq} to be the L^{eq}-theory axiomatized by:
(i) T,
(ii) for each ϕ as above, the sentence $\forall x_1,..,x_n,y_1,..,y_n$ (of sort $S_=$)
$(\phi(x_1,..,x_n,y_1,..,y_n) \leftrightarrow f_\phi(x_1,..,x_n) = f_\phi(y_1,..,y_n))$.
(iii) for each ϕ as above, the sentence expressing that f_ϕ is surjective (as a map from $(S_=)^n$ to S_ϕ).
Finally for M a model of T, M^{eq} will denote the L^{eq}-structure, such that $S_=(M^{eq})$ is the universe of M, $S_\phi(M^{eq}) = M^n/E$ (where E is the equivalence relation on M^n defined by ϕ), $f_\phi(M^{eq})(a_1,.,a_n) = (a_1,..,a_n)/E$ (for $a_1,..,a_n \in M^n$), and such that the old relation and function symbols of L are interpreted on $S_=(M^{eq})$ as they were in the L-structure M.

We may often notationally identify $S_=(M^{eq})$ with M, where hopefully there is no confusion. So we have our original L-structure, living, equipped with all its original L-structure as a sort in M^{eq}. The first question a model-theorist asks in this kind of situation is whether M gets any more "induced structure" this way. We will point out below, among other things, that M does NOT get any "new structure".

Lemma 1.7 *(i) The models of T^{eq} are precisely the structures M^{eq} for M a model of T.*
(ii) If M, N are models of T then any isomorphism between M and N extends to a (unique) isomorphism between M^{eq} and N^{eq}.
(iii) T^{eq} is a complete theory.
(iv) If M, N are models of T, $\bar{a}$ is a tuple (maybe infinite) from M and $\bar{b}$ a tuple from N, and $tp_M(\bar{a}) = tp_N(\bar{a})$ then $tp_{M^{eq}}(\bar{a}) = tp_{N^{eq}}(\bar{b})$.

Proof. (i) and (ii) are obvious. For (iii) to make life a bit easy let us assume GCH so there are arbitrarily large cardinals in which any given theory has a saturated model. Let M^{eq}, N^{eq} be saturated models of T^{eq} of cardinality some $\lambda \geq |T|$. So the same is true of M, N. So as T is complete M and N are isomorphic. Hence by (ii), M^{eq} and N^{eq} are isomorphic, so elementarily equivalent.
(iv) Same proof as (iii).

Fix n. Let us consider two Stone spaces. First $W_1 = S_n(T)$, the space of complete n-types of T. Secondly $W_2 = S_{(S_=)^n}(T^{eq})$, the space of complete types of n-tuples from sort $S_=$ in models of T^{eq}. (Sorry for the double meaning of "S" here.) We have a natural "restriction" map $\pi : W_2 \to W_1$ which is continuous. On the other hand by Lemma 1.7 (iv) and a saturation argument, π is a a bijection. Thus:

Lemma 1.8 *π is a homeomorphism.*

We conclude:

Proposition 1.9 *Let $\phi(x_1, .., x_k)$ be an L^{eq}-formula, where x_i is of sort S_{E_i} say. Then there is an L-formula $\psi(\bar{y}_1, .., \bar{y}_k)$ such that*
$T^{eq} \models \forall \bar{y}_1..\bar{y}_k(\psi(\bar{y}_1, .., \bar{y}_k) \leftrightarrow \phi(f_{E_1}(\bar{y}_1), .., f_{E_k}(\bar{y}_k))$.

Proof. Let n be the length of the tuple $\bar{y} = (\bar{y}_1, .., \bar{y}_k)$. By the previous lemma $\{p(\bar{y}) \in S_n(T) : \phi(f_{E_1}(\bar{y}_1), .., f_{E_k}(\bar{y}_k)) \in \pi^{-1}(p)\}$ is clopen, which is enough.

Corollary 1.10 *(i) If $M \models T$ is an elementary substructure of N then M^{eq} is an elementary substructure of N^{eq}.*
(ii) $\bar{M}^{eq}$ is also $\bar{\kappa}$-saturated and $\bar{\kappa}$-strongly homogeneous.

The proof is left as an exercise.

Any definable set in $\bar{M}$ "correponds" to an element of $\bar{M}^{eq}$: Suppose $X \subset \bar{M}^n$ is defined by $\phi(x, b)$. Let $E(y_1, y_2)$ be the L-formula $\forall x(\phi(x, y_1) \leftrightarrow \phi(x, y_2))$. Then b/E is a element of sort S_E in $\bar{M}^{eq}$. Let us note a couple of things:
Firstly, for any automorphism σ of $\bar{M}$ (or equivalently of $\bar{M}^{eq}$), $\sigma(X) = X$ iff $\sigma(b/E) = (b/E)$.
Secondly, X can be defined over b/E in the structure $\bar{M}^{eq}$. This can either be seen using Lemma 1.2 in the many sorted structure $\bar{M}^{eq}$, or directly by considering the formula $\exists y(f_E(y) = b/E \wedge \phi(x, y))$.
Thirdly, let $\psi(x, (b/E))$ be the L^{eq}-formula with parameter b/E which defines X. Then b/E is the *unique* element z of sort S_E such that $\psi(x, z)$ defines X.

Sometimes we say that the imaginary b/E above "codes" the definable set X. In fact for an arbitrary $\emptyset$-definable equivalence relation E' say (on n-tuples from $\bar{M}$ say) and $c \in \bar{M}^n$, c/E' "codes" the set defined by $E'(x, c)$. So we can also think of $\bar{M}^{eq}$ as adjoining codes for all definable sets in $\bar{M}$. Actually the code for X is not unique, as it depends on the shape of the formula used to define X. But it is unique *up to interdefinability.*

Let us look back at Lemmas 1.2 and 1.5 in this light. Let X be a definable set in $\bar{M}$, let $e \in \bar{M}^{eq}$ be a code for X. Then X is definable over A in $\bar{M}$ iff $e \in dcl(A)$ in $\bar{M}^{eq}$. Likewise, X is definable almost over A in $\bar{M}$ iff $e \in acl(A)$ in $\bar{M}^{eq}$. Let us make this precise:

Lemma 1.11 *Let $X \subseteq \bar{M}^n$ be definable, and let $A \subset \bar{M}$. Then X is definable almost over A if and only if X is $acl^{eq}(A)$-definable in $\bar{M}^{eq}$.*

Proof. Let e be the code for X. So X is definable over e in $\bar{M}^{eq}$. So if X is almost over A, then $e \in acl(A)$ and X is $acl^{eq}(A)$-definable in $\bar{M}^{eq}$. Conversely, suppose X is $acl^{eq}(A)$-definable. As e is a code of X,

$e \in dcl^{eq}(acl^{eq}(A)) = acl^{eq}(A)$. So e has finitely many images under A-automorphisms of $\bar{M}^{eq}$, as does therefore X too. So by 1.5, X is almost over A.

Definition 1.12 *Let a, b be n-tuples from $\bar{M}$. a and b are said to have the same strong type over A (we write $stp(a/A) = stp(b/A)$) if whenever E is an A-definable equivalence relation on $\bar{M}^n$ with only finitely many classes then $E(a, b)$.*

Note that $stp(a/A) = stp(b/A)$ implies $tp(a/A) = tp(b/A)$. With the above notation:

Lemma 1.13 *The following are equivalent:*
(i) $stp(a/A) = stp(b/A)$,
(ii) a and b satisfy the same formulas which are almost over A.
(iii) $tp(a/acl^{eq}(A)) = tp(b/acl^{eq}(A))$ (in $\bar{M}^{eq}$).

Proof. (i) and (ii) are trivially equivalent. The equivalence of (ii) and (iii) follows immediately from Lemma 1.11.

Now it may happen that we can already find codes for definable sets in $\bar{M}$ itself. When this happens we say that T admits elimination of imaginaries.

Definition 1.14 *T has elimination of imaginaries if for any definable $X \subseteq \bar{M}^n$, and "code" $e \in \bar{M}^{eq}$ for X, there is some finite tuple c from $\bar{M}$ such that $e \in dcl(c)$ and $c \in dcl(e)$ (in $\bar{M}^{eq}$). (Alternatively, without mentioning $\bar{M}^{eq}$ we simply require c to exist in $\bar{M}$ such that an automorphism σ of $\bar{M}$ fixes X setwise iff it fixes the tuple c.)*

Exercise 1.15 *The following are equivalent:*
(i) T has elimination of imaginaries.
(ii) If $M \models T$ and $e \in M^{eq}$ then there is a finite tuple c from M such that $dcl(c) = dcl(e)$ in the sense of M^{eq}.
(iii) Let $M \models T$, $\phi(x, y) \in L$ and b in M. Then there is a formula $\psi(y) \in tp(b)$, and a partial $\emptyset$-definble function f with range contained in some M^k such that f is defined on $\psi(M)$ and for all b_1, b_2 satisfying $\psi(y)$, $f(b_1) = f(b_2)$ iff $\phi(x, b_1)(M) = \phi(x, b_2)(M)$.
(iv) Let again $M \models T$ and $\phi(x, y) \in L$ where y is an n-tuple. Then M^n can

be partitioned into finitely many $\emptyset$-definable sets $Y_1 \cup .. \cup Y_r$ say and there are $\emptyset$-definable functions $f_i : Y_i \to M^{k_i}$ for $i = 1, .., r$ such that for each i and $b_1, b_2 \in Y_i$, $f_i(b_1) = f_i(b_2)$ iff $\phi(x, b_1)(M) = \phi(x, b_2)(M)$.
(v) For any definable set X in a model M of T there is an L-formula $\phi(x, y)$ and b such that b is the unique tuple from M such that $\phi(x, b)(M) = X$.

Example 1.16 *Let T be the theory of an equivalence relation with exactly 2 classes both infinite. Then T does not eliminate imaginaries.*

In the above we defined T^{eq} for T 1-sorted, but exactly the same thing can be done for T many-sorted. Similarly, the notion of eliminating imaginaries makes sense for T many-sorted.

Lemma 1.17 *T^{eq} eliminates imaginaries.*

Proof. Let E' be a $\emptyset$-definable (in L^{eq}) equivalence relation on some sort S_E say in a model M^{eq} of T^{eq}. By Proposition 1.9, there is an L-formula $\psi(y_1, y_2)$ (where y_i are of appropriate length) such that for any a_1, a_2, $M \models \psi(a_1, a_2)$ iff $M^{eq} \models E'(f_E(a_1), f_E(a_2))$. So clearly ψ defines an equivalence relation, and we have a $\emptyset$-definable bijection between the sorts S_E/E' and S_ψ.

Note that it follows, for example, that in $\bar{M}^{eq}$, $stp(a/A) = stp(a/A)$ iff $tp(a/acl(A)) = tp(b/acl(A))$, where a, b are tuple from any sorts and A is a set of elements from arbitrary sorts.

Often we will work in T^{eq}.

Many "algebraic" theories/structures such as algebraically closed fields, real closed fields, differentially closed fields, eliminate imaginaries. But some very basic theories such as the theory of an infinite set with no structure do not: the point being only that finite sets (which *are* definable) need not be coded.

Definition 1.18 *T has weak (geometric) elimination of imaginaries if for some model M of T and $e \in M^{eq}$ there is a finite tuple c from M such that $e \in dcl(c)$ ($e \in acl(c)$) and $c \in acl(e)$ (in M^{eq}).*

Exercise 1.19 *Show that the theory of an infinite set (in the language with only equality) has weak elimination of imaginaries, but does not have (full) elimination of imaginaries.*

Hint. Unravelling things, we have to show that for any definable set X in a model, there is a finite tuple c such that c is fixed by any automorphism fixing the set X, and that X has only finitely many images under the group of automorphisms fixing c. Use quantifier-elimination.

2 Stability, totally transcendental theories, and forking

T is as in section 1, one-sorted for convenience. We work in the big model $\bar{M}$. At some point we will pass to T^{eq}. In [1] we defined stability in terms of counting types. We will take a different approach here but will soon show the various definitions to be equivalent.

We will use the "ω-stable" theories defined in [1] as providing somewhat more concrete examples/applications of the machinery developed here. ω-stability was defined only for countable theories. As we are no longer assuming T to be countable, the appropriate notion is that of a *totally trascendental* (or *t.t*) theory. Remember that we defined the Morley rank of an arbitrary formula (with parameters) and this still makes sense for uncountable T. If $\phi(x)$ is a formula (with parameters, where x is an n-tuple of variables), and $X = \phi(\bar{M})$ the subset of $\bar{M}^n$ defined by ϕ we will use $RM(X)$ and $RM(\phi)$ interchangeably.

Definition 2.1 *T is t.t if for every definable set X, $RM(X) < \infty$. (Here X is a definable subset of $\bar{M}^n$ for some n.)*

Exercise 2.2 *(i) If $RM(x = x) < \infty$ (x a single variable), then T is t.t.*
(ii) If T is t.t then T^{eq} is t.t.. Namely for every definable subset X of a finite product of sorts, $RM(X) < \infty$ (computed in $\bar{M}^{eq}$).

Recall also that $RM(tp(c/A))$ is by definition $min\{RM(\phi(x)) : \phi(x) \in tp(c/A))\}$, and if this is ordinal valued and witnessed by $\phi(x)$ then $dM(tp(c/A)) = dM(\phi(x))$.

We will be defining a notion of "independence": for c, d finite tuples and A a set of parameters, we will make sense out of the expression "c is independent from d over A" (at least for T stable). In the case where T is $t.t$, this will reduce to "$RM(tp(c/A, d)) = RM(tp(c/A))$". In the even more special case where T is the theory of algebraically closed fields and $A = k$ is a subfield it will reduce to "$tr.deg.(k(c,d)/k(d)) = tr.deg.(k(c)/k)$", that is the fields $k(c)$ and $k(d)$ are algebraically disjoint over k. In fact in this last example we will see that $RM(tp(c/k)) = tr.deg.(k(c)/k)$.

The independence notion will come from Shelah's theory of forking. This is also valid for simple theories, but in the stable case there are some special features such as (i) a theory of multiplicity (like Morley degree) and (ii) the theory operates and can be developed on a formula-by-formula basis.

Definition 2.3 *Let $\delta(x,y)$ be a formula, possibly with parameters). We say that $\delta(x,y)$ is stable if there do not exist a_i, b_i for $i < \omega$ such that for all $i,j < \omega$, $\models \delta(a_i, b_j)$ iff $i < j$.*

Remark 2.4 *(i) The stability of $\delta(x,y)$ depends on the choice of a division of the tuple of variables in δ into two subtuples.*
(ii) If $\delta(x,(y,z))$ is stable then so is $\delta(x,y,c)$ for any tuple c of parameters.
(iii) If $\delta_1(x,y)$, $\delta_2(x,z)$ are stable, then so are $\neg\delta(x,y)$, $(\delta_1 \vee \delta_2)(x,(y,z))$, $(\delta_1 \wedge \delta_2)(x,(y,z))$, and $\delta_1^*(y,x) = \delta(x,y)$.
(iv) Suppose $\delta(x,y)$ is stable. Then there is some $N < \omega$ such that there do not exist a_i, b_i for $i \leq N$ such that $\models \delta(a_i, b_j)$ iff $i < j$.
(v) Suppose that $\delta(x,y)$ is unstable. Let $\phi((x_1,y_1),(x_2,y_2))$ be the formula $\delta(x_1,y_2)$. Then there are $(c_i : i < \omega)$ such that $\models \phi(c_i, c_j)$ iff $i < j$.

Proof. (ii) is obvious. (iii) is a exercise, and (iv) follows from compactness. (v) Let a_i, b_i for $i < \omega$ witness the instability of δ. Put $c_i = (a_i, b_i)$.

The following begins to tie up stable formulas with the number of types.

Lemma 2.5 *(i) Suppose that some L-formula $\delta(x,y)$ is unstable. Then for any $\lambda \geq |T|$, there is a model M of T such that there are $> \lambda$ many complete types (in variable x) over M. In fact we can find such a set $\mathbf{P}$ of types which are distinguished by formulas of the form $\delta(x,b)$ for $b \in M$*
(ii) If T is t.t. then every L-formula is stable.

Proof. Let μ be the least cardinal such that $2^\mu > \lambda$. So $mu \leq \lambda$. Let ${}^\mu 2$ be the set of functions from μ to 2, and order it by: $f < g$ if there is $\alpha < \mu$ such that $f|\alpha = g|\alpha$ and $f(\alpha) = 0$ and $g(\alpha) = 1$. By compactness there are a_f, b_f for $f \in^\mu 2$ such that $\models \delta(a_f, b_g)$ iff $f < g$. Let $X \subset^\mu 2$ the set of eventually

constant functions. Then $|X| \leq \lambda$, and for each $f_1 \neq f_2 \in^\mu 2$ there is $g \in X$ such that $\models \neg(\delta(a_{f_1}, b_g) \leftrightarrow \delta(a_{f_2}, b_g))$. Let M be a model of cardinality λ containing $\{b_g : g \in X\}$, and we see that the set $\{tp(a_f/M) : f \in^\mu 2\}$ satisfies our requirements.
(ii) If T is $t.t.$ then as in the proof of 5.19 of [1] there are at most λ many complete types over any model of cardinality $\lambda \geq |T|$, so we can use (i).

The next lemma is crucial:

Lemma 2.6 *Let $\delta(x, y)$ be stable. Let M be a model, and $a \in \bar{M}$ a tuple of length $l(x)$. Then there is a finite set $\{a_i^j\}_{i,j<N}$ of tuples contained in M such that*
(*) for any $b \in M$, $\models \delta(a, b)$ iff $\models \vee_j(\wedge_i \delta(a_i^j, b))$.
Moreover
(i) N is determined only by δ (not by the choice of a or M),
(ii) if $\psi(x, c)$ is any formula over M satisfied by a we can choose the a_i^j to satisfy $\psi(x, c)$, and
(iii) if M is λ-saturated and C is a subset of M of cardinality $< \lambda$, we may choose the a_i^j to realize $tp(a/C)$.

Proof. The proof will be in two steps:
First let N_1 be as given by 2.4 (iv).
Claim. For any formula $\psi(x) \in tp(a/M)$ there are $a_1, .., a_n \in M$ (all satisying $\psi(x)$) for some $n \leq N_1$ such that for any $b \in M$, if $M \models \wedge_{i=1,..,n}\delta(a_i, b)$ then $\models \delta(a, b)$.
Proof. We start to build $a_1, a_2, ...$ in M satisfying $\psi(x)$ and $b_1, b_2, ..$ in M such that $\models \neg\delta(a, b_i)$, and such that $\models \delta(a_i, b_j)$ iff $i \leq j$. Suppose we have already found such $a_1, .., a_n, b_1, .., b_n$. Then the formula $\psi(x) \wedge \wedge_{i=1,..,n} \neg\delta(x, b_i)$ is over M, and true of a, so realized in M by a_{n+1} say. Now either $(a_1, .., a_{n+1})$ satisfies the claim, and we stop, or there is $b \in M$ such that $\models \wedge_{i=1,..,n+1}\delta(a_i, b)$ but $\models \neg\delta(a, b)$. Let $b_{n+1} \in M$ be such a b. So the construction continues. So we have to stop for some $n \leq N_1$.

We now pass to Step 2. Let $\chi((x_1.., x_{N_1}), y)$ be the formula $\delta(x_1, y) \wedge ... \wedge \delta(x_{N_1}, y)$. By Remark 2.4, χ is stable (with the given division of its variables). Let N_2 be as given by 2.4 for $\neg\chi$. Let $\bar{x}$ denote $(x_1, .., x_{N_1})$. We construct $\bar{a}^1, \bar{a}^2, ..$ in M satisfying the claim above, and $b_1, b_2, ...$ in M such that
(i) $\models \chi(\bar{a}^i, b_j)$ iff $i > j$,

(ii) $\models \delta(a, b_i)$ for all i.
Suppose we already have constructed $\bar{a}^i, b_i$ for $i = 1, .., n$. By the claim there is $\bar{a}^{n+1}$ in M such that $\models \chi(\bar{a}^{n+1}, b_i)$ for $i = 1, .., n$, and such that for any $b \in M$, if $\models \chi(\bar{a}^{n+1}, b)$ then $\models \delta(a, b)$. So if $\vee_{i=1,..,n+1}\chi(\bar{a}^i, y)$ does not satisfy the requirements of the lemma then there is $b_{n+1} \in M$ such that $\models \delta(a, b_{n+1})$ but $\models \neg\chi(\bar{a}^i, b_{n+1})$ for $i = 1, .., n+1$. So the construction continues. It follows that we must stop at some $n \leq N_2$. This proves the existence of the $\{a_i^j\}$. (i), (ii) and (iii) follow fron the construction and the claim.

Exercise 2.7 *Let $\delta(x, y) \in L$ and let M be a model such that there is no infinite ordered set $(I, <)$ and a_i, b_i in M for $i \in I$ such that $\models \delta(a_i, b_j)$ iff $i < j$. Prove that for any $a \in \bar{M}$ there are $a_1, .., a_n \in M$ such that for any $b \in M$ whether or not $\delta(a, b)$ holds depends only on which $\delta(a_i, b)$ hold. Namely there is a formula $\psi(y)$ equivalent to some Boolean combination of the $\delta(a_i, y)$ such that for any $b \in M$, $\models \delta(a, b)$ iff $M \models \psi(b)$.*

Hint. Assume not. Then construct a_i, b_i, c_i in M for $i < \omega$ such that
(1) for each i, $\models \delta(a, b_i) \leftrightarrow \neg\delta(a, c_i)$.
(2) for each $i \geq j$, $\models \delta(a_i, b_j) \leftrightarrow \delta(a, b_j)$ and $\models \delta(a_i, c_j) \leftrightarrow \delta(a, c_j)$, and
(3) for each $i < j$, $\models \delta(a_i, b_j) \leftrightarrow \delta(a_i, c_j)$.
By thinning out the sequence and applying Ramsey's theorem get a contradiction to our assumptions.

Definition 2.8 *(i) Let $\delta(x, y) \in L$. By a complete δ-type over a model M, we mean a maximal consistent set of formulas of the form $\delta(x, b)$ or $\neg\delta(x, b)$ where $b \in M$. $S_\delta(M)$ denotes the set of such types when there is no ambiguity.*
(ii) The complete δ-type $p(x) \in S_\delta(M)$ is said to be definable if there is an L_M-formula $\psi_\delta(y)$ such that $\psi_\delta(M) = \{b \in M : \delta(x, b) \in p(x)\}$. Note that $\psi_\delta(y)$ is unique up to equivalence. If $A \subseteq M$ we say that $p(x) \in S_\delta(M)$ is definable over A if $\psi_\delta(y)$ is over A (that is equivalent to an L_A-formula) and definable almost over A, if $\psi_\delta(y)$ is almost over A.
(iii) The complete type $p(x) \in S(M)$ is definable if for every $\delta(x, y) \in L$, $p|\delta$ is definable, and definable (almost) over A if each $p|\delta$ is.
(iv) Suppose $p(x) \in S(M)$ is definable, and for each $\delta(x, y) \in L$ let ψ_δ be as in (ii). We call the map $\delta \to \psi_\delta$ a defining scheme for $p(x)$.

Corollary 2.9 *Let $\delta(x, y) \in L$. The following are equivalent:*
(i) δ is stable,

(ii) Every complete δ-type over a model is definable,
(iii) For each $\lambda \geq |T|$, and model M of cardinality λ, $|S_\delta(M)| \leq \lambda$.

Proof. (i) implies (ii) is given by 2.6. (But note that we showed there that the defining formula can be chosen to be of a very special form.)
(ii) implies (iii): Given $M \models T$, any $p(x) \in S_\delta(M)$ is clearly determined by a "defining formula" $\psi(y)$ for p (as in the definition). But there are at most $|M| + |T|$-many such formulas.
(iii) implies (i): By Lemma 2.5 (i).

Corollary 2.10 *The following are equivalent:*
(i) every formula $\delta(x, y)$ (any variables x, y) is stable,
(ii) every complete type over a model is definable,
(iii) For any $\lambda \geq |T|$ and $M \models T$ of cardinality λ, the set of complete types over M has cardinality at most $\lambda^{|T|}$,
(iv) There is some $\lambda \geq |T|$ such that for every model M of cardinality λ, $S(M)$ has cardinality λ.
(v) every formula $\delta(x, y)$ with x a single variable (or a single variable from some sort) is stable.

Proof. This is almost immediate from Corollary 2.9. For (ii) implies (iii), note that the number of possible defining schema for types over M is $|M|^{|T|}$.

Definition 2.11 *We will say that T is stable if it satisfies the equivalent conditions of Corollary 2.10.*

Exercise 2.12 *If T is stable, then so is T^{eq}.*

We will now specialize somewhat to $t.t$ theories and return to the general theory later.

First some notation: let $p(x) \in S(M)$ be a definable type, and let d denote the corresponding defining schema, that is $d(\delta(x, y)) = \psi_\delta(y)$ as in 2.8 (iv). Let N be a model containing M. By $d(N)$ we mean $\{\delta(x, b)$: for $\delta(x, y) \in L$ and $b \in N$ such that $\models d(\delta)(b)\}$.,

Exercise 2.13 *With this notation, $d(N)$ is a complete type over N, extending $p(x)$.*

Lemma 2.14 *Let $\theta(x) \in L_A$ be a formula such that $RM(\theta(x)) = \alpha < \infty$ and $dM(\theta(x)) = 1$. Let $\delta(x, y) \in L$ be stable. Then $\{b \in \bar{M} : RM(\theta(x) \wedge \delta(x, b)) = \alpha\}$ is definable, over A.*

Proof. We may assume A to be finite. Let M be an ω-saturated model containing A. Let $p(x) \in S(M)$ be such that $\theta(x) \in p(x)$ and $RM(p(x)) = \alpha$. (Why does such p exist?) By 2.10, $p(x)|\delta$ is definable, by a formula $\psi(y) \in L_M$ say. But clearly for $b \in M$, $\delta(x, b) \in p(x)$ iff $RM(\theta(x) \wedge \delta(x, b)) = \alpha$. Thus for $b \in M$, $M \models \psi(b)$ iff $RM(\theta(x) \wedge \delta(x, b)) = \alpha$. As M is ω-saturated, the equivalence is valid for all $b \in \bar{M}$. (Why?) Also $\psi(x)$ is equivalent to a formula over A, by Lemma 1.2.

Corollary 2.15 *Let $\theta(x)$ be a formula of Morley rank $\alpha < \infty$ and Morley degree d possibly greater than 1. Let $\delta(x, y) \in L$ be stable. Then $\{b \in \bar{M} : (RM, dM)(\theta(x) \wedge \delta(x, b)) = (\alpha, d)$ is definable over A.*

Proof. Let $\theta_1(x), .., \theta_d(x)$ be formulas of Morley rank α and Morley degree 1 whose disjuction is equivalent to $\theta(x)$. Then $(RM, dM)(\theta(x) \wedge \delta(x, b)) = (\alpha, d)$ if and only if $RM(\theta_i(x) \wedge \delta(x, b)) = \alpha$ for $i = 1, .., d$. So we can use lemma 2.14 (and Lemma 1.2).

Proposition 2.16 *Suppose T is t.t, and $p(x) \in S(M)$ (M a model). Then p has Morley degree 1.*

Proof. Suppose not. So let $\theta(x) \in p(x)$ be such that $(RM, dM)(p(x)) = (RM, dM)(\theta(x)) = (\alpha, d)$ with $d > 1$. For each $\delta(x, y) \in L$ let $\psi_\delta(y)$ be the formula over M given by Corollary 2.15. Then one can see that actually the schema $d(\delta) = \psi_{\delta(y)}$ defines $p(x)$. Let $\theta'(x, d)$ be a formula over $\bar{M}$ which implies $\theta(x)$, and has Morley rank α and Morley degree 1. Let N be a model containing M and d. Let $q(x) = d(N)$, which is by 2.13 a complete type over N extending $p(x)$. Now both $\theta(x) \wedge \theta'(x, d)$ and $\theta(x) \wedge \neg\theta'(x, d)$ have $(RM, dM) < (\alpha, d)$, so $\models \neg\psi_{\theta'}(d)$ and $\models \neg\psi_{\neg\theta}(d)$, a contradiction (why?).

Exercise 2.17 *Let T be t.t and $p(x) \in S(M)$. Then there is a finite subset A of M such that $p(x)$ is definable over A.*

Corollary 2.18 *Let T be t.t., let $M < N$ be models, $p(x) \in S(M)$ and $p(x) \subseteq q(x) \in S(N)$. Then $q(x)$ is definable over M if and only if $RM(q(x)) = RM(p(x))$.*

Proof. Suppose q is definable over M, with defining schema d. So clearly $d(M) = p(x)$. Let $p(x)$ have Morley rank α and degree 1 (by 2.16), and let this be witnessed by $\theta(x) \in p(x)$. So for $\delta(x, y) \in L$, $d(\delta)(y)$ is equivalent to the formula "$RM(\theta(x) \wedge \delta(x, y) = \alpha$" which is given by 2.14 (in M so also in $\bar{M}$). As $q = d(N)$ too we see that for each formula $\delta(x, b) \in q(x)$, $RM(\theta(x) \wedge \delta(x, b)) = \alpha$, hence $RM(q) = \alpha$.
Conversely, suppose $RM(p) = RM(q) = \alpha$. Let (by 2.16) $\theta(x) \in p(x)$ have Morley rank α and Morley degree 1. For each $\delta(x, y)$ let $\psi_\delta(y)$ be "$RM(\theta(x) \wedge \delta(x, y)) = \alpha$". Then clearly $(\psi_\delta)_\delta$ is a defining schema (over M) for q.

Exercise 2.19 *Assume that $p(x) \in S(A)$ and $RM(p) = \alpha < \infty$. Then for any $B \supseteq A$ there is $q(x) \in S(B)$ such that $p(x) \subseteq q(x)$ and $RM(q) = \alpha$.*

Hint. By Exercise 5.18 of [1], $p(x) \cup \{\neg\theta(x) : \theta(x) \in L_B, RM(\theta(x)) < \alpha\}$ is consistent.

Lemma 2.20 *Suppose T is t.t.*
(i) Suppose $A \subseteq M$, $p(x) \in S(A)$ and $p(x) \subseteq q(x) \in S(M)$ with $RM(p) = RM(q) = \alpha$. Then q is definable almost over A.
(ii) Suppose $p(x) \in S(A)$ and $A \subseteq M$. Then there is some $q(x) \in S(M)$ with $p(x) \subseteq q(x)$ such that q is definable almost over A.

Proof. (i) We may assume that M is $(|T| + |A|)^+$-saturated and strongly homogeneous. Note that if f is an A-automorphism of M, then $f(q)$ is an extension of $p(x)$ of Morley rank α hence there are only finitely many possibilities for $f(q)$. (Why?) In particular, for each $\delta(x, y) \in L$, the δ-definition, $\psi_\delta(y)$ say, of q has only finitely many images under A-automorphisms of M, hence is almost over A.
(ii) Suppose $p(x)$ has Morley rank α. By Exercise 2.19, $p(x)$ has an extension in $S(M)$ of RM α. Now use part (i).

Lemma 2.21 *Let T be stable. Let $p(x), q(y)$ be complete types over M which are definable over A. Let $\delta(x, y) \in L$, and write it also as $\epsilon(y, x)$. Let $\psi(y)$ be the $\delta(x, y)$-definition of $p(x)$, and $\chi(x)$ the $\epsilon(y, x)$-definition of $q(y)$. (So $\psi(y), \chi(x)$ can be assumed to be L_A-formulas.) Then $\psi(y) \in q(y)|A$ if and only if $\chi(x) \in p(x)|A$.*

Proof. We may assume M to be saturated enough. Suppose for a contradiction that $\psi(y) \in q(y)$ but $\neg\chi(x) \in p(x)$. Construct a_i, b_i in M for $i < \omega$ such that a_1 realizes $p(x)|A$, b_1 realizes $q(y)|(A, a_1)$, and in general a_{n+1} realizes $p(x)|(A, b_1, .., b_n)$ and b_{n+1} realizes $q(y)|(A, a_1, .., a_{n+1})$. Note that $\models \neg\chi(a_i) \wedge \psi(b_i)$ for all i, so clearly $\models \delta(a_i, b_j)$ iff $i > j$, contradicting stability of $\delta(x, y)$.

We will now use T^{eq}.

Corollary 2.22 *Assume T to be t.t.. Let $p_1(x), p_2(x) \in S(M)$ be both definable over $A \subset M$. Suppose that $p_1|acl^{eq}(A) = p_2|acl^{eq}(A)$. Then $p_1 = p_2$.*

Proof. Let $\delta(x, y) \in L$ and $b \in M$. We want to show that $\delta(x, b) \in p_1(x)$ iff $\delta(x, b) \in p_2(x)$. Namely that $\models \psi_1(b)$ iff $\models \psi_2(b)$ where $\psi_i(y) \in L_A$ is the δ-definition of $p_i(x)$. By 2.20 (ii) $tp(b/A)$ has an extension $q(y) \in S(M)$ which is definable almost over A. In particular if $\chi(x)$ is the $\epsilon(y, x)$-definition of $q(y)$ (where $\epsilon(y, x) = \delta(x, y)$) then $\chi(x)$ is over $acl^{eq}(A)$. By Lemma 2.21 (which is also valid in $\bar{M}^{eq}$), $\chi(x) \in p_i|acl^{eq}(A)$ iff $\psi_i(y) \in q(y)$, for $i = 1, 2$. So by our assumptions, $\models \psi_1(b) \leftrightarrow \psi_2(b)$.

Here is a restatement:

Corollary 2.23 *Assume T to be t.t. Let $A = acl^{eq}(A)$ and $p(x) \in S(A)$. Then for any model M containing A there is a unique extension $q(x) \in S(M)$ of $p(x)$ such that $q(x)$ is definable over A.*

Note in particular that, in the context of 2.23, for any B containing A there is a unique extension $q(x) \in S(B)$ of $p(x)$ such that some extension of $q(x)$ to a model is definable over A. So (again in the context of 2.23) we will denote this extension by $p(x)|B$ (hopefully without ambiguity). Moreover there is a defining schema d over A such that for any $B \supseteq A$, $d(B) = p|B$.

Definition 2.24 *(T stable).*
(i) Let $p(x) \in S(A)$, $A \subseteq B$ and $p(x) \subseteq q(x)$. We say that $q(x)$ is a nonforking extension of $p(x)$ (or $q(x)$ does not fork over A) if $q(x)$ has some extension $q'(x)$ to a complete type over a model, such that $q'(x)$ is definable almost over A. (Here x is a finite tuple of variables.)
(ii) Suppose $A \subseteq B$, and a a finite tuple. We will say that a is independent

from B over A if $tp(a/B)$ does not fork over A.
(iii) If a is a possibly infinite tuple, we say that a is independent from B over A if a' is independent from B over A for all finite subtuples a' of a,
(iv) If a and b are arbitrary tuples, we will say that a is independent from b over A, if a is independent from $A \cup b$ over A in the sense of (iii).

Exercise 2.25 *The notion of nonforking in (i) above is invariant under adding names (constants) for elements of A.*

Proposition 2.26 *Suppose T to be t.t.. Let $p(x) \in S(A)$, $A \subseteq B$ and $p(x) \subseteq q(x) \in S(B)$. Then $q(x)$ is a nonforking extension of $p(x)$ if and only if $RM(p) = RM(q)$.*

Proof. The definitions, as well as Exercise 2.19 allow us to assume that B is a model M. The right to left direction is given by Lemma 2.20 (i).
Left to right: Let $RM(p(x)) = \alpha$. Let $p_1(x) = q|acl^{eq}(A)$. The first observation is that $RM(p(x)) = RM(p_1(x))$. This is because, for any formula $\psi(x) \in p_1$, some disjunction of finitely may A-conjugates of ψ is over A, hence in $p(x)$. Now, by 2.19 and 2.20 (i), there is an extension $q_1(x) \in S(M)$ of $p_1(x)$ which is definable over $acl^{eq}(A)$, and such that $RM(q_1) = RM(p_1) = RM(p)$. By 2.23, $q_1 = q$.

Remark 2.27 *Let us make explicit something observed in the proof above (for T t.t): for any a, A, a is independent from $acl^{eq}(A)$ over A.*

Exercise 2.28 *Let $p(x) \in S(A)$. Let $p_1(x), p_2(x) \in S(acl^{eq}(A))$ be extensions of $p(x)$. Show that there is an elementary $f : acl^{eq}(A) \to acl^{eq}(A)$ such that $f|A = id$ and $f(p_1(x)) = p_2(x)$.*

Proposition 2.29 *(Assume T to be t.t.) Let a, b denote finite tuples, and x a finite tuple of variables.*
(i) (transitivity) Suppose that $A \subseteq B \subseteq C$. Then a is independent from C over A iff a is independent from C over B and from B over A.
(ii) (symmetry) a is independent from b over A iff b is independent from a over A.
(iii) for any a and A there is a finite $A_0 \subseteq A$ such that a is independent from A over A_0,
(iv) For any type $p(x) \in S(A)$ the number of nonforking extensions of $p(x)$

over some (any) model is precisely the Morley degree of p.
(v) (existence) For any $p(x) \in S(A)$ and $B \supset A$, $p(x)$ has a nonforking extension $q(x) \in S(B)$.
(vi) (finite character) Suppose that $A \subseteq B$ and $p(x) \in S(B)$ forks over A. Then there is a finite tuple b from B such that $p(A)|(A \cup b)$ forks over A.

Proof. (i) is immediate from 2.26.
(ii) We may assume $A = acl^{eq}(A)$ (by 2.27). Let $p(x) = tp(a/A)$ and $q(y) = tp(b/A)$. By Exercise 2.25 we may assume that the elements of A are named by constants. Let d_1 be the defining schema corresponding to p given after 2.23, and d_2 the one for q. So a is independent from b over A iff a realizes $d_1(A \cup b)$ and b is independent from a over A if b realizes $d_2(A \cup a)$. Now suppose b to be independent of A over a. Let $\delta(x, y) \in L$, and suppose $\delta(x, b) \in d_1(A, b)$. We want to show that $\models \delta(a, b)$. Now $\models d_1(\delta)(b)$. By 2.21, $\models d_2(\epsilon)(a)$, where $\epsilon(y, x) = \delta(x, y)$. Thus, $\epsilon(y, a) \in d_2(A, a)$, whereby $\models \epsilon(b, a)$, namely $\models \delta(a, b)$.
(iii) Choose a formula $\phi(x) \in tp(a/A)$ with Morley rank $\alpha = RM(p)$. So $\phi(x)$ has parameters from a finite subset A_0 of A. Then $\phi(x) \in p|A_0$, so $RM(p|A_0) = \alpha$. Now use 2.26.
(iv) Let $p(x) \in S(A)$, and suppose $(RM, dM)(p) = (\alpha, d)$. Let $M \supseteq A$ be any model, and let $M' > M$ be suitably saturated. It is easy to show that $p(x)$ has precisely d extensions $p_1, .., p_d$ to complete types over M' of Morley rank α (and degree 1). For each $i = 1, .., d$, $p_i|M$ has Morley rank α and degree 1 (by 2.16).
(v) By 2.20 (ii), or by 2.19 and 2.26.
(vi) If $p(x)$ forks over A then by Proposition 2.6, $RM(p) < RM(p|A)$. Suppose $RM(p) = \alpha$. Choose a formula in p of Morley rank α and let b be the tuple of parameters in that formula.

Remark 2.30 *In the light of Definition 2.24, (i), (ii), and (vi) of 2.29 are also valid for a, b possibly infinite tuples. In fact (v) (existence) is also true for complete types of infinite tuples.*

Proof. Let us just consider existence. Fix an infinite tuple $a = (a_1, a_2, ...)$ say, and a set A. Let M be a model containing A. Let $p_n(x_1, .., x_n) = tp(a_1, .., a_n/acl^{eq}(A))$. Let $q_n(x_1, .., x_n) \in S(M)$ be the unique nonforking extension of p_n to M (by 2.23). Note that $q_n(x_1, .., x_n) \subset q_{n+1}(x_1, .., x_{n+1})$

for all n. (Why?) Let $q(x_1, x_2, ...)$ be the union, a complete type over M. If $b = (b_1, b_2, ...)$ ralizes q, then b is independent from M over A and $tp(b/A) = tp(a/A)$.

Corollary 2.31 *(T t.t.) Assume $A \subseteq C$. Then $tp(a, b/C)$ does not fork over A if and only if $tp(b/C)$ does not fork over A and $tp(a/Cb)$ does not fork over Ab.*

Proof. Enumerate C as c. Assume that (a, b) is independent from c over A. By Remark 2.30 (symmetry), c is independent from (a,b) over A. By transitivity, c is independent from b over A and clearly c is independent from a over $A \cup b$. Apply symmetry again to conclude that b is independent from c over A and a is independent from cb over Ab. The converse goes the same way.

Definition 2.32 *(T t.t.). Let $p(x) \in S(A)$. We say that $p(x)$ is stationary if it has exactly one nonforking extension over any $B \supseteq A$.*

Of course this definition makes sense for an arbitrary theory, but for T *t.t.*, stationarity of $p(x) \in S(A)$, is equivalent, by 2.29 (iv) to p having Morley degree 1. Moreover, any complete type over an *eq*-algebraically closed set is stationary (by 2.23).

Proposition 2.33 *(Conjugacy of nonforking extensions, finite equivalence relation theorem.) Assume T to be t.t.*
(i) Let $p(x) \in S(A)$. Let $M \supseteq A$ be $(|T| + |A|)^+$-saturated and strongly homogeneous. Let $p_1(x), p_2(x) \in S(M)$ be nonforking extensions of $p(x)$. Then there is an A-automorphism f of M such that $f(p_1) = p_2$.
(ii) Let $\theta(x)$ be a formula over A of Morley rank α and Morley degree d. Then there is a formula $\theta'(x)$ over A, which implies $\theta(x)$ and such that $RM(\theta(x) \wedge \neg\theta'(x)) < \alpha$, and there is an A-definable equivalence relation $E(x_1, x_2)$ on $\theta'(\bar{M})$ with precisely d classes, each of Morley rank α.

Proof. (i) First note that if $q(x) \in S(M)$ is a nonforking extension of $p(x)$ then $q(x)$ is determined by $q|acl^{eq}(A)$ (by 2.23). So now, suppose $q_1(x), q_2(x) \in S(M)$ are nonforking extensions of $p(x)$. By Exercise 2.28, let f be an elementary permutation of $acl^{eq}(A)$ which fixes A pointwise and takes $q_1|acl^{eq}(A)$ to $q_2|acl^{eq}(A)$. f then extends to an automorphism f' of

M, and $f'(q_1) = q_2$.
(ii) We begin with
Claim. We can find pairwise inconsistent formulas $\theta_1(x), .., \theta_d(x)$ over $acl^{eq}(A)$, each of which implies $\theta(x)$ and each of which has Morley rank α and Morley degree 1.
Proof. Let $\psi_1(x), .., \psi_d(x)$ be pairwise inconsistent formulas of Morley rank α and degree 1 over some model $M \supseteq A$, each of which implies $\theta(x)$. Let $q_i(x) \in S(M)$ contain $\psi_i(x)$ and have Morley rank α. By 2.26, $q_i(x)$ does not fork over A ($\theta(x) \in q_i(x)$, so $RM(q_i|A) = \alpha$). Let $p_i(x) = q_i(x)|acl^{eq}(A)$. Then $p_i(x)$ is stationary, (so of Morley degree 1) and $i \neq j$ implies $p_i(x) \neq p_j(x)$. So we may find formulas $\theta_i(x)$ in $p_i(x)$ of Morley rank α and degree 1. We may assume that the $\theta_i(x)$ are pairwise inconsistent.

Let Θ_i be the (finite) set of images of $\theta_i(x)$ under A-automorphisms ($\theta_i(x)$ is almost over A). Let Θ be $\cup_i \Theta_i$. So Θ is a finite set of formulas, and moreover Θ is A-invariant (in the obvious sense). Let $E(x_1, x_2)$ be: $\wedge\{\chi(x_1) \leftrightarrow \chi(x_2) : \chi(x) \in \Theta\}$. Then E is definable, A-invariant and so A-definable. Moreover E has finitely many classes. Let us restrict E to $\theta(\bar{M})$. As each formula in Θ has Morley rank α and degree 1, it is clear that each E-class either has Morley rank α and degree 1 or Morley rank $< \alpha$. Let X be the union of the classes of Morley rank α. So X is definable and A-invariant, so A-definable, by $\theta'(x)$ say. Then $\theta'(x)$ and E satisfy the required conditions.

We will discuss a few more issues in the context of $t.t$ theories (such as canonical bases, Morley sequences, dividing, existence of saturated models), and then discuss how the whole theory generalizes to the stable case.

Definition 2.34 *(T stable.) Let* $\mathbf{p}(x)$ *be a "global" type, that is a complete type over* $\bar{M}$*. Let* c *be a small tuple from* $\bar{M}^{eq}$*. We say that* c *is a canonical base of* $\mathbf{p}$ *if for any automorphism* f *of* $\bar{M}$*,* $f(\mathbf{p}) = \mathbf{p}$ *iff* $f(c) = c$*.*

Usually canonical bases are thought of as sets. Here we think of them as (possibly infinite) tuples, although this also causes some clumsy notation. In the case of algebraically closed fields, say, any stationary type is the "generic" type of some irreducible variety, and the canonical base of the type identifies with the smallest field of definition of the variety. Canonical bases play a crucial role in both general stability theory and geometric stability theory. In the $t.t$ case canonical bases will exist as single elements (in $\bar{M}^{eq}$), and given a stationary type p, if c is the canonical base of the global nonforking extenion of p, $RM(tp(c))$ will measure the "size" of the set of the set of conjugates of p (images of p under automorphisms of $\bar{M}$). Algebraically closed fields are an interesting example. Consider the family of algebraic curves over $\mathbf{C}$ given by $y = a_n x^n + a_{n-1}x^{n-1} + .. + a_0$ (as the a_i vary). This is clearly an "$n+1$-dimensional family" of curves. This corresponds to the fact that for "generic independent" $a_0, .., a_n$, the tuple $(a_0, ..., a_n)$ is a canonical base for the "generic type" of the curve, and $RM(tp(a_0, .., a_n)/\emptyset)) = n+1$.

Proposition 2.35 *(T stable) Any global type* $\mathbf{p}(x)$ *has a canonical base, which is unique up to interdefinability. Namely if* c_1, c_2 *are canonical bases of* $\mathbf{p}$ *then* $c_1 \subseteq dcl(c_2)$ *and* $c_2 \subseteq dcl(c_1)$*. In particular,* $\mathbf{p}$ *has a unique definably closed canonical base (in* $\bar{M}^{eq}$*), up to enumeration, which we refer to as* $Cb(\mathbf{p})$*.*

Proof. For each $\delta(x, y) \in L$, let $\psi_\delta(y) \in L_{\bar{M}}$ be a δ-definition for $\bar{p}$ (by 2.10), let c_δ be a code in $\bar{M}^{eq}$ for $\psi_\delta(y)$, and let $c = (c_\delta)_\delta$. It is clear that c works.

If d is another canonical base for $\mathbf{p}$, then by definition, an automorphism of $\bar{M}$ fixes c iff it fixes d. Hence c and d are interdefinable.

Definition 2.36 *(T t.t.) Let* $p(x) \in S(A)$ *be stationary. We define* $Cb(p)$ *to be* $Cb(\mathbf{p}(x))$ *where* $\mathbf{p}(x)$ *is the unique global nonforking extension of* p*.*

Remark 2.37 *(T t.t.) So by the proof of 2.35, if* $p(x) \in S(A)$ *is stationary, then* $Cb(p)$ *is the eq-definable closure of the set of codes of* δ*-definitions of* $\mathbf{p}$ *(the global nonforking extension of* p*) as* $\delta(x, y)$ *ranges over* L*.*

Let me reiterate a remark made earlier: Any complete type $p(x)$ over a set A has a unique extension to a complete type over $dcl^{eq}(A)$. In particular it makes sense to talk about the restrition of $p(x)$ to C for $C \subseteq dcl^{eq}(A)$.

Lemma 2.38 *(Assume T to be t.t.)*
(i) Let $p(x) \in S(A)$ *be stationary. Then* $Cb(p)$ *is the smallest eq-definably closed subset* A_0 *of* $dcl^{eq}(A)$ *such that* $p(x)$ *does not fork over* A_0 *and* $p|A_0$ *is stationary.*
(ii) Given a finite tuple a *and* $A \subseteq B$*,* $tp(a/B)$ *does not fork over* A *if* $Cb(stp(a/B))$ *is contained in* $acl^{eq}(A)$*.*
(iii) If $p(x) \in S(A)$ *is stationary, then* $Cb(p)$ *is the eq-definable closure of a*

single element (of $\bar{M}^{eq}$).
(iv) (normalization) Let $\phi(x)$ be a formula of Morley rank α and degree 1. Then there is another formula $\psi(x)$ of Morley rank α and degree 1, such that $(RM, dM)(\phi(x) \wedge \psi(x)) = (\alpha, 1)$ and for any conjugate $\psi'(x)$ of $\psi(x)$ under $Aut(\bar{M})$, either $\models \forall x(\psi(x) \leftrightarrow \psi'(x))$, or $RM(\psi(x) \wedge \psi'(x)) < \alpha$.

Proof. (i) Let c be $Cb(p)$. Note that any A-automorphism of $\bar{M}$ fixes $\mathbf{p}$ (the global nonforking extension of p) so fixes c pointwise, whereby $c \subseteq dcl^{eq}(A)$. Suppose that $A_0 \subseteq dcl^{eq}(A)$, and p does not fork over A_0 and $p|A_0$ is stationary. Then clearly the global nonforking extension of $p|A_0$ is the same as the global nonforking extension of p. Therefore by the first part of the proof applied to $p|A_0$ we see that $c \subseteq dcl^{eq}(A_0)$. Finally we need to show that $p(x)$ does not fork over c and $p(x)|c$ is stationary. In fact by definition (Definition 2.24 (i)) $p(x)$ does not fork over c. To show that $p(x)|c$ is stationary, we show that $\mathbf{p}$ is its unique global nonforking extension. Let $\mathbf{q}$ be another global nonforking extension of $p|c$. By 2.33 (i) there is a c-automorphism of $\bar{M}$ taking $\mathbf{p}$ to $\mathbf{q}$. But $\mathbf{p}$ is fixed by all c-automorphisms of $\bar{M}$. So $\mathbf{q} = \mathbf{p}$.
(ii) Remember that $stp(a/B)$ denotes $tp(a/acl^{eq}(B))$, a stationary type. Let $c = Cb(stp(a/B))$. Let $\mathbf{p}$ be the global nonforking extension of $stp(a/B)$. Suppose that $c \in acl^{eq}(A)$ where $A \subseteq B$. Then clearly $\mathbf{p}$ is definable almost over A, so does not fork over A. By transitivity $\mathbf{p}|B$ does not fork over$A$. That, is $tp(a/B)$ does not fork over A.

Conversely assume $tp(a/B)$ does not fork over A. So $tp(a/acl^{eq}(B))$ does not fork over A. So $\mathbf{p}$ does not fork over A and hence is definable almost over A. Thus $c \subseteq acl^{eq}(A)$. (We are using transitivity, as well as the definition of nonforking here.)
(iii) This will be something special about *t.t.* theories. Let $p(x) \in S(A)$ be stationary, and $c = Cb(p)$. So we know that $p_0(x) = p|c$ is stationary (by (i)). By 2.29 (iv), p_0 has Morley degree 1. Let $\phi(x, c_0)$ be a formula over c which is in p_0 and has Morley rank $\alpha = RM(p_0)$ and Morley degree 1. In fact there is no harm in allowing c_0 to be a "code" for $\phi(x, c_0)$. Let $\mathbf{p}(x)$ be the global nonforking extension of p_0. So $RM(\mathbf{p}) = \alpha$ and $\phi(x, c_0) \in \mathbf{p}$. Let f be an automorphism of $\bar{M}$ which fixes c_0. Then $f(\mathbf{p})$ has Morley rank α and contains $\phi(x, c_0)$. Hence $f(\mathbf{p}) = \mathbf{p}$. Hence (by definition), $f(c) = c$. So all elements of c are fixed by all c_0-automorphisms of $\bar{M}$. So $c \subseteq dcl^{eq}(c_0)$ as required.

(iv) Let $p(x) \in S(A)$ be a type of Morley rank α which contains $\phi(x)$. So by 2.29(iv) p is stationary. By the proof of (iii) above there is a formula $\psi(x)$ over c of Morley rank α and degree 1 such that if c_0 is a code for $\psi(x)$, then $c = dcl^{eq}(c_0)$. Note that as $\phi(x) \wedge \psi(x) \in p(x)$, $(RM, dM)(\phi(x) \wedge \psi(x)) = (\alpha, 1)$ (hence, in passing, the symmetric difference of ϕ and ψ has Morley rank $< \alpha$). Let $\mathbf{p}$ be the global nonforking extension of $p(x)$. Note that for any automorphism f of $\bar{M}$, $f(\mathbf{p}) = \mathbf{p}$ iff $RM(\psi(x) \wedge f(\psi)(x)) = \alpha$. But also $f(\mathbf{p}) = \mathbf{p}$ if $f(c_0) = c_0$ (as $Cb(\mathbf{p}) = dcl^{eq}(c_0)$). So putting it together we see that for any automorphism f of $\bar{M}$, $RM(\psi(x) \wedge f(\psi(x)) = \alpha$, iff $f(\psi(x))$ is equivalent to $\psi(x)$.

Exercise 2.39 *(i) Suppose that $(b_i : i < \omega)$ is an A-indiscernible sequence (of finite tuples of the same length). Then for all $i, j < \omega$, $stp(b_i/A) = stp(b_j/A)$.*
(ii) (T stable.) Suppose that $(b_i : i < \omega)$ is an A-indiscernible sequence of tules (of some fixed finite length). Then $\{b_i : i < \omega\}$ is totally indiscernible over A, namely for any n and n-element subsets $\{i_1, .., i_n\}$, $\{j_1, .., j_n\}$ of ω, $tp(b_{i_1}, .., b_{i_n}/A) = tp(b_{j_1}, .., b_{j_n}/A)$.

Definition 2.40 *(T t.t.) Let $p(x) \in S(A)$ be stationary. By a Morley sequence (of length ω) of $p(x)$ we mean a sequence $(b_i : i < \omega)$ such that each b_i realizes the (unique) nnonforking extension of $p(x)$ over $A \cup \{b_0, .., b_{i-1}\}$ (or in previous notation b_i realises $p|(A \cup \{b_0, .., b_{i-1}\})$.*

Exercise 2.41 *(T t.t.) Let $(b_i : i < \omega)$ be a Morley sequence of the stationary type $p(x) \in S(A)$. Then $(b_i : i < \omega\}$ is A-indiscernible. Moreover if $(c_i : i < \omega)$ is another Morley sequence of $p(x)$, then $tp((b_i)_{i<\omega}/A) = tp((c_i)_{i<\omega}/A)$.*

Proposition 2.42 *(T t.t) Let $p(x) \in S(A)$ be stationary. Let $(b_i : i < \omega)$ be a Morley sequence of $p(x)$. Then $Cb(p(x)) \subseteq dcl^{eq}(b_i : i < \omega)$ (in fact $Cb(p(x)) \subseteq dcl^{eq}(b_0, .., b_n)$ for some $n < \omega$).*

Proof. Let $\mathbf{p}$ be the global nonforking extension of $p(x)$. By inspecting the proof of Lemma 2.6, we see that for any $\delta(x, y) \in L$, the δ-definition of $\mathbf{p}(x)$ is over $(a_i : i < \omega)$ for some Morley sequence $(a_i)_i$ of $p(x)$. But this δ-definition is also over A. Hence by the second part of Exercise 2.41, it is over any Morley sequence $(b_i : i < \omega)$ of $p(x)$. This shows that $c = Cb(p)$ is in the definable closure of any Morley sequence of $p(x)$. As (by 2.38 (iii)) c is contained in $dcl(c_0)$ for a single imaginary $c_0 \in c$, we obtain the parenthetical remark too.

Definition 2.43 *(T t.t.) Let $\{b_i : i \in I\}$ be a set of tuples, and A a set of parameters. We will say that $\{b_i : i \in I\}$ is independent over A or A-independent, if for each $i \in I$, b_i is independent from $A \cup \{b_j : j \in I, j \neq i\}$ over A.*

Exercise 2.44 *(T t.t) Suppose that $(b_i : i < \omega)$ is a sequence such that b_i is independent from $(b_0, .., b_{i-1})$ over A for each $i < \omega$. Show that $\{b_i : i < \omega\}$ is A-independent. (Forking calculus.) So note in particular that any Morley sequence of a stationary type $p(x) \in S(A)$ will be A-independent.*

Lemma 2.45 *(T t.t). Suppose that A is a set of parameters, $\{b_i : i \in I\}$ is an A-independent set of tuples, c is a finite tuple, and for each $i \in I$, $tp(c/Ab_i)$ forks over A. Then I is finite.*

Proof. Suppose not. Without loss $I = \omega$.
Claim. For each i, $tp(b_{i+1}/A \cup \{b_0, .., b_i\} \cup \{c\})$ forks over $A \cup \{b_0, .., b_i\}$.
If not then by our assumptions, together with transitivity, $tp(b_{i+1}/A \cup \{b_0, .., b_i\} \cup \{c\})$ does not fork over A, hence by transitivity again $tp(b_{i+1}/A \cup \{c\})$ does not fork over A, so by symmetry, $tp(c/A \cup \{b_i\})$ does not fork over A, contradicting the hypotheses.

By the claim and symmetry, $tp(c/A \cup \{b_0, .., b_{i+1}\})$ forks over $A \cup \{b_0, .., b_i\})$ for all $i < \omega$. By Proposition 2.26, we have $RM(tp(c/A)) > RM(tp(c/A \cup \{b_0\})) > ... > RM(tp(c/A \cup \{b_0, .., b_i\})) > RM(tp(c/A \cup \{b_0, .., b_i, b_{i+1}\})) > ...$ which gives a strictly descending infinite sequence of ordinals, a contradiction.

Definition 2.46 *(i) Let $\phi(x, b)$ be a formula with parameter b. $\phi(x, b)$ is said to divide over A if there is an infinite A-indiscernible sequence $(b_i : i < \omega)$ of realizations of $tp(b/A)$ such that $\{\phi(x, b_i) : i < \omega\}$ is inconsistent (namely not realized in $\bar{M}$).*
(ii) A complete type $p(x) \in S(B)$ is said to divide over $A \subseteq B$, if some formula $\phi(x) \in p(x)$ divides over A.

Example 2.47 *(a) Suppose $\phi(x, b)$ is consistent and almost over A. Then $\phi(x, b)$ does not divide ove A.*
(b) Suppose that $b \notin acl(A)$. Then the formula $x = b$ divides over A.

Proof. (a) Let $(b_i : i < \omega)$ be A-indiscernible with $tp(b/A) = tp(b_i/A)$ for some (all) i. The formulas $\phi(x, b_i)$ are images of $\phi(x, b)$ under automorphisms which fix A pointwise. So, as $\phi(x, b)$ is almost over A there must be $i < j$ such that $\models \forall x(\phi(x, b_i) \leftrightarrow \phi(x, b_j))$. So by indiscernibility, all $\phi(x, b_i)$ are equivalent. So clearly $\{\phi(x, b_i) : i < \omega\}$ is consistent (as $\phi(x, b)$ is consistent).
(b) As $b \notin acl(A)$ there are realizations b_i of $tp(b/A)$ for $i < \omega$ such that $i \neq j$ implies $b_i \neq b_j$. By Proposition 5.11 of [1], we may assume that $(b_i : i < \omega)$ is A-indiscernible. But then $x = b_i \wedge x = b_j$ is inconsistent for $i \neq j$. This shows that $x = b$ divides over A.

Remark 2.48 *Shelah's original definition of forking was as follows:*
(i) The formula $\phi(x, b)$ forks over A if there are formulas $\phi_1(x, b_1), ..., \phi_n(x, b_n)$ such that $\models \forall x(\phi(x, b) \rightarrow \vee_i \phi_i(x, b_i))$ and each $\phi_i(x, b_i)$ divides over A.
(ii) $p(x) \in S(B)$ forks over $A \subseteq B$ if there is a formula $\phi(x, b) \in p(x)$ which forks over A.

The next proposition shows that our definition of forking agrees with Shelah's original one (for $t.t$ theories).

Lemma 2.49 *(T $t.t$) Let $p(x) \in S(B)$ and $A \subseteq B$. Then*
(i) $p(x)$ forks over A (in the sense of Definition 2.24) if and only if $p(x)$ divides over A.
(ii) $p(x)$ forks over A (in the sense of Definition 2.24) if and only if $p(x)$ forks over A in Shelah's sense (in Remark 2.48)

Proof. (i) Suppose that $p(x)$ forks over A. By 2.29 (vi) there is finite $b \in B$ such that $p(x)|Ab$ forks over A. Write $p(x)|Ab = q(x, b)$. Let $r(y) = stp(b/A)$. Let $(b_i : i < \omega)$ be a Morley sequence of $r(y)$.
Claim. $\{q(x, b_i) : i < \omega\}$ is inconsistent.
Proof. If not, there is c realizing it. By automorphism $tp(c/Ab_i)$ forks over A for all i. Now $(b_i : i < \omega)$ is A-independent (why?), so Lemma 2.45 gives a contradiction.

By the claim and compactness there is a formula $\phi(x, b) \in q(x, b)$ such that $\{\phi(x, b_i) : i < \omega\}$ is inconsistent. By 2.41, $(b_i : i < \omega)$ is A-indiscernible. So $\phi(x, b)$ divides over A. (Note that $phi(x, b)$ may contain hidden parameters from A, say a. Write $\phi(x, b)$ as $\phi'(x, b, a)$. So $\phi(x, b, a) \in p(x)$, $(b_i, a) : i < \omega)$ is an A-indiscernible sequence of realizations of $tp(ba/A)$, and $\{\phi'(x, b_i, a) : i < \omega\}$ is inconsistent, hence $\phi'(x, b, a)$ divides over A.)

Conversely, suppose that $p(x)$ does not fork over A. Let $\phi(x,b) \in p(x)$, and let $(b_i : i < \omega)$ be an A-indiscernible sequence of realizations of $tp(b/A)$. We may assume that $b = b_0$. Let a realize $p(x)$, let $p'(x)$ be $tp(a/acl^{eq}(B))$, and let $\mathbf{p}(x)$ be the unique global nonforking extension of p'. Then $\mathbf{p}$ does not fork over A (by transitivity) hence is definable over $acl^{eq}(A)$. In particular there is a formula $\psi(y)$ over $acl^{eq}(A)$ such that for any $b' \in \bar{M}$, $\phi(x,b') \in \mathbf{p}$ iff $\models \psi(b')$. By Exercise 2.39 (i), $tp(b_i/acl^{eq}(A)) = tp(b/acl^{eq}(A))$ for all $i < \omega$. As $\phi(x,b) \in \mathbf{p}(x)$, we have $\models \psi(b)$, and thus $\models \psi(b_i)$ for all $i < \omega$, and so $\phi(x,b_i) \in \mathbf{p}(x)$ for all i. So $\{\phi(x,b_i) : i < \omega\}$ is consistent. So $\phi(x,b)$ does not divide over A.

(ii) Note that if $\phi(x)$ divides over A then $\phi(x)$ forks over A in the sense of Shelah, so $LHS \rightarrow RHS$ follows from (i).
Conversely, suppose $p(x) \in S(B)$ and $p(x)$ forks over $A \subseteq B$ in the sense of Shelah. So there are finitely many formulas $\phi_i(x)$ for $i = 1,..,n$ say, such that $p(x) \models \vee_i \phi_i(x)$, and each $\phi_i(x)$ divides over A. Let $C \supseteq B$ contain the parameters from the ϕ_i. So clearly any extension of $p(x)$ to a complete type over C contains one of the ϕ_i hence divides over A. On the other hand $p(x)$ has, by 2.29, a nonforking extension (in sense of 2.24) extension $q(x) \in S(X)$. By what we just said, $q(x)$ divides over A. By part (i), $q(x)$ forks over A in the sense of 2.24. By 2.29 (transitivity), $p(x)$ forks over A (in sense of 2.24).

Here is another characterization of forking in *t.t.* theories. The proof makes use of 2.40 although it could be given without it.

Lemma 2.50 *(T t.t) Let $p(x) \in S(B)$ and $A \subseteq B$. The following are equivalent:*
(i) $p(x)$ does not fork over A,
(ii) for any formula $\phi(x) \in p(x)$ and model $M \supseteq A$, there is $c \in M$ such that $\models \phi(c)$.

Proof. Let us remark first that the model M in (ii) need not contain B.
(i) implies (ii). Assume $p(x)$ does not fork over A. Let $\phi(x,b) \in p(x)$ (where we exhibit the parameters) and let M be a model containing A. Let $p'(x) \in S(M \cup B)$ be a nonforking extension of $p(x)$. Then by transitivity, $p'(x)$ does not fork over M. Let a realizes $p'(x)$. By transitivity and symmetry, $tp(b/M,a)$ does not fork over M. We then know that $tp(b/M,a)$ is precisely $d(M \cup a)$ where d is a defining schema for $tp(b/M)$. Let $\psi(x) = d(\phi(x,y))$.

So $\models \psi(a)$, hence $\psi(x)$ is consistent, hence realized in M by a' say (as $\psi(x)$ is over M). Thus $\models \phi(a', b)$.
Conversely, assume (ii). We will show that $p(x)$ does not divide over A. Let $\phi(x, b) \in p(x)$. Let $(b_i : i < \omega)$ be an A-indiscernible sequence of realizations of $tp(b/A)$, where we may assume that $b_0 = b$. Let $M \supseteq A$ be a model such that $(b_i : i < \omega)$ is also M-indiscernible. (We leave it as an exercise to be added to assignment 3, that such M can be found.) By assumption there is $c \in M$ such that $\models \phi(c, b_0)$. But then $\models \phi(c, b_i)$ for all i, hence $\{\phi(x, b_i) : i < \omega\}$ is consistent.

Finally let us return to the question of the existence of saturated models of $t.t$ theories. We proved in Lemma 5.20 of [1] that if T is countable and ω-stable, then for any infinite regular cardinal, T has a saturated model of cardinality λ. The proof goes over to the $t.t.$ case: T has a saturated model of cardinality λ for every regular $\lambda \geq |T|$. We can now deal with the general case. First:

Remark 2.51 *Suppose T is t.t. Then for every $\lambda \geq |T|$, T is λ-stable, that is, for every model M of T of cardinality λ, there are only λ-many complete n-types over M for all M.*

Proof. A complete type $p(x)$ over a model M is determined by the choice of a formula $\phi(x) \in p$ such that $RM(p(x)) = RM(\phi(x))$ and $dM(\phi(x)) = 1$.

Proposition 2.52 *Suppose T is t.t. Then for every $\lambda \geq |T|$, T has a saturated model of cardinality λ.*

Proof. By Remark 2.51, we can build a continuous elementary chain $(M_\alpha : \alpha < \lambda)$ of models of T of cardinality λ, such that all complete (finitary) types over M_α are realized in $M_{\alpha+1}$. Let M be the union of the chain. So M is of cardinality λ. We claim that M is λ-saturated. Let $A \subset M$ have cardinality $< \lambda$, and $p(x) \in S(A)$. We must show that p is realized in M. By adding a finite set of parameters to A we may assume that $p(x)$ is stationary (why?). Let A_0 be a finite subset of A such that $p(x)$ does not fork over A_0 and $p|A_0$ is stationary. (For example, let $\phi(x) \in p(x)$ have least Morley rank, and Morley degree 1 and let A_0 be the parameters from ϕ.) Then A_0 is contained in M_α for some $\alpha < \lambda$. Let $p_0 = p|A_0$. By choice of the M_β we can find $c_{\beta+1} \in M_{\beta+1}$ for $\alpha \leq \beta < \lambda$, such that $c_{\beta+1}$ realizes the unique nonforking extension of $p_0(x)$ over M_β. Then $(c_\beta)_\beta$ is a Morley sequence in $p_0(x)$ (of length λ) in particular, is A_0-independent. By 2.45, for each finite tuple b from A all but finitely many c_β's are independent from b over A_0. By the finite character of forking, and the fact that A has cardinality $< \lambda$ there is some c_β which is independent from M' over A_0. But then $tp(c_\beta/A)$ is the unique nonforking extension of p_0 over A which is $p(x)$. So $p(x)$ *is* realized in M.

In the next section we will discuss structures/definable sets of finite Morley rank, especially of rank 1, and there will be many examples. So we conclude this section with some infinite Morley rank examples.

Example 2.53 *Let E_i for $i < \omega$ be binary relation symbols. Let $L_n = \{E_i : i \leq n\}$. Let T_n be the L_n-theory with axioms:*
- for $i \leq n$, E_i is an equivalence relation with infinitely many classes,
- E_0 is equality,
- if $i < n$ then every E_{i+1}-class is a union of infinitely many E_i-classes.
Let $T_\omega = \cup_n T_n$ in language $L_\omega = \cup_n L_n$.
Then in T_n, $(RM, dM)(x = x) = (n+1, 1)$, and in T_ω, $(RM, dM)(x = x) = (\omega, 1)$.

Explanation The proof of this is left to you. First show that each T_n is complete with quantifier-elimination. Show that a formula $E_i(x, a)$ has Morley rank i and degree 1. How can you describe forking for 1-types in these theories?

Before the next example, recall that we proved in 3.6 of [1] that ACF_p is complete with QE in the language of rings (for p a prime or zero). It follows from QE that any definable subset of an algebraically closed field is finite or cofinite, hence $x = x$ has Morley rank 1 and degree 1. In particular ACF_p is ω-stable (why??).

Example 2.54 *DCF_0 was introduced in Example 3.17 of [1], where we said (without proof) that DCF_0 is complete with quantifier-elimination. We use this to sketch a proof that DCF_0 is ω-stable, hence t.t..*

Let $(K, +, -, \cdot, 0, 1, \partial)$ be a countable model of DCF_0 and let K' be a saturated elementary extension. We consider complete 1-types over K realized in

K' and show there are only countably many of them. Let L be the language of differential rings and L_r the language of rings. $(K, +, -, \cdot, 0, 1)$ will be an algebraically closed field of characteristic 0. Fix $p(x) \in S_1(K)$, and let $a \in K'$ realize it.
We have two cases:
Case 1. $(a, \partial(a), \partial^2(a), ...)$ is algebraically independent over K. (Namely for any polynomial $P(x_0, ..., x_n) \in K[x_0, .., x_n]$, $P(a, \partial(a), .., \partial^n(a)) \neq 0$. By QE, $p(x)$ is uniquely determined. That is, there is at most one $p(x) \in S_1(K)$ such that Case 1 holds.

Case 2. Otherwise. Let n be least such that $(a, \partial(a), .., \partial^n(a))$ is algebraically dependent over K. So n depends only on $p(x)$ and we call n the order of p.
Claim. $p(x)$ is determined by $n = ord(p)$ together with $tp_{L_r}(a, \partial(a), .., \partial^n(a)/K)$.
Let $P(x)$ be the minimal polynomial over $\partial^n(a)$ over $K(a, \partial(a), .., \partial^{n-1}(a))$. Then applying ∂ to $P(\partial^n(a)) = 0$, and using minimality of $P(X)$, one finds a K-rational function $s(x_0, .., x_n)$ such that $\partial^{n+1}(a) = s(a, \partial(a), .., \partial^n(a))$. Continuing, one finds K-rational functions $s_i(x_0, .., x_n)$ such that $\partial^{n+i}(a) = s_i(a, \partial(a), .., \partial^n(a))$.
The conclusion is that if $b \in K'$, $ord(tp(b/K)) = n$ and $tp_{L_r}(b, \partial(b), .., \partial^n(b)/K) = tp_{L_r}(a, \partial(a), .., \partial^n(b)/K)$ then for all $i \geq 1$, $\partial^{n+i}(b) = s_i(b, ..., \partial^n(b))$. It follows that for all m, $tp_{L_r}(b, \partial(b), .., \partial^m(b)/K) = tp_{L_r}(a, \partial(a), .., \partial^m(a)/K)$, and so by QE, $tp(b/K) = tp(b/K) = p(x)$, proving the claim.

By the claim together with ω-stability of ACF_0, there are only countably many 1-types over K of finite order. Together with Case 1 giving a unique 1-type we see that there are only countably many complete 1-types over K. This proves ω-stability of DCF_0.

Let us point out a few additional things about 1-types in DCF_0 without proof:
- $RM(x = x) = \omega$,
- $k_1 \subset k_2 \subset K$ are differential subfields of K, and $p(x)$ is a complete 1-type over k_2, then $p(x)$ does not fork over k_1 iff $ord(p(x)) = ord(p(x)|k_1)$.

3 Strongly minimal sets and "geometry"

Let T be a (complete) theory in a language L, and $\phi(\bar{x})$ an L-formula. We say that $\phi(\bar{x})$ is strongly minimal if $RM(\phi) = 1$ and $dM(\phi) = 1$. If X is an A-definable set in some structure M we say that X is strongly minimal, if the formula ϕ defining X is a strongly minimal formula in the theory $Th(M,a)_{a\in A}$.

Remark 3.1 *Let $\bar{M}$ be an ω-saturated structure, and $X \subseteq \bar{M}^n$ a set which is A-definable in $\bar{M}$, where A is a finite set of parameters from M. Then X is strongly minimal if X is infinite, and for every definable (with parameters) subset Y of $\bar{M}^n$, $X \cap Y$ is finite or $X \setminus Y$ is finite.*

By a strongly minimal theory we mean a (complete) 1-sorted theory T in which the formula "$x = x$" is strongly minimal.

Strongly minimal formulas and sets are important for $t.t$ theories of finite Morley rank. Any structure of finite Morley rank can be more or less built out of strongly minimal sets. For groups of finite Morley rank this can be made more precise: if G is a group of finite Morley rank, then there are normal definable subgroups $\{1\} < N_1 < .. < N_k = G$ such that each quotient N_i/N_{i-1} is "almost strongly minimal".

Studying strongly minimal sets amounts to the same thing as studying strongly minimal theories. Let us first elaborate on this, as it gives us an opportunity to discuss "induced structure" and "stable embeddedness". First:

Remark 3.2 *Let T be t.t. Then any complete type over any set A is definable: for each $\phi(x,y) \in L$ there is $\psi(y) \in L_A$ such that for $b \in A$, $\phi(x,b) \in p(x)$ iff $\models \psi(b)$.*

Proof. Let $\theta(x) \in p(x)$ have least (RM, dM), say (α, d). So for any $\delta(x,y) \in L$ and $b \in A$, $\delta(x,b) \in p(x)$ iff $(RM, dM)(\theta(x) \wedge \delta(x,b)) = (\alpha, d)$. Now use Corollary 2.15.

Definition 3.3 *Let M be an L-structure, possibly many-sorted. Let S be a sort of L, and X a definable subset of $S(M)$, defined with parameters A from M. Then by the "induced structure over A on X" we mean the following: The 1-sorted language L' consists of an n-place relation symbol R_Y for each A-definable subset Y of $S(M)^n$. Make X into an L'-structure with universe X by interpreting each R_Y as $Y \cap X^n$.*

Lemma 3.4 *Assume T to be t.t. Let M be a model of T (maybe saturated). Let X be an A-definable set in M. Let $\mathcal{X}$ be the L'-structure on X given by*

the definition above. Then
(i) $Th(\mathcal{X})$ has quantifier-elimination in L'.
(ii) The subsets of X^n definable (with parameters) in $\mathcal{X}$ are the same as those definable (with parameters) in M.

Proof. (i) is obvious.
(ii). By (i) any subset of X^n definable with parameters in $\mathcal{X}$ is definable (with the same parameters) in M. Conversely, let $Y \subset X$ say be definable by the formula $\phi(x, b)$ in M where b may be outside X. By Remark 3.2, $tp(b/X)$ is definable (over X). So there is a formula $\psi(x, c)$ with parameters c from X, such that for $a \in X$, $M \models \phi(a, b)$ iff $M \models \psi(a, c)$. Let Z be the set in M defined by $\psi(x, z)$. Then Y is defined in $\mathcal{X}$ be the formula $R_Z(x, c)$.

Remark 3.5 *Note that, in Lemma 3.4, if M is 1-sorted and X is M itself, then the structure $\mathcal{X}$ is essentially just the Morleyization of M.*

Let us consider some examples. The first has been mentioned before:

Example 3.6 *The theory of algebraically closed fields of some fixed characteristic is strongly minimal.*

Proof. ACF_p has quantifier-elimination in the language of rings. So if K is an algebraically closed field, every definable (with parameters) subset of K is a Boolean combination of sets defined by things of the form $P(x) = 0$ where $P \in K[x]$ is a polynomial over K in single indeterminate x. The set of solutions of $P(x) = 0$ is either finite or everything. So any definable subset of K is finite or cofinite.

Example 3.7 *The formula $\partial(x) = 0$ is strongly minimal in DCF_0. Moreover the induced structure on the corresponding strongly minimal set in a model of DCF_0 is just that of an algebraically closed field.*

Proof. Because of 3.4(ii) and 3.6 it is enough to show the second part. Let K be a differentially closed field. The solution set of $\partial(x) = 0$ in K is usually called the constants of K and sometimes dentoted C_K. I will be using the fact that C_K is algebraically closed. (Proof: A derivation on a field F extends uniquely to a derivation on the algebraic closure of F. Thus every existentially closed differential field K is algebraically closed. By the uniqueness assertion above the algebraic closure in K of C_K must also consist of constants.) As C_K is defined without parameters, we take A as in 3.3 to be $\emptyset$. Let X now be a $\emptyset$-definable subset of K^n, and consider $X \cap C_K^n$, and we want to show that this latter set is $\emptyset$-definable in $(C_K, +, \cdot, 0, 1)$. By quantifier-elimination in DCF_0 we may assume that X is given by $P(\bar{x}, \partial\bar{x}, ...) = 0$, where P is a polynomial over $\mathbf{Z}$. As ∂ is 0 on C_K, $X \cap C_K^n$ is defined by $P(\bar{x}, \bar{0}, \bar{0}, ..) = 0$.

(The point we are making here is that the subsets of C_K^n which are $\emptyset$-definable in the structure $(K, +, \cdot, -, 0, 1, \partial)$ are the same as those which are $\emptyset$-definable in the structure $(C_K, +, \cdot, -, 0, 1)$. The theory of the latter structure is strongly minimal, by 3.6. By 3.4 (ii), and $t.t.$-ness of DCF_0, C_K is a strongly minimal set in K.)

Example 3.8 *Let F be a division ring. Let V be an infinite-dimensional vector space over F. Consider V as a structure in the language containing $+, -, 0$, as well as unary functions μf for $f \in F$ (representing scalar multiplication by elements of F). Then $Th(V)$ is strongly minimal.*

Proof. A back-and-forth argument shows that $Th(V)$ has quantifier-elimination in the given language. So, up to Boolean combination, a formula $\phi(x, a_1, .., a_n)$ is of the form $\mu x + \sum_{i=1,..,n} \mu_i a_i = \lambda x + \sum_{i=1,..,n} \lambda_i a_i$ which is equivalent to $x = b$ for some $b \in V$. Good.

(Note that every definable subset of V^n is, up to Boolean combination, of the $\sum_{i=1,..,n} \mu_i x_i = b$.)

Let us now fix a $t.t.$ theory T, a saturated model $\bar{M}$ of T and a strongly minimal set D $\emptyset$-definable in $\bar{M}$. The reader can assume D to be precisely $\bar{M}$ if he or she so wishes. We will study definability and independence in D. x will denote a variable of the sort of which D is a subset. Here is a special case of Lemma 2.14, which has an easier proof.

Remark 3.9 *Let $\phi(x, y)$ be an L-formula. Then there is an L-formula $\psi(y)$ such that for all $b \in \bar{M}$, $\models \psi(b)$ iff $\phi(x, b) \wedge D(x)$ is finite.*

Proof. If not, then for arbitrarily large n there is $b \in \bar{M}$ such that both $\phi(x, b) \wedge D(x)$ and $\neg\phi(x, b) \wedge D(x)$ have at least n solutions. By compactness there is b such that both $\phi(x, b) \wedge D(x)$ and $\phi(x, b) \wedge D(x)$ have infinitely many solutions, contradicting strong minimality of D.

Insofar as definable subsets of D itself are concerned, the infinite ones have Morley rank 1 and the finite ones have Morley rank 0. Let us take an informal look at definable subsets of $D \times D$. Let $X \subseteq D \times D$ be definable. Let π be the projection on the first coordinate. For $a \in D$, let X_a be the "fibre" $\{b : (a,b) \in X\}$. Let $Y = \{a \in D : X_a \neq \emptyset\}$, $Y_1 = \{a \in Y : X_a \text{ is finite}\}$ and $Y_2 = \{a \in Y : X_a \text{ is infinite}\}$. Y is clearly definable, and Y_1, Y_2 are definable by Remark 3.9. Let $X_i = \pi^{-1}(Y_i) \cap X$ for $i = 1, 2$. So X is the disjoint union of X_1 and X_2.

If Y_1 is finite, then so is X_1. If Y_1 is infinite (so cofinite), then X_1 is intuitively something "1-dimensional". If Y_2 is finite and nonemmpty, then again X_2 is something "1-dimensional". If Y_2 is infinite (so cofinite) then intuitively X_2 is "2-dimensional". As we shall see these dimensions correspond exactly to Morley rank.

We will need the following which is related to Exercise 2.2 (ii).

Exercise 3.10 *Suppose that a, b are finite tuples and $b \in acl(A, a)$. Then $RM(tp(a/A)) \geq RM(tp(b/A))$ and $RM(tp(a/A)) = RM(tp(a, b/A))$.*

Let p_0 be the unique complete type over $\emptyset$ which contains $D(x)$ and has Morley rank 1 (and degree 1). Note that $p_0(x)$ is "axiomatized" by $\{D(x)\} \cup$ "$x \notin acl(\emptyset)$". Also p_0 is stationary.

Remark 3.11 *Let $a \in D$ and $A \subseteq \bar{M}$. The following are equivalent:*
(i) $a \notin acl(A)$,
(ii) a realizes $p_0|A$.

Proof. The nonforking extensions of p_0 are by 2.26 the extensions of p_0 of Morley rank 1. But the types of Morley rank 0 are precisely the algebraic types.

Corollary 3.12 *Let $a, b \in D$ and A a set of parameters. Suppose $b \in acl(A, a) \setminus acl(A)$. Then $a \in acl(A, b) \setminus acl(A)$.*

Proof. This follows from forking symmetry for $t.t$ theories, bearing in mind 3.11.

Of course Corollary 3.12 can be proved directly and easily, without recourse to forking symmetry in $t.t.$ theories.

Let us make a quick definition, which makes sense in any structure: Let $\{b_i : i \in I\}$ be a set of finite tuples in a structure $\bar{M}$, and A a set of parameters. We call $\{b_i : i \in I\}$ *algebraically independent over* A if for each i, $b_i \notin acl(A \cup \{b_j : j \in I, j \neq i\})$.

Back in our situation:

Corollary 3.13 *Let $\{b_i : i < \omega\} \subseteq D$. Assume that $b_i \notin acl(A \cup \{b_j : j < i\})$ for all $i < \omega$. Then*
(i) $\{b_i : i < \omega\}$ is an A-independent set of realizations of p_0/A, and
(ii) $\{b_i : i < \omega\}$ is algebraically independent over A

Proof. (i) is a direct application of 3.11 and 2.44. (ii) follows from (i) and 3.11.

The next result will show that Morley rank equals "dimension" inside strongly minimal sets.

Proposition 3.14 *Let $\bar{a} = (a_1, .., a_m)$ be an m-tuple of elements of D. Assume that $(a_1, .., a_n)$ is algebraically independent over A, and that $\bar{a} \subseteq acl(A \cup \{a_1, .., a_n\})$. Then $RM(tp(\bar{a}/A)) = n$.*

Proof. We prove this by induction on n. For $n = 1$, $RM(tp(a_1/A)) = 1$ by 3.11, and then $RM(tp(\bar{a})) = 1$ by 3.10. Now assume true for $n' < n$, and we prove it for n. By 3.10 it is enough to prove that $RM(tp(a_1, .., a_n/A)) = n$. Now by induction hypothesis $RM(tp(a_1, .., a_n/A \cup \{a_1\})) = n - 1$ and by 2.47(b) and symmetry, $tp(a_1, .., a_n/A \cup \{a_1\})$ forks over A. So by 2.26, $RM(tp(a_1, .., a_n/A)) \geq n$. Suppose for a contradiction that $RM(tp(a_1, .., a_n/A)) > n$. So $RM(D^n) > n$. It follows that there are at least two complete types $q_1(x_1, .., x_n)$ and $q_2(x_1, .., x_n)$ over some set B, which contain "$\bar{x} \in D^n$", and both have Morley rank n. Let $\bar{c}$ realise q_1 and $\bar{d}$ realize q_2. By the induction hypothesis, each of the sets $\{c_1, .., c_n\}$, and $\{d_1, .., d_n\}$ is algebraically independent over A. By 3.11, each of $\bar{c}$, $\bar{d}$ is an B-independent set of realizations of $p_0|B$ and is thus a Morley sequence in $p_0|B$. By 2.41, $tp(\bar{c}/B) = tp(\bar{d}/B)$, a contradiction. This proves that $RM(tp(a_1, .., a_n/A)) = n$, completing the proof of the proposition.

Definition 3.15 *Let $A \subset \bar{M}$ and $B \subseteq D$. By a basis of B over A we mean a maximal subset B_0 of B such that B_0 is algebraically independent over A.*

Remark 3.16 *Let B_0 be a basis of B over A. Then $B \subseteq acl(A \cup B_0)$.*

Proof. If not let $b \in B$ such that $b \notin acl(A \cup B_0)$. By Corollary 3.13, $B_0 \cup \{b\}$ is algebraically independent over A, contradicting maximality of B_0.

Lemma 3.17 *(With the notation of Definition 3.15.) Let B_0 and B_1 be two bases of B over A. Then B_0 and B_1 have the same cardinality.*

Proof. Suppose first that one of the bases is infinite. Without loss of generality $|B_0| < |B_1| = \kappa$ and κ is infinite. For each element $b \in B_1$ there is a finite tuple c_b from B_0 such that $b \in acl(A \cup c_b)$. It follows that there is some finite tuple c from B_0 and and infinite subset B_0' of B_0 such that $b \in acl(A \cup c)$ for all $b \in B_0'$. So $tp(b/A \cup c)$ forks over A for all $b \in B_0'$ and (by 3.13), B_0' is A-independent. This contradicts Lemma 2.45.

Now assume that both B_0 and B_1 are finite. Let $(b_0^1, .., b_0^n)$ be an enumeration of B_0 and $(b_1^1, .., b_1^m)$ an enumeration of B_1. Let c be the concatenation of these two tuples. Then By 3.14 and 3.16, $n = RM(tp(c/A)) = m$.

Remark 3.18 *(With notation as above.) By $dim(B/A)$ we mean the cardinality of a basis for B over A. If B is finite and b is an enumeration of B, we write $dim(b/A)$. So note, by 3.14 and 3.17, if b is a finite tuple from D then $dim(b/A) = RM(tp(b/A))$.*

Lemma 3.19 *Let b, c be finite tuples from D. Then $dim(bc/A) = dim(b/A \cup c) + dim(c/A)$. Hence by Remark 3.18 and 3.14, $RM(tp(bc/A)) = RM(tp(b/cA)) + RM(tp(c/A))$.*

Proof. If b_0 is a basis for b over $A \cup c$ and c_0 a basis for c over A, then b_0c_0 is a basis for bc over A.

Exercise 3.20 *Morley rank is "definable" in strongly minimal sets. That is, for any formula $\phi(x_1, .., x_n, y)$ (where y is some tuple of variables and ϕ implies that each x_i is in D), and for any m, there is an L-formula $\psi(y)$, such that for all tuples b in $\bar{M}$ (of length that of y), $RM(\phi(x_1, .., x_n, b)) = m$ iff $\models \psi(b)$.*

Definition 3.21 *(i) By D^{eq} we mean $\{c \in \bar{M}^{eq}$: for some finite tuple d from D, $c \in dcl(d)\}$.*
(ii) For $A \subset \bar{M}$, $D^{eq}_A = \{c \in \bar{M}^{eq}$: for some finite tuple d from D, $c \in dcl(A \cup d)\}$.
(iii) D^{aeq} and D^{aeq} are defined as in (i) and (ii) but with $acl(-)$ replacing $dcl(-)$.
(iv) Let X be a definable set in $\bar{M}^{eq}$, defined over A say. We say that X is almost strongly mininimal (with respect to D) if for some $B \supset A$, $X \subset D^{eq}_B$.

Remark 3.22 *Let c be a finite tuple from D and A any set of parameters from $\bar{M}$. Then $Cb(stp(c/A)) \in D^{eq}$.*

Proof. By Proposition 2.42.

Lemma 3.23 *Let $c \in D^{aeq}$. Then there is some set A of parameters and some finite tuple d from D such that c is independent from A over $\emptyset$, and c is interalgebraic with d over A. Hence $RM(tp(c/\emptyset)) = RM(tp(c/A)) = RM(tp(d/A))$.*

Proof. Let $\bar{d} = (d_1, .., d_n)$ be some tuple from D such that $c \in acl(d_1, .., d_n)$. Relabelling, if necessary, let $\bar{d}' = (d_1, .., d_r)$ be a maximal subtuple of $\bar{d}$ which is independent from c over $\emptyset$. It follows that for $i = 1, .., n$, $d_i \in acl(c, \bar{d}')$ (why?). Take A to be $\bar{d}'$, and d to be $\bar{d}$.

Corollary 3.24 *. Let A be any set, and let $b, c \in D^{aeq}$ (or even D^{aeq}_A). Then $RM(tp(bc/A)) = RM(tp(b/cA)) + RM(tp(c/A))$.*

Proof. Let us add names for elements of A to the language. We have to show that $RM(tp(bc/\emptyset)) = RM(tp(b/c)) + RM(tp(c/\emptyset))$. By the lemma above, let B be independent from b such that b is interalgebraic with some finite tuple from D over B. Likewise, let C be independent from c such that c is interalgebraic over C with a finite tuple from D. We may choose B, C such that (b, c) is independent from $B \cup C$ over $\emptyset$. (First let B' realise a nonforking extension of $tp(B/b)$ over (b, c). Then let C' realize a nonforking extension of $tp(C/c)$ over $B' \cup \{b, c\}$. Replace B, C by B', C'.) Let $E = B \cup C$. So Let d_1 be a finite tuple from D interalgebraic with b over E, and d_2 a finite tuple from D interalgebraic with c over E. Then by Exercise 3.10, $RM(tp(b, c/E)) = RM(tp(d_1, d_2/E))$, $RM(tp(c/E)) = RM(tp(d_2/E))$

and $RM(tp(b/cE)) = RM(tp(d_1/d_2E))$. On the other hand, as (b,c) is independent from E over $\emptyset$ (and thus also b is independent from cE over c), $RM(tp(b,c/E)) = RM(tp(b/c))$, $RM(tp(b/cE)) = RM(tp(b/c))$ and $RM(tp(c/E)) = RM(tp(c/\emptyset))$. Together with 3.19, this yields the required result.

The material above, especially 3.17, can be developed at a greater level of generality, that of *pregeometries* and *geometries*. I will say a little about this now. Let S be a set and cl a closure operator on S, namely a map from the power set $P(S)$ of S to itself. The pair $(S, cl(-))$ is said to be a (combinatorial) pregeometry if the following hold (where X, Y denote subsets of S and a, b elements of S):
(i) $X \subseteq cl(X)$,
(ii) $X \subseteq Y$ implies $cl(X) \subseteq cl(Y)$.
(iii) $cl(cl(X)) = cl(X)$,
(iv) If $a \in cl(X \cup \{b\}) \setminus cl(X)$ then $b \in cl(X \cup \{a\})$.
(v) If $a \in cl(X)$ then $a \in cl(X')$ for some finite subset X' of X.

In a pregeometry we obtain a notion of independence: $X \subset S$ is independent if for each $x \in X$, $x \notin cl(X \setminus \{x\})$. Any subset X of S contains a maximal independent subset X_0, which we call a basis of X. All bases of X have the same cardinality, which we call $dim(X)$.

If A is a subset of S, then the localization of S at A is the pregeometry which has the same underlying set S, but a new closure operation $cl_A(-)$ where $cl_A(X) = cl(A \cup X)$. We write $dim(X/A)$ for $dim(X)$ in the localization of S at A. Note that
(vi) $dim(X \cup Y) = dim(X/Y) + dim(Y)$.

There are various rather important properties which a pregeometry (S, cl) may or may not have:
(a) Triviality: $cl(X) = \cup_{x \in X} cl(\{x\})$, for all X.
(b) Modularity: $dim(X) + dim(Y) - dim(X \cap Y) = dim(X \cup Y)$ for all closed X and Y.
(c) Local modularity: The localization of (S, cl) at some singleton of S is modular.
(d) Local finiteness: for finite X, $cl(X)$ is finite.
(e) Homogeneity: for any closed $X \subseteq S$ and $a, b \in S \setminus X$, there is an

automorphism of S (that is a permutation of S preserving $cl(-)$) which fixes X pointwise and takes a to b.

Exercise 3.25 *(i) The pregeometry (S, cl) is modular iff and only if property (b) above holds for all closed $X, Y \subseteq S$ such that $dim(X) = 2$.*
(ii) (S, cl) is modular iff for all closed $X, Y \subseteq S$,
$dim(X/Y) = dim(X/X \cap Y)$.

Hint. For the second part use property (vi) above.

Let us now return to the strongly minimal subset $\emptyset$-definable subset D of the saturated model $\bar{M}$ of the $t.t$ theory T.

Lemma 3.26 *Let $cl(-)$ be algebraic closure restricted to D. Then (D, cl) is a homogeneous pregeometry.*

Proof. (i), (ii), (iii) and (v) in the definition of a pregeometry are clear. (iv) is by 3.12. For homogeneity, we will assume that $D = \bar{M}$ (although it is true in general). Let X be an algebraically closed subset of D (even of big cardinality), and let $a, b \in D \setminus X$. Extend each of a, b to bases A, B of D over X. Then A and B have the same cardinality, by 3.17. Enumerate A as $(a_i : i < \kappa)$ and B as $(b_i : i < \kappa)$ where $a = a_0$ and $b = b_0$. Then these two sequences have the same type over X. The elementary map which fixes X pointwise and takes $(a_i : i < \kappa)$ to $(b_i : i < \kappa)$ extends (by taking algebraic closures) to an automorphism of the structure D.

Remark 3.27 *(D, acl) is modular if and only for for any tuples $a, b \subset D$, a is independent from b over $acl(a) \cap acl(b) \cap D$ (in the sense of nonforking).*

Proof. Suppose (D, acl) is modular. It is enough to prove the RHS for a, b finite tuples from D. By 3.25 (ii), $dim(a/b) = dim(a/acl(a) \cap acl(b) \cap D)$. So by 3.18, $RM(tp(a/b)) = RM(tp(a/acl(b) \cap D)) = RM(tp(a/acl(a) \cap acl(b) \cap D))$, which is enough.

Conversely, in the same way the right hand side implies that $dim(a/b) = dim(a/acl(a) \cap acl(b) \cap D))$ for all tuples from D, and we can use 3.25 again.

Example 3.28 *(i) Let $\bar{M} = D =$ an infinite set in the empty language. Then $(D, acl(-))$ is a trivial pregeometry.*

(ii) Let $\bar{M} = D =$ an (infinite-dimensional) division vector space over a division ring (as in Example 3.8). Then (D, acl) is modular.
(iii) Let $\bar{M} = D =$ an algebraically closed field in the language of rings. Then (D, acl) is not locally modular.

Proof. Exercise.

Definition 3.29 *Let T be t.t, $\bar{M}$ a saturated model and X a $\emptyset$-invariant subset of $\bar{M}^{eq}$. We say that X is 1-based if for any finite tuple a of elements of X and any $A \subset \bar{M}^{eq}$ such that $tp(a/A)$ is stationary, the canonical base of $tp(a/A)$ is contained in $acl^{eq}(a)$.*

Exercise 3.30 *X is 1-based iff for any tuple a from X, and any $A \subset \bar{M}^{eq}$, a is independent from A over $acl^{eq}(a) \cap acl^{eq}(A)$.*

Hint. Use 2.38 (ii).

From Remark 3.27 and Exercise 3.30, we see some kind of formal similarity between the notions of modularity and 1-basedness.

Lemma 3.31 *(i) Suppose X is 1-based. Then so is X^{aeq}.*
(ii) 1-basedness of X is invariant under naming parameters.

Proof. We will use the fact (exercise) that if a, b, are tuples and a is independent from b over C then $acl(a) \cap acl(b) \subseteq acl(C)$.
(i) Let d be a finite tuple from X and let $c \in acl(d)$ and consider a stationary type $tp(c/A)$ where without loss, A is eq-algebraically closed. We may assume that d is independent from A over c. Hence by the fact mentioned above
(*) $acl(d) \cap A \subseteq acl(c)$.
By 1-basedness of X, d is independent from A over $acl(d) \cap A$. As $c \in acl(d)$, c is independent from A over $acl(d) \cap A$ and thus over $acl(c) \cap A$ by (*). So $Cb(tp(c/A)) \subseteq acl(c)$ by 2.38.
(ii) If X is 1-based, it clearly remains 1-based after naming parameters. Conversely, suppose X is 1-based in $Th(\bar{M}, a)_{a \in A}$. Let $c \in X$ and B some set such that $tp(c/B)$ is stationary. We may assume that (c, B) is independent from A over $\emptyset$ in $\bar{M}$. In particular, $tp(c/BA)$ does not fork over B. So $c_0 = Cb(tp(c/B)) = Cb(tp(c/BA))$. Using our assumptions $c_0 \in acl(cA) \cap B$. But B is independent from cA over c, hence (by the fact above) $c_0 \in acl(c)$.

Lemma 3.32 *Suppose D is modular. Then D is 1-based.*

Proof. Assume modularity of D. Let a be a finite tuple from D and A some set from $\bar{M}$, such that $tp(a/A)$ is stationary. Let $c = Cb(tp(a/A))$. We know (2.42) that $c \in D^{eq}$, so there is some $C \subseteq D$ such that $c \in dcl(C)$. We may choose C so that C is independent from a over c and so therefore that a is independent from C over c. We may assume that C is algebraically closed (in D). Note that $c = Cb(tp(a/C))$ too. By 3.27, a is independent from C over $acl(a) \cap C$. By 2.38, $c \in acl^{eq}(acl(a)) = acl^{eq}(a)$.

Definition 3.33 *D is said to be linear, if whenever $a, b \in D$, $A \subset \bar{M}$, $RM(tp(a,b/A)) = 1$, and $c = Cb(stp(a,b/A))$, THEN $RM(tp(c/\emptyset)) \leq 1$.*

A remark: If $a, b \in D$ and $RM(tp(a,b/A)) = 1$, then $dim(a,b/A)) = 1$ and we can consider (a, b) as a “generic point” of a “plane curve” $X \subset D \times D$, which is defined over c. The rank of $tp(c)$ measure the “dimension” of the family $(X_{c'})_{c'}$.

Lemma 3.34 *D is linear if and only if for any $a, b \in D$, and any set A, $Cb(stp(a,b/A)) \in acl(a,b)$.*

Proof. Suppose first that D is linear. Let $c = Cb(stp(a,b/A))$. If $RM(tp(a,b/A)) = 2$ then (a, b) is independent from A over $\emptyset$, so $c \in acl(\emptyset)$. If $RM(tp(a,b)) = 0$, then $c = (a, b)$. So we may assume that $RM(tp(a,b/A)) = 1$. If $RM(tp(a,b/\emptyset)) = 1$ then again $c \in acl(\emptyset)$. So we may assume that $RM(tp(a,b/\emptyset)) = 2$. So (a, b) forks with c over $\emptyset$, whereby $c \notin acl(\emptyset)$, so by linearity, $RM(tp(c)) = 1$. Note that $RM(tp(a,b/c)) = 1$. By 3.24, $RM(tp(a,b/c)) + RM(tp(c)) = RM(tp(c/a,b)) + RM(tp(a,b))$. So $1 + 1 = RM(tp(c/a,b)) + 2$, whereby $c \in acl(a,b)$ as required.

The converse follows by a similar computation.

Theorem 3.35 *The following are equivalent:*
(i) There is some set A such that D is modular in $Th(\bar{M}, a)_{a \in A}$.
(ii) D is 1-based,
(iii) D is linear,
(iv) D is locally modular.

Proof. (i) implies (ii): By 3.32, D is 1-based after adding names for elements of A. By 3.31, D is 1-based.
(ii) implies (iii): By 3.34.
(iii) implies (iv). Let $a_0 \in D \setminus acl(\emptyset)$. We will show that the localization of D at $\{a_0\}$ is modular. This means precisely that D is modular after adding a constant for a_0 to the language. By Exercise 3.25(i) it is enough to show that
if $a_1, a_2 \in D$ are such that $\{a_0, a_1, a_2\}$ is algebraically independent over $\emptyset$, and $A \subset D$ is algebraically closed and is such that $a_0 \in A$ and $dim(a_1, a_2/A) = 1$, and $a_i \notin A$ for $i = 1, 2$ then $acl(a_0, a_1, a_2) \cap A \setminus acl(a_0)$ is nonempty.

Let $c = Cb(stp(a_1, a_2/A))$. So $c \in acl^{eq}(A)$. By linearity and 3.34, $c \in acl(a_1, a_2)$. Note that $dim(a_1, a_2/c) = 1$, and $a_i \notin acl(c)$ for $i = 1, 2$. So $a_2 \in acl(c, a_1)$. As $a_0 \notin acl(a_1, a_2)$, $a_0 \notin acl(c)$. Hence a_0 realizes $p_0|c$. But a_1 also realizes $p_0|c$. Hence $tp(a_0, c) = tp(a_1, c)$. Let a_2' be such that $tp(a_0, a_2', c) = tp(a_1, a_2, c)$. Then $a_2' \in acl(c, a_0)$. Thus $a_2' \in A \setminus acl(a_0)$. Also $a_2' \in acl(a_0, a_1, a_2)$. This completes the proof of (iii) implies (iv).

(iv) implies (i) is immediate.

Let us show that algebraically closed fields are NOT locally modular. Fix an algebraically closed field $(K, +, -, \cdot, 0, 1)$ which we assume to be of uncountable cardinality κ say (hence κ-saturated). We know K to be strongly minimal. Let $a, b \in K$. Let C be the $\{(x, y) \in K \times K : y = ax + b\}$. Then C is also strongly minimal (as it is in definable bijection with K). Let $c \notin acl(a, b)$. Let $d = ac + b$. Then $tp(c, d/\{a, b)\})$ has Morley rank 1 and degree 1. In fact (c, d) is a "generic" point of C.
CLAIM. (a, b) is (namely is interdefinable with) $Cb(tp((c, d)/\{a, b\}))$.
Proof of CLAIM. Let $\mathbf{p}(x, y)$ be the global nonforking extension of $tp((c, d)/\{a, b\})$. We have to show (see Definition 2.34) that for any automorphism f of K, $f(a, b) = (a, b)$ if and only if $f(\mathbf{p}) = \mathbf{p}$. Clearly left to right is true. Now suppose that $f(a, b) = (a', b') \neq (a, b)$. Write C as $C_{(a,b)}$. Then $C_{(a',b')} \cap C_{(a,b)}$ is either empty, or is a singleton. As $\mathbf{p}$ has Morley rank 1, and "$(x, y) \in C_{(a,b)}$" $\in \mathbf{p}$ it follows that $(x, y) \not{i}n C_{(a',b')} \in \mathbf{p}$. Hence $f(\mathbf{p}) \neq \mathbf{p}$.

So choosing a, b algebraically independent, we see that $RM(tp(Cb(tp(c, d/\{a, b\})))) = 2$, so K is not linear, hence not locally modular.

Let us briefly discuss orthogonality and related notions. We are working in the saturated model $\bar{M}$ of the $t.t.$ theory T.

Definition 3.36 *(i) Let $p(x) \in S(A)$ and $q(y) \in S(B)$ be stationary types. p is said to be nonorthogonal to q if there is $C \supseteq A \cup B$, and there are realizations a of $p|C$ and b of $q|C$ such that $tp(a/C, b)$ forks over C.*
(ii) Let D_1 be a strongly minimal set defined over A, and D_2 a strongly minimal set defined over B. Let $p(x) \in S(A)$ be the "generic type" of D_1 (namely $p(x) = \{x \in D_1\} \cup$ "$x \notin acl(A)$", and likewise for $q(y)$. We say that D_1 is nonorthogonal to D_2 if $p(x)$ is nonorthogonal to $q(y)$.

Example 3.37 *Let T be theory of disjoint infinite unary predicates P and Q. Then P and Q are strongly minimal and orthogonal.*

Lemma 3.38 *Let D_1, D_2 be strongly minimal sets. Then D_1 is nonorthogonal to D_2 if and only if there is some definable $R \subset D_1 \times D_2$, such that the projections of R on D_1 and D_2 are infinite, and for every x there are at most finitely many y such that $R(x, y)$ and dually.*

Proof. If D_1 and D_2 are nonorthogonal, then there is some C over which both are defined, and $a \in D_1 \setminus acl(C)$ and $b \in D_2 \setminus acl(C)$ such that a forks with b over C (why?). This implies that $a \in acl(b, C)$ and $b \in acl(C, a)$. Let $\phi(x, y)$ be a formula over C such that $\models \phi(a, b)$, and such that $\models \forall x \exists^{\leq k} y \phi(x, y)$ and $\models \forall y \exists^{\leq l} y \phi(x, y)$ for some k, l. Then $\phi(x, y)$ defines the required relation R.

Conversely, suppose such R exists. Let D_1, D_2, R be defined over C. By the assumptions on R there is $(a, b) \in R$ such that $a \in D_1 \setminus acl(C)$, and $b \in D_2$. Then $b \in acl(C, a)$ and $a \in acl(C, b)$. Hence $tp(a/C, b)$ forks over C.

Corollary 3.39 *Nonorthogonality between strongly minimal sets is an equivalence relation.*

What became known as **Zilber's conjecture** was:
(ZC1) Suppose D is strongly minimal and non locally modular. Then there is a strongly minimal algebraically closed field D_2 definable in $\bar{M}$ such that D_1 is nonorthogonal to D_2.

Zilber's conjecture has been very influential. A counterexample was found by Hrushovski in the late 1980's, and the techniques used to construct it gave rise to a whole subarea of model theory, "Hrushovski constructions". Nevertheless the conjecture is true in many "natural" situations, and often has quite profound mathematical meaning and implications.

An equivalent formulation of Zilber's conjecture is:
(ZC2) Let X be a definable set of finite Morley rank. Suppose that X is not 1-based. Then there is a strongly minimal algebraically closed field definable in X^{eq}.

Recent work has led to a somewhat stronger version of the Zilber conjecture which more or less states that any violation of 1-basedness "lives in" a definable field in some sense. The notion of internality is important here.

Definition 3.40 *Let $p(x) \in S(A)$ be stationary and X some B-definable set, or even some B-invariant set. We say that $p(x)$ is (almost) internal to X if there is $C \supseteq A \cup B$ and a realizing $p(x)|C$ and some finite tuple c of elements of X such that $a \in dcl(C, c)$ ($a \in acl(C, c)$).*

The strong version of Zilber conjecture then says:
(ZC3) Suppose $tp(a)$ has finite Morley rank, $tp(a/A)$ is stationary and $c = Cb(tp(a/A))$. Then there is some definable set X which is a finite union of strongly minimal algebraically closed fields, such that $stp(c/a)$ is almost internal to X.

There is a weaker version of (ZC3) which does not mention fields and may be true.
(NMC) Suppose $tp(a)$ has finite Morley rank, $tp(a/A)$ is stationary and $c = Cb(tp(a/A))$. Then there is some definable set X which is a finite union of non locally modular strongly minimal sets, such that $stp(c/a)$ is almost internal to X.

Let us finish by discussing internality a bit more.

Lemma 3.41 *Suppose $p(x) \in S(A)$ is stationary and internal to the A-invariant set X. Then there is a set $B \supset A$, such that for every realization a of p there is some tuple c from X such that $a \in dcl(B, c)$. So $p(\bar{M}) \subseteq X_B^{eq}$.*

Proof. Let $B \supset A$ and a realize $p|B$ and $c \subset X$ be such that $a \in dcl(Bc)$. We may assume $tp(a, c/B) = q(x, y)$ is stationary. Let $(a_i, c_i)_{i<\omega}$ be a Morley

sequence in $tp(a, c/B)$. Let $q'(x, y)$ be the nonforking extension of $q(x, y)$ over $B \cup \{(a_i, c_i)\}_i$, and let (a', c') realize q'. By 2.42, q' does not fork over $(a_i, c_i)_i$ and the restriction of q' to this set is stationary. It follows (why?) that $a' \in dcl(A \cup (a_i, c_i)_i \cup c')$. Thus there is n and finite $d \subset X$ such that $a' \in dcl(A \cup (a_1, .., a_n) \cup d)$. Note that a' is independent from $(a_1, .., a_n)$ over A (why?) Now let a'' be any realization of $p(x)$. Let $(a''_1, .., a''_n)$ be such that $tp(a'', a''_1, .., a''_n/A) = tp(a', a_1, .., a_n/A)$ and $(a''_1, .., a''_n)$ is independent from $(a_1, .., a_n)$ over $A \cup a'$. In particular
(a) there is $d' \subset X$ such that $a'' \in dcl(a''_1, .., a''_n, d', A)$, and
(b) each a''_i is independent from $(a_1, .., a_n)$ over A, so has the same type over $A \cup \{a_1, .., a_n\}$ as a', whereby there is $d_i \subset X$ such that $a''_i \in dcl(A, a_1, .., a_n, d_i)$.
By (a) and (b), $a'' \in dcl(A, a_1, .., a_n, d', d_1, .., d_n)$.

Note that it follows from 3.41 and compactness that under the same assumptions, there is an A-definable set Y in $p(x)$, of maximal Morley rank, and a B-definable function from some B-definable subset of X^n onto Y.

Internality can give rise to definable groups if additional parameters are really needed to see it. This idea again originates with Zilber. Here is a rather striking case, which for now we state without proof.

Lemma 3.42 *Suppose that $tp(a/A)$ is stationary and internal to the A-invariant set X. Suppose also that a is independent from c over A for all $c \subset X$ (that is, a is independent from X over A). THEN there is an A-definable group G and an A-definable transitive action of G on $p(\bar{M})$.*

4 ω-stable groups

In this section $\bar{M}$ remains a saturated model of the $t.t$ theory T and we will study definable groups in $\bar{M}$, sometimes specializing to definable groups of finite Morley rank.

We will also work in $\bar{M}^{eq}$ without mentioning it.

By a definable group (in $\bar{M}$) we mean a definable set G equipped with a definable group operation $\cdot$. If both the set G and the group operation $\cdot$ are defined over A, we say that the group $(G, \cdot)$ is defined over A. Note then that both the identity of the group as well as group inversion are also A-definable.

We often just write G rather than $(G,\cdot)$ for the group. A special case is when the universe of $\bar{M}$ is equipped with a $\emptyset$-definable group operation.

In any case when we think of a definable group as a structure in its own right, it is not just the group structure but all the structure on G induced from $\bar{M}$, in the sense of Definition 3.3.

This is in accordance with the idea of an *algebraic group*, which is our basic example. An algebraic group (over a given algebraically closed field K) is an algebraic variety G over K together with a morphism $G \times G \to G$ over K which is a group operation. Any algebraic group is a definable group in $(K,+,\cdot)$, and conversely it can be shown that any group definable in $(K,+,.\cdot)$ is definably isomorphic to an algebraic group. There are two extremes of algebraic groups; abelian varieties (whose underlying variety is a Zariski closed subset of some projective space over K) and linear algebraic groups, which are algebraic subgroups of some $GL(n,K)$, the group of invertible $n \times n$ matrices over K.

We may sometimes also consider *type-definable* groups, namely where the underlying set of the group is defined by a partial type.

We will be using throughout the fact that the Morley rank and degree of a definable set are invariant under definable bijection.

Lemma 4.1 *Let $(G,\cdot)$ be a definable group, and H a definable subgroup of G. Then either $RM(H) < RM(G)$, or $dM(H) < dM(G)$.*

Proof. The left cosets $a \cdot H$ of H in G partition G, and are are all in definable bijection with H so have the same Morley rank and degree as H. So if the index of H in G is infinite, then $RM(H) < RM(G)$, and if the index is finite, n say, then $RM(H) = RM(G)$ and $dM(G) = n(dM(H))$.

Corollary 4.2 *A definable group G has the DCC on definable subgroups. (There is no infinite descending chain of definable subgroups.) In particular, G has a smallest definable subgroup of finite index in G, which we call the connected component of G and write as G^0. G^0 is normal in G and is A-definable if G is A-definable. Moreover $RM(G^0) = RM(G)$.*

Proof. The DCC is immediate from Lemma 4.1. It follows that G^0 exists and is the intersection of all definable subgroups of finite index in G. By this property, G^0 is invariant under A-automorphism of G hence A-definable. As any conjugate of a finite index subgroup also has finite index, G^0 is normal.

We will tend to assume that our ambient definable group G is $\emptyset$-definable in $\bar{M}$.

Definition 4.3 *Assume that $RM(G) = \alpha$.*
(i) A definable subset X of G is said to be generic if $RM(X) = \alpha$.
(ii) Let $p(x) \in S(A)$ be a complete type over a set A such that $p(x) \models$ "$x \in G$". $p(x)$ is said to be a generic type of G (over A) if $RM(p(x)) = \alpha$.
(iii) Let A be any subset of $\bar{M}$ (usually a subset of G), and $a \in G$. We say that a is a generic element of G over A if $tp(a/A)$ is a generic type of G.

Let us now fix the $\emptyset$-definable group G with Morley rank α and Morley degree d. Note that there exist generic types of G over any set A. Note that also that if $p(x)$ is a complete type over A of an element of G and $q(x)$ is a nonforking extension of p then p is generic iff q is generic. Also note that as inversion is an $\emptyset$-definable bijection between G and itself, $tp(a/A)$ is generic iff $tp(a^{-1}/A)$ is generic. Finally note that a definable subset X of G is generic if and only if it is contained in some generic type of G.

Let us begin with a characterization of generic types, with can be taken as the definition in the stable (or even simple) case.

Lemma 4.4 *Let $a \in G$ and $A \subset \bar{M}$. The following are equivalent:*
(i) $tp(a/A)$ is generic,
(ii) if $b \in G$ and a is independent from b over A then $b \cdot a$ is independent from b over A,
(iii) If $b \in G$, and a is independent from b over A then $a \cdot b$ is independent from b over A.

Proof. Assume (i). We will prove (ii) and (iii). Let $b \in G$ be such that a is independent from b over A. So $RM(tp(a/A, b)) = \alpha$. So $RM(tp(b \cdot a/A, b)) = \alpha$. So (as $\alpha = RM(G)$), $RM(tp(b \cdot a/A)) = \alpha$, hence $b \cdot a$ is independent from b over A. So we proved (ii). (iii) follows similarly.
Now assume (ii). Choose $b \in G$ generic over $A \cup \{a\}$. Note that b is independent from a over A. So by $(i) \rightarrow (iii)$, $tp(b \cdot a/A)$ is generic. Note that a is independent from b over A. By (ii), $b \cdot a$ is independent from b over A. Hence $tp(b \cdot a/A, b)$ is generic, hence $tp(a/A, b)$ is generic. As a is independent from b over A, $tp(a/A)$ is generic. So we have (i). In a similar fashion (iii) implies (i).

By 2.33 (ii) there are d disjoint $acl(\emptyset)$-definable subsets of G, each of Morley rank α, and Morley degree 1. So we obtain d distinct stationary generic types $p_1(x), .., p_d(x)$ of G, over $acl(\emptyset)$. Note that any stationary generic type of G is a nonforking extension of one of the p_i. We aim towards showing that d is precisely the index of G^0 in G. Let $S = \{p_1, .., p_d\}$. Let us start by defining an action of G on S.

Lemma 4.5 *Let $g \in G$. Let a realize p_i with a independent from g over $\emptyset$. Then $tp(g \cdot a/acl(\emptyset)) = p_j$ for some j. Moreover j depends only on g and i (not on the choice of a). So we write $g \cdot p_i = p_j$.*

Proof. Note that $RM(tp(a/g)) = \alpha$, hence $RM(tp(g \cdot a/g) = \alpha$, hence $RM(tp(g \cdot a/acl(\emptyset)) = \alpha$, so must be p_j for some $j \in \{1, .., d\}$.

As p_i is stationary, it has a unique nonforking extension over $acl(\emptyset) \cup \{g\}$, hence $tp((g, a)/acl(\emptyset))$ is uniquely determined by g and p_i, and therefore also $tp(g \cdot a/acl(\emptyset)$.

Exercise 4.6 *The map from $G \times S$ to S defined by $(g, p_i) \to g \cdot p_i$ is a group action.*

Lemma 4.7 *(i) G acts transitively on S.*
(ii) for any i, j, $\{g \in G : g \cdot p_i = p_j\}$ is definable (over $acl(\emptyset)$).

Proof. Let $p_i, p_j \in S$. Let a, b realizes p_i, p_j respectively such that a is independent from b over $\emptyset$. In particular b is independent from a over $\emptyset$ so independent from a^{-1} over $\emptyset$. By 4.4 and the genericity of $tp(b)$, $b \cdot a^{-1}$ is independent from a over $\emptyset$. Put $g = b \cdot a^{-1}$, and we see that $g.a = b$, hence $g \cdot p_i = p_j$.
(ii) Let $\phi_i(x)$ be a formula over $acl(\emptyset)$ of Morley rank α and Morley degree 1 which implies "$x \in G$", and is contained in $p_i(x)$. Let $\mathbf{p}_i(x)$ be the global nonforking extension of p_i.

Now fix i and j. Note that $\mathbf{p}_j$ is definable (over $acl(\emptyset)$), hence $\{g \in G : \phi_i(g^{-1}.x) \in \mathbf{p}_j\}$, is definable, by a formula $\psi_{i,j}(y)$ over $acl(\emptyset)$.
Claim. Let $g \in G$. Then $\models \psi_{i,j}(g)$ iff $g \cdot p_i = p_j$.
Proof of claim. Suppose first that $\models \psi_{i,j}(g)$. So we can find b realizing $p_j|(acl(\emptyset) \cup \{g\})$ such that $\models \phi_i(g^{-1} \cdot b)$. Let $a = g^{-1} \cdot b$. By 4.4, $tp(a/acl(\emptyset))$ is generic (and independent from g over $\emptyset$). As $\models \phi_i(a)$, $tp(a/acl(\emptyset)) = p_i$. So $g \cdot a = b$, so $g \cdot p_i = p_j$.
The converse is similar.

Proposition 4.8 *The index of G^0 in G is precisely d, the Morley degree of G. Hence G^0 and each of its translates has Morley degree 1, and the generic types $p_1, .., p_d$ of G correspond to the cosets of G^0 in G.*

Proof. Let r be the index of G^0 in G. As G^0 and each of its cosets in G have Morley rank α,
(*) $r \leq d$.

Now consider the action of G on S described above. Let $G_i = Fix(p_i) = \{g \in G : g \cdot p_i = p_i\}$. By transitivity of the action (4.7 (i)), each G_i has index d in G. By 4.7 (ii), each G_i is definable, so contains G^0, and hence by (*), equals G^0.

We worked above with a left action of G on S. Identical results hold for the corresponding right action.

Lemma 4.9 *Let X be a definable subset of G. Then X is generic if and only if finitely many left translates of X cover G if and only if finitely many right translates cover G.*

Proof. Suppose first that finitely many left (or right) translates of X cover G. As each such translate has the same Morley rank as X, it follows that $RM(X) = \alpha$ hence X is generic.

Now for the converse. Suppose X to be generic. To prove that finitely many left translates of X cover G, we may clearly replace X by a translate of a definable subset of X. Note that some generic definable subset of X is contained in a single coset of G^0 in G. So after translating, we may assume that X is contained in G^0. It is clearly enough to show that finitely many translates of X cover G^0. Assume X is A-definable. Let p_0 be the unique generic type of G over A containing "$x \in G^0$". (So p_0 is stationary.) Note that "$x \in X$" $\in p_0$. (Why?) Let $\{a_i : i < \omega\}$ be a Morley sequence in p_0. Let $g \in G^0$. By 2.45, there is a_i which is independent from g over A. Then $tp(a_i^{-1} \cdot g/A)$ is generic, and $a_i^{-1} \cdot g \in G^0$, hence $tp(a_i^{-1} \cdot g/A) = p_0$ so contains "$x \in X$". So $a_i^{-1} \cdot g \in X$, so $g \in a_i \cdot X$.
We have shown that every $g \in G^0$ is contained in $a_i \cdot X$ for some $i < \omega$. By compactness, finitely many of the $a_i \cdot X$ cover G^0.

Remark 4.10 *In the same way as we defined generic types, connected components,... of G, we can do the same thing for any subgroup of G which is* *definable in $\bar{M}$. If H is such, we call H connected if $H = H^0$. So if H is connected then H has a unique generic type (over any set of parameters over which H is defined).*

Definition 4.11 *Let H be a definable subgroup of G and X a left translate of H in G (namely a left coset $c \cdot H$ of H in G). Assume that both H and X are defined over A. By a generic type of X over A we mean some $p(x) \in S(A)$ such that $p(x) \models$ "$x \in X$" and $RM(p) = RM(X)(= RM(H))$. Likewise, $a \in X$ is said to be generic in X over A if $tp(a/A)$ is a generic type of X. (Nte that if H is connected then X has a unique generic type over A.)*

Lemma 4.12 *Let H, X, A be as in Definition 4.11, and let $a \in X$. Then a is generic in X over A if there is $b \in X$ such that a is independent from b over A and $b^{-1} \cdot a$ is generic in H over $A \cup \{b\}$.*

Proof. Suppose $RM(X) = RM(H) = \beta$. First suppose a to be generic in X over A. Let b be any element of X such that a is indepependent from b over A. So $RM(tp(a/A \cup \{b\})) = \beta$. So $RM(tp(b^{-1} \cdot a/A \cup \{b\})) = \beta$. But $b^{-1} \cdot a \in H$, so is generic in H over $A \cup \{b\}$).
The converse is similar.

Let us now discuss *stabilizers* which were already implicit in the material above.

Definition 4.13 *Let $p(x) = tp(a/A)$ be stationary, where $a \in G$. By the left stabilizer of $p(x)$ we mean $\{g \in G :$ for some a' realizing $p(x)|(A \cup \{g\})$, $tp(g \cdot a'/A) = p(x)\}$. We write $Stab(p(x))$ for this left stabilizer.*

Remark 4.14 *Suppose $p(x) = tp(a/A)$ is stationary, with $a \in G$.*
(i) Let $g \in G$ and a' realize $p(x)|(A \cup \{g\})$. Then $tp(g \cdot a/A) = p(x)$ if and only if $tp(g \cdot a/A \cup \{g\}) = p(x)|(A \cup \{g\})$.
(ii) Let $\phi(x)$ be a formula in $p(x)$ of Morley rank β equal to $RM(p)$ and of Morley degree 1. Then $g \in Stab(p)$ iff $g \cdot (\phi(G)) \cap \phi(G)$ has Morley rank β.
(iii) The left stabilizer of $p(x)$ is equal to the right stabilizer of $p^{-1}(x) = tp(a^{-1}/A)$.

Proof. (i) Right to left is clear. Suppose now that $tp(g \cdot a'/A) = p(x)$. Clearly $RM(tp(g \cdot a'/A \cup \{g\})) = RM(tp(a'/A \cup \{g\})) = RM(p(x))$. So $g \cdot a'$ is independent from g over A.
(ii) and (iii) are left to the reader.

Lemma 4.15 *Let $p(x) \in S(A)$ be stationary and contain "$x \in G$". Then*
(i) $Stab(p)$ is an A-definable subgroup of G.
(ii) if $p'(x) \in S(B)$ is a nonforking extension of $p(x)$, then $Stab(p') = Stab(p)$,
(iii) $RM(Stab(p)) \leq RM(p)$.

Proof. (i) Let $\phi(x) \in p(x)$ have Morley rank $\beta = RM(p)$ and Morley degree 1. Let $g \in G$. As in the proof of 4.7(ii), we see that $g \in Stab(p)$ iff $\phi(g^{-1} \cdot x)$ is contained in the global nonforking extension of $p(x)$, and that this is an A-definable condition on g. By Remark 4.14(i), $g \in Stab(p)$ iff $g^{-1} \in Stab(p)$. So to see that $Stab(p)$ is a subgroup it suffices to see that it is closed under multiplication. Suppose $g, h \in Stab(p)$. Let a realize p independent of $h \cdot g$ over A. We may assume a is independent from $\{g, h\}$ over A. So $g \cdot a$ realises $p(x)$. As in the proof of 4.7(i), $g \cdot a$ is independent from h over A. Hence $h \cdot (g \cdot a)$ realizes p.
(ii) By the proof of (i), $Stab(p)$ depends only on the global nonforking extension $\mathbf{p}$ of p.
(iii) Let $g \in Stab(p)$. Let a realize $p(x)|(A \cup \{g\})$. Then $RM(tp(g/A)) = RM(tp(g/A \cup \{a\})) = RM(tp(g \cdot a/A \cup \{a\}))$. But the latter is $\leq RM(p(x))$ as $tp(g \cdot a/A)) = p(x)$.

Lemma 4.16 *Let $a \in G$ and $p = tp(a/A)$ be stationary Let $H = Stab(p)$. The following are equivalent:*
(i) $RM(H) = RM(p)$.
(ii) The right coset $H \cdot a$ is $acl(A)$-definable.
(iii) H is connected, $X = H \cdot a$ is A-definable, and $p(x)$ is the generic type of X over A.

Proof. First we know by the previous lemma that H is A-definable. Let $X = H \cdot a$, and let $\beta = RM(H) = RM(X)$.
(i) implies (ii). Let $g \in H$ be generic over $A \cup \{a\}$. So $RM(tp(g/A \cup \{a\})) = RM(tp(g \cdot a/A \cup \{a\})) = \beta$. But as $H = Stab(p)$, $tp(g \cdot a/A) = p$ and so has Morley rank β. It follows that
(*) $g \cdot a$ is independent from a over A.
But (as $g \cdot a \in H \cdot a$), H is definable over $A \cup \{a\}$ as well as over $A \cup \{g \cdot a\}$. Thus by (*) H is definable over $acl(A)$. (Why?)
(ii) implies (i). Assuming (ii) we see that the formula "$x \in X$" is in

$tp(a/acl(A))$, and thus $RM(X) \geq RM(tp(a/acl(A)) = RM(p)$. By the previous lemma we get equality.
(ii) implies (iii). Assume X to be $acl(A)$-definable. To show that X is A–definable we must show that X is invariant under A-elementary permutations of $acl(A)$. Let f be such. As $p(x) = tp(a/A)$ is stationary, $tp(a/acl(A)) = tp(f(a)/acl(A))$. But $a \in X$, and so $a \in f(X)$. But $f(X)$ is clearly also a right translate of H, so must equal X.
Clearly $p(x)$ is a generic type of X (we know by (ii) implies (i) that $RM(p) = RM(X)$). So all that is left is to show that H is connected. Suppose not, and we will contradict the stationarity of $p(x)$. Then H^0 is a proper subgroup of H, also defined over A. So we can find $h_1, h_2 \in H$ generic over A, with $h_1 \in H^0$, and $h_2 \notin H^0$, and moreover with h_i independent from a over A for $i = 1, 2$. Then $h_1 \cdot a$ and $h_2 \cdot a$ both realize $p(x)$ (as $H = Stab(p)$), and moreover a rank computation shows that $h_i \cdot a$ is independent from a over A for $i = 1, 2$. So $tp(h_1 \cdot a/A \cup \{a\}) = tp(h_2 \cdot a/A \cup \{a\}) = p|(A \cup \{a\})$. But the formula "$xa^{-1} \in H^0$" is in the first type but not the second, a contradiction. So H is connected as claimed, and the proof is complete.

The theory above, of generics, stabilizers,.. in groups definable in a model of a $t.t$ theory extends easily to definable homogeneous spaces. By an A-definable homogeneous space we mean an A-definable group G together with an A-definable transitive action on an A-definable set X. Note that after naming a point $a \in X$, the action of G on X becomes isomorphic to the action of G on G/H where H is $Fix(a)$.

Let us now bring canonical bases into the picture.

Lemma 4.17 *Let X be an A-definable left translate of a connected definable subgroup H of G. Let $p(x) \in S(A)$ be the generic type of X. Let u be a canonical parameter for X. Then u is interdefinable with $Cb(p)$.*

Proof. Left to the reader.

Before the next lemma we will introduce a bit more notation. Suppose $p(x) \in S(A)$ is a stationary type of an element in G and let $H = Stab(p)$. Let $\mathbf{p}(x) \in S(\bar{M})$ be the global nonforking extension of p. Let a' realize $\mathbf{p}$ in some elementary extension of $\bar{M}$. For $d \in G$, let $d \cdot \mathbf{p} = tp(d \cdot a'/\bar{M})$. Note that
(*) $H = \{d \in G : d \cdot \mathbf{p} = \mathbf{p}\}$.

Lemma 4.18 *Let $p(x) \in S(A)$ be a stationary type implying "$x \in G$". Let H be the left stabilizer of $p(x)$. Let $d \in G$ and a realize $p(x)|(A \cup \{d\})$. Then $tp(d \cdot a/A \cup \{d\})$ is stationary, and the left coset $d \cdot H$ is (as an imaginary) interdefinable with $Cb(tp(d \cdot a/A \cup \{d\}))$ over A. (Likewise with right stabilizers and cosets in place of left ones.)*

Proof. First as $tp(a/A \cup \{d\})$ is stationary and a and $d \cdot a$ are interdefinable over $A \cup \{d\}$, also $tp(d \cdot a/A \cup \{d\})$ is stationary.

Note that the global nonforking extension of $tp(d \cdot a/A \cup \{d\})$ is precisely $d \cdot \mathbf{p}$ where $\mathbf{p}$ is the global nonforking extension of p. By the definition of canonical bases, we must show that for any A-automorphism f of $\bar{M}$, $f(d \cdot H) = d \cdot H$ iff $f(d \cdot \mathbf{p}) = d \cdot \mathbf{p})$. But as f fixes A pointwise and $\mathbf{p}$ is definable over A, $f(\mathbf{p}) = \mathbf{p}$. Thus, for $f \in Aut_A(\bar{M})$, $f(d \cdot \mathbf{p}) = f(d) \cdot f\mathbf{p} = f(d) \cdot \mathbf{p}$. Hence $f(d \cdot \mathbf{p}) = d \cdot \mathbf{p}$ iff $f(d) \cdot \mathbf{p} = d \cdot \mathbf{p}$ iff $(d^{-1} \cdot f(d)) \cdot \mathbf{p} = \mathbf{p}$ iff (by (*) above) $d^{-1} \cdot f(d) \in H$ iff $d \cdot H = f(d) \cdot H$ (using the fact that $f(H) = H$ as H is A-definable).

Lemma 4.19 *Let $a \in G$ be such that $tp(a/A)$ is stationary. Let $H = Stab(p)$. Let $c \in G$ be generic over $A \cup \{a\}$. Then $H \cdot a$ (as an imaginary) is interdefinable with $Cb(stp(c/A \cup \{a \cdot c\}))$ over $A \cup \{c\}$*

Proof. To make notation easier we assume $A = \emptyset$. By genericity of $tp(c)$, a is independent from $a \cdot c$ (over $\emptyset$). So a^{-1} is independent from $a \cdot c$. Now by 4.14(iii), H is the right stabilizer of p^{-1}. Hence by Lemma 4.18, $H \cdot (a \cdot c)$ is interdefinable with $Cb(tp(a^{-1} \cdot (a \cdot c)/a \cdot c))$. As H is $\emptyset$-definable, $H \cdot (a \cdot c)$ is clearly interdefinable with $H \cdot a$ over c. Hence we get the conclusion of the lemma.

We can now obtain some consequences for 1-based groups.

Proposition 4.20 *Suppose G is 1-based (and $\emptyset$-definable). Then*
(i) Any stationary type $tp(a/A)$ ($a \in G$) is the generic type of some right (left) translate of a connected definable subgroup of G.
(ii) Any connected definable subgroup H of G is definable over $acl(\emptyset)$.
(iii) G^0 is commutative.

Proof. (i) Let $H = Stab(p)$ which is A-definable. Let c be generic in G over $A \cup \{a\}$. By 4.19, $H \cdot a$ (as an imaginary) is interdefinable with $e = Cb(tp(c/A \cup \{a \cdot c\})$ over $A \cup \{c\}$. By 1-basedness of G, $e \in acl(c)$. Hence

the imaginary $H \cdot a$ is in $acl(c)$. So (as c is independent from a over A, so independent from $H \cdot a$ over A), $H \cdot a \in acl(A)$. That is $H \cdot a$ is, as a set, $acl(A)$-definable. Now apply Lemma 4.16
(ii) Let H be connected and definable over A say. Let a be generic in H over A. Let $g \in G$ be generic over $A \cup \{a\}$. So a is generic in H over $A \cup \{g\}$. Note that $X = g \cdot H$ is defined over $A \cup \{g\}$, and $tp(a/A \cup \{g\})$ and $tp(g \cdot a/A \cup \{g\})$ have the same Morley rank ($= RM(H) = \beta$ say) and that $g \cdot a \in X$. Hence $g \cdot a$ is generic in X over $A \cup \{g\}$, and note $q(x) = tp(g \cdot a/A \cup \{g\})$ is stationary.
By Lemma 4.17, (the canonical parameter of) X is interdefinable with $Cb(q)$. By 1-basedness of G, $Cb(q) \in acl(g \cdot a)$. So X is defined over $acl(g \cdot a)$. It then follows that H is defined over $acl(g \cdot a)$. By generic choice of g, $g \cdot a$ is independent from A over $\emptyset$, hence (as H is A-definable) H is defined over $acl(\emptyset)$.
(iii) It is enough to assume that G is connected and prove that G is commutative. Note that the group $G \times G$ is also 1-based. For $g \in G$, let $H_g = \{(a, a^g) : a \in G\}$. Then each H_g is a connected definable subgroup of $G \times G$. By (ii), each H_g is $acl(\emptyset)$-definable. It follows easily that there are only finitely many groups among the H_g. However the set of groups H_g is in bijection with $G/Z(G)$ (via $H_g = H_h$ iff $g^{-1} \cdot h \in Z(G)$). Hence $Z(G)$ has finite index in G. As G is connected this means that $Z(G) = G$ and so G is commutative.

Corollary 4.21 *The following are equivalent (where remember that G is a $\emptyset$-definable group in the saturated model $\bar{M}$ of the t.t theory T):*
(i) G is 1-based.
(iii) For every $n < \omega$, every definable subset of $G^n = G \times G \times .. \times G$ is a finite Boolean combination of left (right) translate of $acl(\emptyset)$-definable subgroups of G^n.

Proof. (i) implies (ii): Note that as G is 1-based, so is the group G^n. We first prove:
Claim. Let $\mathbf{p}$ and $\mathbf{q}$ be complete types over $\bar{M}$ containing $x \in G^n$ and containing the same right cosets of $acl(\emptyset)$-definable subgroups of G^n. Then $\mathbf{p} = \mathbf{q}$.
Proof of claim. By 4.20 there are connected $acl(\emptyset)$-definable subgroups H_1, H_2 of G^n and right translates X_1, X_2 of H_1, H_2 respectively, such that $\mathbf{p}$

is THE generic type of X_1 over $\bar{M}$, and $\mathbf{q}$ is THE generic type of X_2 over $\bar{M}$. By the assumptions of the claim, $x \in X_1 \wedge x \in X_2$ is in both $\mathbf{p}$ and $\mathbf{q}$. But $RM(\mathbf{p}) = RM(X_1) = RM(H_1)$ and $RM(\mathbf{q}) = RM(X_2) = RM(H_2)$. It follows that $RM(X_1 \cap X_2) = RM(X_1) = RM(X_2) = \alpha$, say. So $\mathbf{p}$ has Morley rank α and contains "$x \in X_2$". That is $\mathbf{p}$ is also the generic type over $\bar{M}$ of X_2, whence $\mathbf{p} = \mathbf{q}$. The claim is proved.

We can now deduce (ii) from the claim together with a standard compactness argument as in the proof of 2.25 from [1].

Remark 4.22 *(ii) in Corollary 4.21 can be replaced by: for any n, any $\emptyset$-definable subset of G^n is a finite Boolean combination of translates of definable subgroups.*

Exercise 4.23 *Let $M = (M, \cdot, ...)$ be an arbitrary saturated structure (not necessarily t.t), where $\cdot$ is a group operation on M. Suppose that every definable subset of M is a Boolean combination of translates of $acl(\emptyset)$ definable subgroups of M. Then $Th(M)$ is stable.*

We will complete this section with a few scattered but important facts about groups in *t.t.* theories. The first is a result of Poizat regarding type-definable subgroups of ω-stable groups(which was subsequently generalized in a far reaching way by Hrushovski). T remains a complete *t.t* theory and $\bar{M}$ a saturated model of T. First:

Definition 4.24 *Let $\Sigma(x)$ be a partial type, namely a possibly infinite but small set of formulas with fee variable x. Assume Σ is closed under finite conjunctions. Then $(RM, dM)(\Sigma) = min\{RM(\phi(x)), dM(\phi(x))) : \phi(x) \in \Sigma(x)\}$.*

Exercise 4.25 *Suppose that $\Sigma(x)$ is a partial type over A and that $(RM, dM)(\Sigma) = (\alpha, d)$. Then over $acl^{eq}(A)$ there are preciselt d complete types which contain $\Sigma(x)$ and have Morley rank α.*

Let G be our $\emptyset$-definable group. We will say that a subgroup H of G is *type-definable* if it is defined by (i.e. is the solution set) of a partial type $\Sigma(x)$.

Proposition 4.26 *Any type-definable subgroup of G is definable, namely defined by a single formula.*

Proof. Let H be a subgroup of G defined by the partial type $\Sigma(x)$. Without loss of generality Σ is closed under finite conjunctions and consists of formulas without parameters. (Just add names for parameters in Σ.) Let $(RM, dM)(\Sigma) = (\alpha, d)$. By Exercise 4.25, let $p_1, .., p_d$ be the complete types over $acl^{eq}(\emptyset)$ which contain Σ and have Morley rank α. (So the p_i can be considered as the "generic types" of H.) For $g \in G$, let $g \cdot p_i = tp(g \cdot a/acl^{eq}(\emptyset))$ for some (any) a realizing the nonforking extension of p_i over $\{g\} \cup acl^{eq}(\emptyset)$. As remarked earlier, this is well-defined, by stationarity of p_i. Let $H_1 = \{g \in G :$ for each $i = 1, .., d$, $g \cdot p_i = p_j$ for some $j = 1, ., d\}$. (So H_1 is the "stabilizer" of $\{p_1, .., p_d\}$.) As in the proof of Lemma 4.7, H_1 is definable (over $acl^{eq}(\emptyset)$), and it is rather easy to see that H_1 is a subgroup of G.
Claim. $H_1 = H$.
Proof. First let $g \in H_1$. Let a realize p_1 with a independent of g. Then $g \cdot a$ realizes p_j for some j. But both p_i and p_j contain $\Sigma(x)$, hence a and $g \cdot a$ are in H. Thus $g \in H$.
Conversely, let $g \in H$. Let a realize some p_i, independent from g over $\emptyset$. Then $RM(tp(a/g)) = \alpha$ and also $RM(tp(g \cdot a/g)) = \alpha$ (why?). Hence by the choice of α, $RM(tp(g \cdot a/\emptyset)) = \alpha$, and so $tp(g \cdot a/acl^{eq}(\emptyset)) = p_j$ for some $j = 1, .., d$. Thus $g \in H_1$.
The claim is proved. As H_1 is definable, the proposition is also proved.

A somewhat stronger result than Proposition 4.26 is true. By a type-definable group (in a saturated model) we mean a pair $(G, \cdot)$ such that $\cdot$ is a group operation on G, G is a type-definable set, and the graph of $\cdot$ is also type-definable. It is not hard to see that there is sone *definable* function f whose restriction to $G \times G$ is precisely the group operation. In any case, the result is that if T is $t.t$, then any such type-definable group is definable. We will leave this as an exercise.

Finally we point out how the "generic" elements of a definable group G control G.

Lemma 4.27 *For any $a \in G$, there are $b, c \in G$, each generic in G over a such that $a = b \cdot c$.*

Proof. Let $\alpha = RM(G)$. Let $p \in S(acl^{eq}(\emptyset))$ be a generic type of G. Let b realize the nonforking extension of p over a. Then $RM(tp(b^{-1}/a)) = \alpha$, so $RM(tp(b^{-1} \cdot a/a) = \alpha$. Putting $c = b^{-1} \cdot a$, we see that c is generic over a and $a = b \cdot c$.

Corollary 4.28 *Suppose that X is a $\emptyset$-invariant set, and that some generic type of G is X-internal: namely there is a generic type $p(x) \in S(acl^{eq}(\emptyset))$ of G, a small set $A \supseteq acl^{eq}(\emptyset)$ of parameters, a realization a of $p(x)|A$ and some tuple c from X such that $a \in dcl(A, c)$. THEN there is a small set B of parameters such that $G \subseteq dcl(B \cup X)$ (namely $G \subset X_B^{eq}$). Likewise replacing internal by almost internal and dcl by acl.*

Proof. Let $e_1, .., e_d$ be representatives of cosets of G^0 in G. Let $B = A \cup \{e_1, .., e_d\}$. Let $a \in G$. Let $b, c \in G$ be given by Lemma 4.27. Without loss of generality, (b, c) is independent from $\{a\} \cup B$ over $\{a\}$. There are e_i and e_j such that $b' = b \cdot e_i$ realizes $p(x)|B$ and $c' = c \cdot e_j$ realizes $p(x)|B$. By assumptions each of b', c' is in the definable closure of A together with some tuple from X. It follows that $a = b \cdot c$ is in the definable closure of B together with a tuple from X.

5 Theories and groups of finite Morley rank

The aim of this section is to give some kind of coherent picture of the "category" of definable sets in a (many-sorted saturated) structure in which all definable sets have finite Morley rank.

For now T remains a complete *t.t.* theory and $\bar{M}$ a saturated model of T. There is no harm in working in $\bar{M}^{eq}$.

Definition 5.1 *We define the U-rank on complete types. $p(x), q(x), ..$ denote complete types over subsets of $\bar{M}$.*
(i) "$U(p) \geq 0$" for all p.
(ii) "$U(p) \geq \alpha+1$" if $p(x)$ has a forking extension $q(x)$ such that "$U(q) \geq \alpha$".
(iii) For limit δ, "$U(p) \geq \delta$" if "$U(p) \geq \alpha$ for all $\alpha < \delta$".
(iv) $U(p) = \alpha$ if α is the least ordinal such that "$U(p) \geq \alpha$". (v) $U(p) = \infty$ if "$U(p) \geq \alpha$" for all ordinals α.

Lemma 5.2 *For any complete type $p(x)$, $U(p) \leq RM(p)$.*

Proof. We prove by induction on α that $U(p) \geq \alpha$ implies $RM(p) \geq \alpha$ for all cmplete types). The only thing to prove is the induction step: Suppose $U(p) \geq \alpha+1$, so p has a forking extension $q(x)$ with $U(q) \geq \alpha$. By Proposition 2.26, $RM(q) < RM(p)$, and by induction hypothesis, $RM(q) \geq \alpha+1$. Hence $RM(p) \geq \alpha$.

So in our context (T *t.t.*) every complete type has ordinal-valued U-rank. Note that if $p(x) \subseteq q(x)$ then $U(p) \geq U(q)$ (from the definitions). Note also that if if $p \in S(A)$ and f is an automorphism of $\bar{M}$ then $U(p) = U(f(p))$. Also it is rather easy to see that $U(p) =)$ iff p is algebraic (has only finitely many realizations).

We will need the following "forking" exercise.

Exercise 5.3 *Suppose that $A \subseteq B$ and $A \subseteq C$. Let $p(x) \in S(A)$, $q(x) \in S(B)$ and $r(x) \in S(C)$ be such that $q(x)$ is a nonforking extension of $p(x)$ and $r(x)$ is an extension of $p(x)$. Then there is an A-automorphism f of $\bar{M}$ and a complete type $q'(x) \in S(B \cup f(C))$ such that $q(x) \subseteq q'(x)$ and $q'(x)$ is a nonforking extension of $f(r) \in S(f(C))$.*

Lemma 5.4 *For any complete types $p(x) \subset q(x)$, $U(p) = U(q)$ if and only if q is a nonforking extension of p.*

Proof. Left to right is immediate from the definition. Now suppose that $p(x) \in S(A)$, $A \subseteq B$ and $q(x) \in S(B)$ is a nonforking extension of $p(x)$. Suppose that $U(p) = \alpha$. We want to prove that $U(q) = \alpha$, that is,
(*) for any $\beta < \alpha$, q has a forking extension with U-rank $\geq \beta$.
Now $p(x)$ has a forking extension $r(x) \in S(C)$ such that $U(r) \geq \beta$. By the exercise above, we may assume that $r(x)$ has a nonforking extension $q'(x) \in S(B \cup C)$ which extends $q(x)$. Note that $U(r) = \beta'$ for some $\beta \leq \beta' < \alpha$ So by induction hypothesis, $U(q') = \beta'$. On the other hand, clearly q' is a forking extension of q (why?). We have proved (*), and hence the lemma.

Here is a nice additivity property for types of finite U-rank, which is proved using Corollary 2.31.

Exercise 5.5 *Suppose that $U(tp(b/A)$ and $U(tp(a/A \cup \{b\})$ are both finite. Then $U(tp(a,b/A)) = U(tp(a/A \cup \{b\})) + U(tp(tp(b/A))$.*

Complete types of U-rank 1 will be important for us.

Remark 5.6 *Let $p(x) \in S(A)$ be a stationary type. The following are equivalent:*
(i) $U(p) = 1$,
(ii) For any $B \supseteq A$, $p(x)$ has a unique nonalgebraic extension $q(x) \in S(B)$.
(iii) p is nonalgebraic, and for any formula $\phi(x)$ (maybe with parameters), the set of realizations of $p(x) \cup \{\phi(x)\}$ is either finite, or is cofinite in the set of realizations of $p(X)$.

Types satisfying the equivalent conditions of Remark 5.6 are called *minimal types*. For example, if $(RM(p), dM(p)) = (1, 1)$ then p is minimal. But there are examples of minimal types with Morley rank > 1: Let the language L consist just of unary predicates P_i for $i \in \omega$. Let T say that the P_i are infinite and pairwise disjoint. T has quantifier-elimination (by a back-and-forth argument) from which we can also conclude that each P_i is strongly minimal, and the universe $(x = x)$ has Morley rank 2 and Morley degree 1. Let $p(x) \in S(\emptyset)$ be the "generic type" of $x = x$, namely $p(x)$ is the unique 1-type over $\emptyset$ with Morley rank 2. $p(x)$ is (axiomatized by) $\{\neg P_i(x) : i < \omega\}$. p is stationary (as it has Morley degree 1). We claim that p is minimal: Let B be any set of parameters, and $q(x) \in S(B)$ a nonalgebraic extension of $p(x)$. By quantifier-elimination we see that $q(x)$ is axiomatized by $p(x) \cup \{x \neq b : b \in B\}$. So $q(x)$ is the unique nonalgebraic extension of $p(x)$ over B.

Exercise 5.7 *Let $p(x) \in S(A)$ be a minimal type. Let X be the set of realizations of $p(x)$ in $\bar{M}$. Then X together with algebraic closure over A, is a homogeneous pregeometry in the sense of section 3. Moreover, if a is a finite sequence of elements of X, then $dim(a) = U(tp(a/A))$. All the results of section 3 on strongly minimal sets and their pregeometries are valid for X (with U-rank replacing Morley rank).*

On the other hand we have:

Lemma 5.8 *Suppose that $p \in S(A)$ is minimal, and that the corresponding pregeometry is nontrivial. Then $p(x)$ has Morley rank 1 (so is the "generic type" of a strongly minimal set over A).*

Proof. It is enough to find some nonforking extension of $p(x)$ which has Morley rank 1 (by 2.26). Thus, by the definition of nontriviality, we may find realizations a, b, c of $p(x)$ which are pairwise independent over A but such that each of a, b, c is in the algebraic closure of A together with the other two. Let $\phi(x, y, z)$ be a formula over A witnessing this (that is $\models \phi(a, b, c)$ and $\models \forall x, y\exists^{\leq k} z(\phi(x, y, z))$ etc.) We may also assume that $\phi(x, b, c)$ isolates $tp(a/A, b, c)$.

Now suppose that $RM(p) = \alpha$ and let $\theta(x) \in p(x)$ be chosen of Morley rank α and Morley degree 1. (So for any $B \supseteq A$, the nonforking extension of p over B is the unique type over B which contains θ and has Morley rank α. By definability of the stationary type $p(x)$, we may find a formula $\psi(x)$ over A, such that for any a', $\psi(a')$ iff for b' realizing $p(x)|(A, a')$, $\exists z(\phi(a', b', z) \wedge \theta(x))$. Note that $\models \psi(a)$, hence $\psi(x) \in p(x)$ and $\theta(x) \wedge \psi(x) \in p(x)$. Let $\psi'(x) = \theta(x) \wedge \psi(x)$. Then $\psi'(x)$ has Morley rank α and Morley degree 1.

Claim. For any $B \supseteq A$, there is a unique complete nonalgebraic type over B which contains $\psi'(x)$ (so has to be the unique nonforking extension of $p(x)$ over B).

Proof of claim. Let $B \supseteq A$. Let a' realize $\psi'(x)$ with $a' \notin acl(B)$. Let b' realize $p(x)|(B, a')$. By choice of ψ', there is c' such that $\models \phi(a', b', c',) \wedge \theta(c')$. Now by choice of ϕ, $b' \in acl(A, a', c')$, hence $RM(tp(b'/B, a')) \leq RM(tp(c'/B, a')) \leq RM(tp(c'/B) \leq \alpha$ (as $\models \theta(c')$. On the other hand, $RM(tp(b'/B, a')) = \alpha$. Hence we conclude that:

(*) $RM(tp(c'/B) = \alpha$, and so c' realizes $p(x)|B$.

Now if $c' \in acl(B, b')$ then (by choice of ϕ), $a' \in acl(B, b')$, but a' is independent from b' over B, so $a' \in acl(B)$, a contradiction. So $c' \notin acl(B, b')$, so by (*) realizes $p(x)|(B, b')$. Forking calculus implies newline (**) (b', c') is independent from B over A, and also

(***) $tp(b', c'/A) = tp(b, c/A)$.

As $\models \phi(a', b', c')$, the choice of ϕ together with (***) implies that $tp(a'/A) = p(x)$. Also by (**) and the fact that $a' \in acl(A, b', c')$ we see that a' is independent from B over A. Hence a' realizes $p(x)|B$. The claim is proved.

It follows from the claim that $\psi'(x)$ is strongly minimal. (If $\psi'(x)$ were the disjoint union of two infinite definable sets, then we obtain two nonalgebraic complete types containing $\psi'(x)$ over some set of parameters.)

Types of U-rank 1 play an important role in that they "coordinatize" definable sets of finite Morley rank, in a sense we will now make precise.

Proposition 5.9 *Let $p(x) \in S(A)$ be nonalgebraic and of finite U-rank. Then $p(x)$ is nonorthogonal to some stationary type p_0 of U-rank* 1 *(in the sense of Definition 3.36). (Moreover if $RM(p) < \omega$ then $RM(p_0) < \omega$.)*

Proof. Suppose $U(p) = n$ and we may suppose that $n > 1$. By definition of U-rank, there is $B \supset A$ and an extension $q(x) \in S(B)$ of $p(x)$ such that $U(q) = n - 1$. We may assume that $B = acl^{eq}(B)$ and so q is stationary. Let $c \in \bar{M}^{eq}$ be the canonical base of q (or rather an element such that $Cb(q) \subseteq dcl^{eq}(c)$ as in Lemma 2.38). By 2.42, c is in the definable closure of a finite sequence of realizations of p. By 5.5, $U(tp(c/A))$ is finite (and clearly nonzero). Let a realize q and note that $U(tp(a/A, c)) = n - 1$.

Now, we can again find some set D of parameters containing A such that $U(tp(c/D)) = 1$. Without loss D is independent from a over $A \cup \{c\}$ (why?).
Claim. a is independent from D over A.
Proof of Claim. By choice of D and forking calculus, $tp(a/Dc)$ does not fork over Ac, and so has U-rank $n - 1$. But $U(a/D) \geq U(a/Dc)$, so if a forked with D over A, we would have to have that $U(a/D) = n-1$, and so $tp(a/Dc)$ is a nonforking extension of $tp(a/D)$. But c is the canonical base of the latter type, so by 2.38 (ii), we would deduce that $c \in acl(D)$, a contradiction to $U(tp(c/D)) = 1$. The claim is proved.

Note that a forks with c over D. (Otherwise, by transitivity of nonforking and the claim, it would follow that a is independent from Ac over A.) Thus (by the claim), $tp(a/A) = p$ is nonorthogonal to $tp(c/D)$, proving the proposition. Note that $c \in acl(D, a)$ so $RM(tp(c/D)) \leq RM(tp(a/D))$.

Remark 5.10 *A similar kind of argument, together with a suitable generalization of 5.5 to the infinite U-rank case, yields that any stationary type is nonorthogonal to a type whose U-rank is of the form ω^α for some ordinal α.*

Definition 5.11 *Let $p(x) \in S(A)$ be stationary. Then p is said to be semi-minimal if there are $B \supseteq A$, a realising $p|B$ and $c_1, .., c_n$ each of whose type over B is minimal, such that $a \in dcl(B, c_1, .., c_n)$.*

Proposition 5.12 *Suppose that $a \notin acl(A)$, and $U(tp(a/A)) < \omega$. Then there is $d \in dcl(a, A) \setminus acl(A)$ such that $stp(d/A)$ is semi-minimal. (Again* *if $RM(tp(a/A)) < \omega$ then $stp(d/A)$ is semi-minimal with respect to minimal types of finite Morley rank.)*

Proof. By 5.9 there are $B \supseteq A$, such that a is independent from B over A, and c such that $tp(c/B)$ is minimal, and a forks with c over B (so $c \in acl(B, a)$). We may assume that $B = A \cup \{b\}$ for some element $b \in \bar{M}^{eq}$ (why?). Let $d = Cb(stp(b, c/A, a))$. By 2.42, there is a Morley sequence $(b_i, c_i)_{i<\omega}$ such that $d \in dcl(b_0, c_0, .., b_n, c_n)$ for some n. Let $B' = acl(A \cup \{b_0, .., b_n\}$. Forking calculus yields that a is independent from B' over A. Thus, as $d \in acl(A, a)$, d is independent from B' over A. For each i, $tp(c_i/A, b_i)$ is minimal, and either $tp(c_i/B')$ is the unique nnforking extension of the latter type, or $c_i \in B'$. Thus, d is contained in the definable closure of B' together with some realizations of minimal types over B'. That is, $tp(d/A)$ is semi-minimal. Now we only know that $d \in acl(A, a)$. Let d' be the imaginary $\{d_1, .., d_m\}$ where the d_j are the conjugates of d over A, a. Then $d' \in dcl(A, a)$ and we leave it to the reader to check that $tp(d'/A)$ is still semi-minimal.

Let us see what 5.12 means at the level of definable sets. Suppose that X is a definable set of finite Morley rank and of Morley degree 1, definable over A say. Let a realize the "generic type" of X over A. So there is an A-definable function f such that f is defined on a and $stp(f(a)/A)$ is semi-minimal. In fact, as $tp(a/A)$ is stationary, so is $tp(f(a)/A)$. So $f(a)$ realizes the generic type of some A-definable set Z of finite Morley rank and Morley degree 1. By semiminimality and compactness, after refining Z, there are $B \supset A$, B-definable sets $Y_1, .., Y_k$ each of finite Morley rank and Morley degree 1, whose generic types are minimal, and such that $Z \subset (Y_1 \times Y_2 \times ..Y_k)^{eq}_B$. Again by compactness, after throwing away from X an A-definable set of small Morley rank, $f(X) \subseteq Z$. So putting it altogether, there is an A-definable function f from X onto an A-definable set Z which lives inside a product of definable sets whose generic types are minimal.

For $b \in Z$ over A, we have the fibre $f^{-1}(b)$, defined over A, b. We have an A, b-definable equivalence relation on $f^{-1}(b)$ whose classes have Morley degree 1 and we can again apply 5.9 to these classes. By compactness the resulting maps are uniformly definable (as b varies in Z).

The process must eventually stop. So we have a "fibration" of X by definable sets contained in products of definable sets with minimal generic types. If it so happens that all minimal types are nontrivial, then by 5.8, the definable sets whose generic types are minimal are precisely the strongly minimal sets.

Let us now pass to the "fine" structure of groups of finite Morley rank.
Assumption. T is a complete *t.t.* theory, $\bar{M}$ is saturated model of T, and G is a group definable in $\bar{M}$ with $RM(G) < \omega$.

We want to explain how G is built up from almost strongly minimal(or semi-minimal) groups, and then discuss the "socle" (greatest connected semi-minimal subgroup) of G.

There are two approaches, from below, using the so-called Zilber indecomposability theore, or from above, using 5.12. We will discuss both.

Definition 5.13 *Let X be a type-definable subset of G (that is X is the set of realizations of some partial type $\Sigma(x)$). We say that X is indecomposable if for each definable subgroup H of G, either X is contained in a single left coset $a \cdot H$, or infinitely many left cosets of H meet X.*

Note that if X is indecomposable, then so is $a \cdot X$ for any $a \in G$. In any case here are some examples:

Lemma 5.14 *Suppose that X is the set of realizations of a complete stationary type $p(x) \in S(A)$ (where G is defined over A, and $p(x) \models$ "$x \in G$"). Then X is indecomposable.*

Proof. We may assume that $A = \emptyset$. Let $RM(p) = m$. Suppose by way of contradiction, that H is a definable subgroup of G and $|X/H|$ is finite with cardinality > 1. It follows that there is a unique coset, say $a \cdot H$ such that $RM(X \cap a \cdot H) = m$. (Let B be such that H and each coset of H meeting X are defined over B. Then the nonforking extension of $p(x)$ over B must contain exactly one of the "$x \in a_i \cdot H$" where $a_1 \cdot H, .., a_r \cdot H$ are the cosets meeting X.) Now, using the DCC on definable subgroups of G, let $K < H$ be the (unique) smallest definable subgroup of G such that for some coset $b \cdot K$ of K in G, $RM(X \cap b \cdot K) = m$. Note that $b \cdot K \subseteq a \cdot H$.
Claim. $b \cdot K$ is $\emptyset$-definable (and so K is too).
Proof of Claim. Let f be an automorphism of $\bar{M}$. Then $f(b \cdot K) = f(b) \cdot f(K)$ and $RM(X \cap f(b) \cdot f(K)) = m$. It follows that $RM(X \cap b \cdot K \cap f(b) \cdot f(K)) = m$

(why?). Hence by choice of K, and stationarity of p, $f(b \cdot K) = b \cdot K$. This proves the claim.

But p is the set of realizations of a complete type over $\emptyset$, so by the claim $p(x) \models x \in b \cdot K$, hence X is contained in $b \cdot K$ and so also in $a \cdot H$, contradiction. The proof of the lemma is complete.

Note that we actually proved above that if $p(x)$ is a stationary type, and $p(x) \models x \in G$ and some nonforking extension of p contains $x \in a \cdot H$ for some definable subgroup H of G, then all realizations of p are contained in $a \cdot H$.

Lemma 5.15 *Suppose that G is connected. Then for any $a \in G$, the conjugacy class a^G is indecomposable.*

Proof. Suppose again for a contradiction that H is a definable subgroup of G and a^G/H has cardinality finite and > 1. For each $g \in G$, conjugation by g is a (group) automorphism of G which leaves a^G invariant setwise, and takes H to the definable subgroup H^g. Hence the same is true for a^G/H^g for all $g \in G$. By the DCC on definable groups, the intersection of all conjugates H^g of H is a finite subintersection, so if K is this intersection, then K is normal in G and again a^G/K is finite of cardinality > 1.

So we see that working in the group G/K, the conjugacy class $(a/K)^{G/K}$ is finite and of cardinality > 1. But this conjugacy class is in definable bijection with $((G/K)/(C_{G/K}(a/K)))$, the centralizer of a/K in G/K. Thus this centralizer has finite index > 1 in G/K. But G/K is connected (as G is), and we have a contradiction.

The following is proved using the DCC on definable groups.

Exercise 5.16 *Any definable subset X of G can be partitioned into a finite number of definable indecomposable sets.*

Proposition 5.17 *Suppose that $\{X_i : i \in I\}$ is a set of indecomposable type-definable subsets of G such that the identity element $e \in G$ is in each X_i. Then the (abstract) subgroup of G generated by the X_i's is definable and connected. Moreover $H = X_{i_1} \cdot .. \cdot X_{i_n}$ for some $i_1, .., i_n \in I$.*

Proof. Again we may assume G to be defined over $\emptyset$. For $i_1, .., i_n \in I$, let $X_{i_1} \cdot ... \cdot X_{i_n} = \{a_1 \cdot ... \cdot a_n : a_j \in X_{i_j}\}$, a type-definable subset of G. As G has finite Morley rank, we can find such $Y = X_{i_1} \cdot ... \cdot X_{i_n}$ (for some $i_i, .., i_n \in I$) such that $RM(Y) = m$ is maximized. Suppose Y is type-definable over $A = acl(A)$, and let $p(x) \in S(A)$ be (stationary) and of Morley rank m such that $p(x) \models$ "$x \in Y$". Let H be the (left) stabilizer of $p(x)$.
Claim. $X_j \subseteq H$ for all $j \in I$.
Proof of claim. If not, then by indecomposability of X_i, and as $e \in X_j$, it follows that X_j/H is infinite. Let $b_1, b_2, ...$ be elements in X_j such that $b_r \cdot H \neq b_s \cdot H$ for $r \neq s$. Let $B \supseteq A$ be algebraically closed such that X_i is defined over B and each of $b_1, b_2, ...$ is in B. Let $p'(x) \in S(B)$ be the nonforking extension of $p(x)$ over B, realized by c say. Then for each $r < \omega$, $tp(b_r \cdot c/B)$ has Morley rank m. But for each $r \neq s$, $b_s^{-1} \cdot b_r \notin H$, hence $tp(b_s^{-1} \cdot b_r \cdot c/B) \neq tp(c/B)$, hence $tp(b_r \cdot c/B) \neq tp(b_s \cdot c/B)$. But maximal choice of $RM(Y)$, $RM(X_j \cdot Y) = m$. As each $b_r \cdot c \in X_j \cdot Y$, we see that $X_j \cdot Y$ contains infinitely many distinct types of Morley rank m, a contradiction. This proves the claim.

By the claim,
(*) H contains the subgroup of G generated by all the X_i.
In particular $Y \subseteq H$, and so $RM(p) \leq RM(H)$. By 4.15 and 4.16, H is connected, and $p(x)$ is the generic type of a translate of H. But by (*), $p(x) \models$ "$x \in H$". Hence $p(x)$ is the generic type of H. By Lemma 4.27, every element of H is a product of two realizations of p and thus $H \subseteq Y \cdot Y$. Together with (*) this proves the proposition.

I will now give a series of consequences of Proposition 5.17. Remember that G is a group of finite Morley rank definable in the model $\bar{M}$ of the $t.t$ theory T.

Corollary 5.18 *Suppose that G is infinite, noncommutative and has no infinite definable normal proper subgroups. Then*
(i) G has no infinite normal proper subgroup,
(ii) $Z(G)$ is finite and $G/Z(G)$ is simple as an abstract group.

Proof. (i) Clearly G is connected. By our assumptions, $Z(G)$ (the centre of G) is finite. Suppose H is an infinite normal subgroup of G. Let $a \in H \setminus Z(G)$. Then a^G is infinite (otherwise $C(a)$ has finite index in G

so $C(a) = G$ so $a \in Z(G)$, contradiction). By 5.15, a^G is indecomposable. So $a^G \cdot b^{-1}$ is indecomposable and contains e, for all $b \in a^G$. By 5.17, the subgroup of G generated by all the sets $a^G \cdot b^{-1}$ for $b \in a^G$, is definable. Let H_1 be this subgroup. Then it is normal, infinite, and is clearly contained in H. By our assumptions, $H_1 = G$, hence $H = G$.
(ii) We already know that $Z(G)$ is finite (and normal).
Claim I. $G/Z(G)$ is noncommutative and connected.
Proof. Clearly $G/Z(G)$ is connected (for otherwise the preimage in G of a proper definable subgroup of finite index of $G/Z(G)$ would contradict connectedness of G. If $G/Z(G)$ were commutative, then G' (the subgroup of G generated by all commutators $a \cdot b \cdot a^{-1} \cdot b^{-1}$) would be contained in $Z(G)$. But G' is generated by all $a^G \cdot a^{-1}$ for $a \in G$, and by 5.15 and 5.17, G' is definable, normal (and infinite). So $G' = G$, contradiction.
Claim II. $G/Z(G)$ has no finite normal subgroups.
Proof. The preimage in G of a finite normal subgroup of $G/Z(G)$ would be a finite normal subgroup H of G, containing $Z(G)$. Note that G acts definably on H by conjugation. Hence $G/C_G(H)$ acts faithfully on H, whereby $G/C_G(H)$ is finite. So $G = C_G(H)$ and $H = Z(G)$.

Note that $G/Z(G)$ has no proper definable infinite normal subgroups. Hence by Claims I, II, and part (i), we see that $G/Z(G)$ has no proper nontrivial normal subgroup. This completes the proof of 5.18.

We now want to “decompose” G into almost strongly minimal groups.

Corollary 5.19 *Suppose that G is infinite and has no proper infinite normal definable subgroups. Then G is almost strongly minimal (there is strongly minimal $X \subset G$ and some finite set of parameters B over which G and X are defined, such that $G \subseteq X_B^{eq}$).*

Proof. As $RM(G) < \infty$, there is some strongly minimal set $X \subset G$. Let A be such that G and X are A-definable. Let $p(x) \in S(A)$ be stationary of Morley rank 1 containing X. Let a realize p and let $Y = a^{-1} \cdot p(\bar{M})$. For each $b \in G$, let $Y_b = b \cdot Y \cdot b^{-1}$. Then by 5.14 each Y_b is type-definable, indecomposable, and contains e. Let H be the subgroup of G generated by all the Y_b's. H is clearly normal, and infinite. By part (i) (or 5.17 and the hypotheses), $H = G$. By 5.17, $G = Y_{b_1} \cdot \ldots \cdot Y_{b_n}$ for some $b_1, .., b_n \in G$. Then clearly, for $B = A \cup \{a, b_1, .., b_n\}$, $G \subseteq dcl(X \cup B)$.

Now fix G, connected and of finite Morley rank. Then there is greatest n such that there exist $\{e\} = H_0 < H_1 < .. < H_n = G$, where each H_i is a normal definable connected subgroup of G (and H_i is a proper subgroup of H_{i+1}). In fact by Lemma 4.1, $RM(H_i) < RM(H_{i+1})$, so $n \leq RM(G)$. Note that for each i

For future purposes we denote n by $n(G)$.

Corollary 5.20 *With G, n and the H_i as above, each H_{i+1}/H_i is almost strongly minimal.*

Proof. This is almost given by Corollary 5.19. As there, let X be a strongly minimal subset of H_{i+1}/H_i, $p(x) \in S(A)$ its "generic type" and Y some translate of $p(\bar{M}$ by a^{-1} for a realizing p. Now consider the subgroup H/H_i of G/H_i generated by all Y^b for $b \in G/H_i$. So by 5.17, H/H_i is definable, infinite, normal (in G/H_i), connected, and contained in H_{i+1}/H_i. By choice of the H_i, we see that $H = H_{i+1}$.

Exercise 5.21 *(G a definable group of finite Morley rank.) There are stationary types $p_1, .., p_n$ of Morley rank 1 (maybe in $\bar{M}^{eq}$) such that*
(i) each stationary $tp(a/A)$ wth $a \in G$ is nonorthogonal to one of the $p_1, .., p_n$,
(ii) for any stationary type p of U-rank 1, if p is nonorthogonal to $tp(a/A)$ for some $a \in G$, then p is nonorthogonal to one of $p_1, .., p_n$ (and hence $RM(p) = 1$ too).

Example 5.22 *(i) Let T be strongly minimal (e.g. $T = Th(ACF_p)$. Then any group definable in $\bar{M}$ is almost strongly minimal.*
(ii) Let $T = Th(\mathbf{Z}_4^{(\omega)}, +)$. Let G denote this group. Them G is not almost strongly minimal. However both $2G$ and $G/2G$) are strongly minimal.

We want to say a little more about the H_{i+1}/H_i's.

Lemma 5.23 *Let G be connnected. Let H be a minimal infinite normal definable subgroup of G. Then either H is commutative, or H is noncommutative with no proper infinite normal subgroup.*

Proof. Note that H is connected. We proceed by induction on $RM(G)$. We may assume that $H \neq G$ (using also 5.18). Let us also assume that H is noncommutative. So $Z(H)$ is finite (otherwise the connected component of

$Z(H)$ is a proper infinite normal subgroup of H, normal in G). As in the proof of 5.18, $H/Z(H)$ is centreless. So (quotienting G by the finite normal subgroup $Z(H)$) we may assume that H is centreless.

It is enough (by 5.18) to show that H has no infinite proper normal definable subgroups. Suppose for a contradiction that N is such. So N is connected. By 5.17 the subgroup of G generated by $\{N^g : g \in G\}$ is definable and connected. This group is clearly infinite, normal in G and contained in H. So it must equal H. Thus (by 5.17 again) $H = N_1 \cdot N_2 .. \cdot N_k$ for distinct conjugates N_i of N (and note each N_i is normal in H. Now each $N_i \cap N_j$ is normal in H, so finite, so central in H, so trivial. Thus H is a direct product $N_1 \times .. \times N_k$. If N were commutative, then each N_i would be too, so H would be commutative, cntradiction. Thus each Ni (a minimal normal definable infinite subgroup of H) is noncommutative. We can apply induction ($RM(H) < RM(G)$) to conclude that each N_i has no proper infinite normal subgroup. Note that $Z(N_i) < Z(H) = \{e\}$, so each N_i is also centreless. Thus H is a direct product of finitely many noncommutative (abstractly) simple definable groups, $N_1, .., N_k$. An elementary grou-theoretic fact implies that any simple nontrivial normal subgroup of H is among the N_i. Thus G acts (by conjugation) on the finite set $\{N_1, .., N_k\}$. As G is connected, the action is trivial. That is each N_i is normal in G, as is N. Contradiction. The lemma is proved.

Corollary 5.24 *In the context of 5.20, each H_{i+1}/H_i is either commutative, or noncommutative with no infinite normal subgroup (so by 5.18 abstractly simple modulo its finite centre).*

Proof. H_{i+1}/H_i is clearly a minimal infinite normal subgroup of G/H_i so we can use 5.23.

Corollary 5.18 raises the question: if G is connected and infinite, does it follow that G has no abstract proper subgroup of finite index. This is not true in general: the additive group of an algebraically closed field of characteristic $p > 0$ is connected, but (as it is an infinite dimensional vector space over $\mathbf{F}_p$) it has proper subgroups of finite index. This is essntially the only obstacle.

Corollary 5.25 *Suppose that for every connected commutative group G of finite Morley rank (definable in $\bar{M}$), G has no proper subgroup of finite index.*

Then the same is true for every connected definable group of finite Morley rank.

Proof. Let $n = n(G)$ and let H_{n-1} be as in the discussion before 5.20. Suppose for a contradiction that H is a proper subgroup of finite index in G. So $H \cap H_{n-1}$ has finite index in H_{n-1}. By induction ($RM(H_{n-1}) < RM(G)$), G contains H_{n-1} whereby H/H_{n-1} is a proper subgroup of finite index in G/H_{n-1}. Note that any proper subgroup of finite index contains a proper normal subgroup of finite index. So we have a contradiction to 5.24, using our assumptions.

In particular if all commutative connected definable groups of finite Morley are divisible, then any connected definable group of finite Morley rank is abstractly connected.

Finally we discuss the socle argument. G remains a group of finite Morley rank definable in $\bar{M} \models T$.

Definition 5.26 *The socle of G, $s(G)$ is the subgroup of G generated by all the connected definable almost strongly minimal subgroups of G.*

Remark 5.27 *$s(G)$ is connected, definable and normal in G and definable over the same set of parameters as G.*

Proof. If H is a connected definable subgroup of G, then clearly H is indecomposable and contains the identity. If also H is almost strongly minimal, then every conjugate H^g of H is almost strongly minimal (as well as being connected). Hence $s(G)$ is normal. By 5.17 it is definable and connected, and equals $K_1 \cdot .. \cdot K_n$ for some almost strongly minimal definable connected subgroups $K_1, .., K_n$ of G.

Lemma 5.28 *$s(G)$ is the unique maximal connected definable subgroup H of G such that there are strongly minimal sets $Y_1, .., Y_d$ in $\bar{M}^{eq}$ with $H \subseteq acl(Y_1, .., Y_d, B)$ for some small set B of parameters.*

Proof. Suppose for a contradiction that H is a connected definable subgroup of G properly containing $S = s(G)$ such that H is contained in the algebraic closure of a finite number $Y_1, .., Y_d$ of strongly minimal sets (plus a small set of parameters). Let B be the set of parameters (over which we may assume that G, S, H and the Y_i are defined). Let $a \in H$ with $tp(a/B)$ generic. Then $a/S \notin acl(B)$. Let $b_i \in Y_i$ for $i = 1, .., d$,such that $a \in acl(B, b_1, .., b_d)$. We may assume that for each i $b_i \notin acl(B \cup \{b_j : j \neq i\})$. Note that $a/S \in acl(B, b_1, .., b_d)$ too. After including in B some of the b_i's, we may assume that each of a, a/S is interalgebraic with $(b_1, .., b_r)$ over B and that $a/S \notin acl(B)$. (We still have that $(b_1, .., b_r)$ is A-independent, but $tp(a/B)$ need no longer be generic in H.) In particular, both a/S and a are interalgebraic with b_r over $B, b_1, .., b_{r-1}$. So both $stp((a/S)/B, b_1, .., b_{r-1})$ and $q(x) = stp(a/B, b_1, .., b_{r-1})$ are stationary of Morley rank 1. In particular the set of realizations X of $q(x)$ intersects infinitely many cosets of S in G. The same is true for $a^{-1} \cdot X$. By 5.17 the latter set generates a connected definable almost strongly minimal subgroup of G which is not contained in S, contradiction.

We will restrict our attention to commutative groups below, although everything works in the noncommutative case too. Assume G to be defined over $\emptyset$.

Proposition 5.29 *Let G be connected and commutative (and of finite Morley rank). Let $S = s(G)$. Let $p(x) = tp(a/A)$ be stationary, with $a \in G$. Assume also*
(i) every connected definable subgroup of S is defined over some c such that c is independent from d over A for all $d \in G/S$.
(ii) $Stab(p)$ is finite.
Then all realizations of $p(x)$ are contained in a single translate of S.

Before proving the proposition, let is discuss the hypotheses and conclusion. Hypothesis (i) is a "rigidity" condition. It will be satisfied in the following special situations: (a) every definable connected subgroup of G is defined over $acl(\emptyset)$, (b) S is "fully orthogonal" to G/S (whenever $p(x)$ and $q(y)$ are complete stationary types extending $x \in S$, $y \in G/S$ respectively, then p is orthogonal to q). (a) will be true of algebraic tori (finite products of the multiplicative group) as well as abelian varieties, in the case where $\bar{M}$ is an algebraically closed field, although in this case every definable connected group is equal to its socle. A more interesting example is where G as a structure in its own right is a compact complex Lie group equipped with predicates for analytic subvarieties of G^n for all n. (Such a structure is not saturated but we can work with a saturated elementary extension).

Hypothesis (ii) is equivalent to $Stab(p) \cap S$ is finite. For suppose the latter holds and $Stab(p)$ is infinite. Then the connected component of $Stab(p)$ contains an almost strongly minimal subgroup (why?), which is not contained in S, a contradiction.

The conclusion of the Proposition could possibly be better expressed in terms of definable sets. For $X \subseteq G$ a definable set of Morley rank m and Morley degree 1, define $Stab(X)$ to be $\{g \in G : RM(X \cap g \cdot X) = m\}$. Then $Stab(X)$ is the same as $Stab(p)$ where p is the unique stationary type of Morley rank m extending "$x \in X$". Then the conclusion can be stated as: if $Stab(X)$ is finite, then, up to translation, and a definable set of Morley rank $< m$, X is contained in S.

Together with the truth of the "Zilber conjecture" in the many sorted structure of compact complex manifolds (which will be discused later), the Proposition yields the following result of Ueno: if X is an irredicible analytic subvariety of a compact complex Lie group A, and $Stab(X)$ is finite, then X is biholomorphic with a complex algebraic variety. (However there are simpler proofs, even of a model-theoretic nature.)

Proof of Proposition 5.29. The proof is a little bit involved. The general strategy is to assume the conclusion is false, and then produce a connected definable nontrivial subgroup of G which is contained in the algebraic closure of finitely many strongly minimal sets (in $\bar{M}$) and is NOT contained in S, contradicting 5.28.

We may assume that $A = \emptyset$. Let $\pi : G \to G/S$ be the canonical surjective homomorphism. Let X be the set of realizations of p. Fix $a \in X$ and let $b = \pi(a)$. Note that $tp(b/\emptyset)$ is also stationary, so if it is algebraic then $b \in dcl(\emptyset)$, and we see that all realizations of p are contained in the single translate $\pi^{-1}(b)$ of S. So we may assume that $b \notin acl(\emptyset)$. Let $X_b = X \cap \pi^{-1}(b)$. Note that X_b is precisely the set of realizations of $tp(a/b)$ (why?). Let $G_b = \pi^{-1}(b)$. So G_b is a translate of S. In particular S acts on G_b by addition.

Now let Z be a subset of G_b which is definable with parameters from $S \cup \{b\}$ and is of of least (RM, dM) subject to this condition.

Claim I. Z is of the form $H + c$ for some $c \in Z$ and some definable subgroup H of S.

Proof of Claim I. Note that if $s \in S$, then either $s + Z = Z$ or $s + Z$ is disjoint from Z. (If both $(s+Z) \cap Z$ and the symmetric difference of $s+Z$ and Z were nonempty then one would have strictly smaller (RM, dM) than Z, a contradiction, as both are definable over $S \cup \{b\}$.) Thus $H = \{s \in S : s+Z = Z\}$ is a definable subgroup of S, and $Z = H + c$ for some (any) $c \in Z$.

Note that the sets $s+Z$ for $s \in S$ cover G_b and each is a coset (translate) of H.

Claim II. Any translate $s+Z$ of Z by $s \in S$ is either contained in X_b or is disjoint from X_b. Thus X_b is invariant under translation by H.

Proof. Suppose that $s+Z$ meets X_b. Now X_b is an intersection of b-definable sets $\{Y_i\}_i$ say, so as $s+Z$ is defined over $S \cup \{b\}$ and has the same Morley rank and degree as Z, $s+Z$ must be contained in every Y_i (otherwise we again contradict choice of Z). So $s+Z$ is contained in X_b.

Now let H^0 be the connected component of H (a connected definable subgroup of S). By Claim II, X_b is invariant under translation by H^0.

Claim III. H^0 is contained in $Stab(p)$.

Proof. By assumption, H^0 is defined over some parameter t such that t is independent from b over $\emptyset$. Fix $d \in H^0$. We will show that $d \in Stab(p)$. We may assume that d is independent from b over t, hence $\{t, d\}$ is independent from b over $\emptyset$. Let $a \in X_b$ be independent from $\{t, d\}$ over b. Then by forking calculus we see that a is independent from d over $\emptyset$ (and a realizes p). As X_b is invariant under translation by H^0, $d + a \in X_b$ so realizes p too. This proves that $H_0 \subseteq Stab(p)$.

From Claim III and assumption (ii), H^0 is trivial hence H is finite, so Z is finite. Z is defined by a formula $\phi(x, b, e)$ for some e from S. We may assume that for all b', e', $\phi(x, b', e')$ defines a finite (maybe empty) set. As $tp(b)$ is nonalgebraic, there is an infinite definable subset Y of G/S such that for all $b' \in Y$, there is e' from S such that $\phi(x, b', e')$ defines a (finite) nonempty subset of the fibre $G_{b'}$. As Y is infinite, it contains a strongly minimal definable subset Y' say. Let W be the the union of all sets defined by $\phi(x, b', e')$ for $b' \in Y'$ and e' in S. Thus W is a definable subset of G contained in the algebraic closure of S and Y'. As S itself is contained in the definable closure of a finite set of strongly minimal sets (plus some parameters), W is contained in the algebraic closure of finitely many strongly minimal sets. Note that W meets infinitely many cosets of S in G. By Exercise 5.15, there is some indecomposable definable subset W' of W which meets infinitely many cosets of S in G. After translating W' we obtain an indecomposable definable subset W'' of G which contains the indentity and meets infinitely many cosets of S. Let K be the subgroup of G generated by W''. By 5.17, K is definable, connected, not contained in S. The (definable connected) subgroup of G generated by S and K is then contained in the algebraic closure of finitely many strongly minimal sets, and properly contains S. This contradicts Lemma 5.28. Proposition 5.29 is proved.

It is an open question whether Proposition 5.29 is true without the "rigidity" assumption (i).

References

[1] A. Pillay, Lecture Notes - Model Theory.