AMENABILITY AND DEFINABILITY

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Abstract. We study amenability of definable groups and topological groups, as well as a new notion of *first order amenability* of a theory T, and prove various results, briefly described below.

Among our main technical tools, of interest in its own right, is an elaboration on and strengthening of the Massicot-Wagner version [30] of the stabilizer theorem [15], and also some results about measures and measure-like functions (which we call means and pre-means).

As an application we show that if G is an amenable topological group, then the Bohr compactification of G coincides with a certain "weak Bohr compactification" introduced in [27]. In other words, the conclusion says that certain connected components of G coincide: $G^{00}_{\mathrm{top}} = G^{000}_{\mathrm{top}}$. We also prove wide generalizations of this result, implying in particular its extension to a "definable-topological" context, confirming the main conjectures from [27].

Secondly, we study the relationship between definability of an action of a definable group on a compact space (in the sense of [12]), weakly almost periodic (wap) actions of G (in the sense of [10]), and stability. We conclude that any group G definable in a sufficiently saturated structure is "weakly definably amenable" in the sense of [27], namely any definable action of G on a compact space supports a G-invariant probability measure. This gives negative solutions to some questions and conjectures raised in [23] and [27].

Thirdly, we introduce the notion of *first order [extreme] amenability*, as a property of a first order theory T: every complete type over $\emptyset$, in possibly infinitely many variables, extends to an automorphism-invariant global Keisler measure [type] in the same variables. [Extreme] amenability of T will follow from [extreme] amenability of the (topological) group $\mathrm{Aut}(M)$ for all sufficiently large $\aleph_0$-homogeneous countable models M of T (assuming T to be countable), but is radically less restrictive. A further adaptation of the technical tools mentioned above is used to prove that if T is amenable, then T is G-compact, namely Lascar strong type and KP-strong type over $\emptyset$ coincide. This extends and essentially generalizes results in [27].

In the second and third part of the paper, stability in continuous logic will play a role in some proofs.

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0. Introduction

The general motivation standing behind this research is to understand relationships between dynamical and model-theoretic properties of definable [topological] groups or between dynamical properties of groups of automorphisms of first order structures and model-theoretic properties of the underlying theories. More specifically, similarly to [27], in this paper our goal is to understand model-theoretic consequences of various notions of amenability.

The consequences that we consider in this paper are versions of G-compactness: the equality of Lascar strong type and KP-strong type, or for definable groups the equality of G^{000} and G^{00} as well as a number of other contexts including arbitrary topological groups. Although Sections 2, 3, and 4 of this paper are thematically connected, they can be read relatively independently.

The notions of amenability considered in [27] come from certain natural categories of flows, e.g.:

(1) the classical notion of amenability of a topological group is equivalent to the existence of a left-invariant, Borel probability measure on the universal topological ambit;
(2) definable amenability of a group $G(M)$ definable in M means that there is such a measure on the $G(M)$-ambit $S_G(M)$;
(3) definable topological amenability of a topological group $G(M)$ definable in a first order structure M so that the members of a basis of open neighborhoods of 1 are definable means that there is such a measure on the $G(M)$-ambit $S_G(M)/{\sim_\mu}$, where $p \sim_\mu q \iff \mu \cdot p = \mu \cdot q$, where μ is the subgroup consisting of infinitesimal elements;
(4) weak definable [topological] amenability of a definable [topological] group $G(M)$ means that there is such a measure on the universal definable [continuous] $G(M)$-ambit (see Section 1).

In each of the contexts mentioned in the above list (topological, definable, definable topological), we have two associated notions of components of G: G^{00}_{top}, G^{000}_{top}; G^{00}_M, G^{000}_M; and $G^{00}_{\mathrm{def,top}}$, $G^{000}_{\mathrm{def,top}}$ (see Section 1).

The following statement is Conjecture 0.4 in [27].

Conjecture 0.1. *Let $G(M)$ be a topological group and assume that the members of a basis of open neighborhoods of the identity are definable. If G is definably topologically amenable, then $G^{00}_{\mathrm{def,top}} = G^{000}_{\mathrm{def,top}}$.*

One of the main results of [27] is Theorem 0.5 in there saying that Conjecture 0.1 is true if $G(M)$ has a basis of open neighborhoods of the identity consisting of definable, open subgroups. This implies that Conjecture 0.2 from [27] holds under the assumption that $G(M)$ has a basis of open neighborhoods of the identity consisting of open subgroups, namely, if $G(M)$ is a topological group satisfying this assumption, then $G^{00}_{\mathrm{top}} = G^{000}_{\mathrm{top}}$.

In Subsection 2.6, we will prove Conjecture 0.1 in full generality (see Corollary 2.36). Similarly to [27], the proof is based on the Massicot-Wagner argument from [30], but here we use *means* on certain lattices instead of measures on Boolean algebras. Moreover, in Subsection 2.3, we give a less numerical variant of the argument from [30], using a general notion of largeness, discussed in Subsection 2.2, which coincides with non-forking in stable theories and seems interesting also outside stable context. In fact, using these arguments, we obtain in Subsection 2.6 much more general results (namely, Theorems 2.34 and 2.35) than Conjecture 0.1, which do not assume any topology on $G(M)$. All of this also requires some extension results concerning pre-means, means, and measures — established in Subsections 2.4 and 2.5 — which may prove to be useful also in other situations. In Subsection 2.7, we apply these kind of arguments to topological groups equipped with the so-called $\bigvee$-*definable group topologies* (including group topologies induced by type-definable subgroups as well as uniformly definable group topologies). The key property of a $\bigvee$-definable group topology on a $\emptyset$-definable group G is that for any model M the group $G(M)$ is also a topological group. We prove (using our version of the Massicot-Wagner theorem) that the existence of a left-invariant mean on the lattice of closed, type-definable subsets of the group $G = G(M^*)$ (where $M^* \succ M$ is a monster model and G is a $\emptyset$-definable group) equipped with a $\bigvee$-definable group topology, such that the projections of closed, type-definable sets are closed, implies that $\mathrm{cl}(G^{00}_M) = \mathrm{cl}(G^{000}_M)$, where cl denotes closure with respect to the $\bigvee$-definable topology; this is Proposition 2.51.

The notion of a *definable action* of a definable group $G(M)$ on a compact space X comes from [12], namely for any $x \in X$, the map taking $g \in G(M)$ to gx is *definable*, that is induced by a continuous map from the type space $S_G(M)$ to X. In [27, Definition 3.1(i)], $G(M)$ was defined to be weakly definably amenable if any definable action of $G(M)$ on a compact space X supports a $G(M)$-invariant Borel probability measure on X. In [27, Definition 3.1(ii)], this notion was also generalized to the context of topological (not necessarily discrete) groups definable in a structure M. The following generalization of Conjecture 0.1 is stated as Conjecture 0.3 of [27].

Conjecture 0.2. *Let $G(M)$ be a topological group definable in an arbitrary structure M. If G is weakly definably topologically amenable, then $G^{00}_{\mathrm{def,top}} = G^{000}_{\mathrm{def,top}}$.*

In Section 3, we will refute this conjecture by showing that it is already false in the "discrete case". In fact, we show that a definable group $G(M)$, defined in an $\aleph_1$-saturated structure M, is always weakly definably amenable. Our methods are as interesting as the refutation of the conjecture: under the saturation assumption, definable actions are weakly almost periodic, so support invariant measures. Our proofs involve stable group theory in a continuous logic setting. This will also give us the negative answer to the question stated in [23, Problem 4.11(1)], namely whether the assignment $S_G(M)/E \to G/G^{000}_M$ given by $\mathrm{tp}(a/M)/E \mapsto a/G^{000}_M$ is well-defined, where E is the equivalence relation on $S_G(M)$ such that $S_G(M)/E$ is the universal definable $G(M)$-ambit. In [23, Proposition 4.10], it was noted that an analogous assignment to G/G^{00}_M is a well-defined continuous semigroup epimorphism (with the natural semigroup structure on $S_G(M)/E$ coming from the fact that this is the universal definable $G(M)$-ambit). We also provide a description of the universal definable $G(M)$-ambit as the "Gelfand space" of the algebra of stable continuous functions from $S_G(M)$ to $\mathbb{R}$, and describe the universal minimal definable $G(M)$-flow as G/G^{00}_M. In this section, we also discuss definable actions when M is not necessarily saturated, and make the connection between weakly almost periodic actions and continuous logic stability *in a model.*

In Subsection 4.2, we introduce the notions of *amenable* and *extremely amenable* first order theory. This is part of our attempt to extract the model-theoretic content of the circle of ideas around [extreme] amenability of automorphism groups of countable structures, which we discuss further below. We say that T is amenable if for every $p \in S_{\bar{x}}(\emptyset)$, in any (possibly infinite) tuple of variables $\bar{x}$, there exists an $\mathrm{Aut}(\mathfrak{C})$-invariant, Borel probability measure on $S_p(\mathfrak{C}) := \{q \in S_{\bar{x}}(\mathfrak{C}) : p \subset q\}$, where $\mathfrak{C}$ is a monster model. Extreme amenability of T means that the invariant measure above can be chosen to be a *Dirac*. Namely, every p extends to a global $\mathrm{Aut}(\mathfrak{C})$-invariant complete type. We study properties of [extreme] amenability, showing for example that they are indeed properties of the theory (i.e. do not depend on $\mathfrak{C}$) and providing several equivalent definitions. We will discuss here amenability, leaving the extreme version to the final paragraph. One of the equivalent definitions of amenability of T is that $\mathrm{Aut}(\mathfrak{C})$ is *relatively definably amenable* (i.e. there is an $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive, probability measure on the Boolean algebra of relatively definable subsets of $\mathrm{Aut}(\mathfrak{C})$ treated as a subset of $\mathfrak{C}^{\mathfrak{C}}$). Relative definable amenability of $\mathrm{Aut}(\mathfrak{C})$ (or, more generally, of the group of automorphisms of any model) is a natural counterpart of definable amenability of a definable group. The above observations work for any $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model M in place of $\mathfrak{C}$. For such an M, if $\mathrm{Aut}(M)$ is amenable as a topological group (with the pointwise convergence topology), then T is amenable. We point out in a similar fashion that (for countable T) if $\mathrm{Aut}(M)$ is amenable for all sufficiently large $\aleph_0$-homogeneous countable models, then T is amenable. In the NIP context, we get a full characterization of amenability of T in various terms, e.g. by saying that $\emptyset$ is an extension base, which also yields a class of examples of amenable theories, e.g. all stable theories are amenable. The main result of Section 4 is contained in Subsection 4.4, namely, we prove the following

Theorem. *Every amenable theory is G-compact.*

We should mention here that this result generalizes Theorem 0.7 from [27] which says that whenever M is a countable, ω-categorical structure and $\mathrm{Aut}(M)$ is amenable as a topological group, then $\mathrm{Th}(M)$ is G-compact. [27, Theorem 0.7] was deduced (by a non-trivial argument which is interesting in its own right) from

Conjecture 0.1 for groups possessing a basis of open neighborhoods of the identity consisting of open subgroups (note that $\mathrm{Aut}(M)$ has this property). In the general context of Subsection 4.4, we do not have an argument showing that the main theorem follows from Conjecture 0.1; instead we give a direct proof working with relatively definable subsets of the group of automorphisms. The engine of our proof is once again the argument from [30]. In Subsection 4.3, we give a simpler proof of the above main result of Section 4, but under the stronger assumption of the existence of $\emptyset$-definable Keisler measures on all $\emptyset$-definable sets and using stability theory in continuous logic. This also includes the ω-categorical context from [27, Theorem 0.7], yielding yet another proof of [27, Theorem 0.7].

Extreme amenability of automorphism groups of (arbitrary) countable structures M was studied in detail by Kechris, Pestov, and Todorcević. Their paper [21] inspired a whole school, connecting to structural Ramsey combinatorics and dynamics. When $\mathrm{Th}(M)$ is ω-categorical, then extreme amenability of $\mathrm{Aut}(M)$ is a property of this first order theory, so is a model-theoretic notion (in the sense of model theory being the study of first order theories rather than arbitrary structures). Some of this extends to homogeneous models of arbitrary theories and to continuous logic (thanks to Todor Tsankov for a conversation about this with one of the authors).

Let us comment on the relation between extreme amenability of the automorphism group of an ω-categorical countable structure M as considered in [21], which we call KPT-extreme amenability), and extreme amenability of $\mathrm{Th}(M)$ in our sense. KPT-extreme amenability concerns *all* flows of the topological group $\mathrm{Aut}(M)$ and says that the universal flow (or rather ambit) has a fixed point. Our first order extreme amenability (of $\mathrm{Th}(M)$) can also be read off from flows of $\mathrm{Aut}(M)$ and says that a *particular flow* $S_{\bar{m}}(M)$ has a fixed point (where $\bar{m}$ is an enumeration of M and $S_{\bar{m}}(M)$ here denotes the space of complete extensions of $\mathrm{tp}(\bar{m})$ over M). The class of KPT-extremely amenable ω-categorical theories T is not at present explicitly classified, but appears to be very special (analogous to monadic stability in the stable world). It follows from their definition that whenever $\mathcal{L}'$ is a language extending the language $\mathcal{L}$ of T and T' is a universal $\mathcal{L}'$-theory consistent with T, then the countable model M of T has an expansion to a model of T' where the new symbols in $\mathcal{L}'$ are interpreted as certain $\emptyset$-definable sets in M. Note in particular that KPT-amenability of an ω-categorical structure M implies the existence of a $\emptyset$-definable linear ordering on M. By contrast, our first-order extreme amenability is a quite common property; in particular, all Fraïssé classes with free (or, more generally, canonical) amalgamation enjoy it; so does T expanded by constants for a model, or, when T is stable, for an algebraically closed set in T^{eq}, and often also when T is NIP. Although not explicitly named or identified, this property has also been useful in various situations, such as for elimination of imaginaries.

Keisler measures play a big role in this paper (especially in the notion of first order amenability) and we generally assume that the reader is familiar with them. A Keisler measure on a sort (or definable set) X over a model M is simply a finitely additive (probability) measure on the Boolean algebra of definable (over M) subsets of X. As such it is a natural generalization of a complete type over M containing the formula defining X. As pointed out at the beginning of Section 4 of [17], a Keisler measure on X over M is the "same thing" as a regular Borel probability measure on the space $S_X(M)$ of complete types over M containing the formula defining X. Keisler measures are completely natural in model theory, but it took some time for them to be studied systematically. They were introduced in Keisler's seminal paper [22] mainly in a stable environment, and later played an important role in [16] in the solution of some conjectures relating o-minimal groups to compact Lie groups.

1. Some notions and definitions

We recall here model-theoretic definitions of certain components of groups in some categories, and also the relevant variants of the notion of amenability; for more details, see Section 2 of [27]. The new notions which we introduce in this paper will appear in the relevant sections.

As usual, by a monster model of a given theory we mean a κ-saturated and strongly κ-homogeneous model for a sufficiently large cardinal κ (typically, $\kappa > |T|$ is a strong limit cardinal). Where recall that the (standard) expression "strongly κ-homogeneous" means that any partial elementary map between subsets of the model of cardinality $< \kappa$ extends to an automorphism of the model. A set [tuple] is said to be small [short] if it is of bounded cardinality (i.e. $< \kappa$). When G is a $\emptyset$-definable group (in the monster model) and A a (small) set of parameters, then G^{00}_A denotes the smallest, A-definable subgroup of G of bounded index; and G^{000}_A — the smallest A-invariant subgroup of G of bounded index.

Let $G(M)$ be a topological group $\emptyset$-definable in a structure M. Assume for a moment that all open subsets of $G(M)$ are also $\emptyset$-definable. By G we denote the interpretation of $G(M)$ in a monster model M^*. Define μ to be the intersection of all $U = U(M^*)$ with $U(M)$ ranging over all open neighborhoods of the identity. So μ is the subgroup of infinitesimals of G; it is not necessarily normal, but it is normalized by $G(M)$.

Definition 1.1. 1) $G^{00}_{\mathrm{top}} := \mu G^{00}_M$; equivalently, this is the smallest M-type-definable subgroup of G of bounded index which contains μ.
2) $G^{000}_{\mathrm{top}} := \langle \mu^G \rangle G^{000}_M$; equivalently, this is the smallest normal, invariant over M subgroup of G of bounded index which contains μ.

It turns out that G^{00}_{top} is a normal subgroup of G and the map $G(M) \to G/G^{00}_{\mathrm{top}}$ is the classical Bohr compactification of $G(M)$ as a topological group (i.e. the universal group compactification). For a description of G/G^{000}_{top} as the initial object in a certain category see [27, Proposition 2.18]. In particular, one gets that both quotients G/G^{00}_{top} and G/G^{000}_{top} are independent as topological groups (equipped with the logic topology) of the choice of the language (provided that all open subsets of $G(M)$ are $\emptyset$-definable) and of the choice of the monster model in which they are computed. Moreover, the closure of the identity in G/G^{000}_{top} is exactly $G^{00}_{\mathrm{top}}/G^{000}_{\mathrm{top}}$, so the property $G^{00}_{\mathrm{top}} = G^{000}_{\mathrm{top}}$ is also independent from the choice of the language and the monster model.

There is also a model-theoretic description of the universal (left) $G(M)$-ambit as the quotient $S^{\mu}_G(M) := S_G(M)/{\sim_\mu}$, where $p \sim_\mu q \iff \mu \cdot p = \mu \cdot q$ with the distinguished point $\mathrm{tp}(1/M)/{\sim_\mu}$ and the action of $G(M)$ given by $g * (\mu \cdot p) := \mu \cdot (g \cdot p) = g \cdot (\mu \cdot p)$. It is clear that this ambit is isomorphic to $S_{\mu\backslash G}(M)$ – the space of complete types over M of hyperimaginary elements from $\mu\backslash G$.

Recall the classical definition of amenability.

Definition 1.2. The topological group $G(M)$ is *amenable* if for every $G(M)$-flow (equivalently $G(M)$-ambit) X there is a $G(M)$-invariant, Borel probability measure on X; equivalently, there is a $G(M)$-invariant, Borel probability measure on the universal ambit $S^{\mu}_G(M)$.

The following is [27, Conjecture 0.2].

Conjecture 1.3. *Let $G(M)$ be a topological group. If $G(M)$ is amenable, then $G^{00}_{\mathrm{top}} = G^{000}_{\mathrm{top}}$.*

In the above discussion, we are looking at the classical "topological" categories of topological flows and compactifications via model-theory. In Section 2.2 of [27], we proposed to look at more general "definable-topological" categories. It is a bit subtle, so we try to be precise about the notions and definitions (although a full account is given in [27]). So we start with an $\mathcal{L}$-structure M, and a group $G(M)$ $\emptyset$-definable in M. We assume that $G(M)$ is also a topological group, although this is not necessarily "seen" by the structure M. Let M' be an expansion of M in a language $\mathcal{L}'$ containing $\mathcal{L}$ such that we have predicates for all open subsets of the topological group $G(M)$. Let $(M')^* \succ M'$ be a monster model of $\mathrm{Th}(M')$ whose reduct $M^* \succ M$ to $\mathcal{L}$ is also a monster model. So the dynamics of $G(M)$ as a *topological group* is seen through the model theory of M' and $(M')^*$ as discussed earlier in this section. But we are more interested in what is definable in M. So as to avoid too much unnecessary notation, we will rather talk about M, M^* and distinguish between definability in $\mathcal{L}$ (which we just call definable) and definability in the richer language $\mathcal{L}'$. G^{00}_M and G^{000}_M are computed in $\mathcal{L}$, and $S_G(M)$ denotes the space of complete types in the sense of $\mathcal{L}$.

Definition 1.4. 1) $G^{00}_{\mathrm{def,top}} := \mu \cdot G^{00}_M = G^{00}_{\mathrm{top}} \cdot G^{00}_M$; equivalently, this is the smallest M-type-definable (in the sense of $\mathcal{L}$) subgroup of G of bounded index which contains μ.

2) $G^{000}_{\mathrm{def,top}} := \langle \mu^G \rangle \cdot G^{000}_M = G^{000}_{\mathrm{top}} \cdot G^{00}_M$; equivalently, this is the smallest normal, invariant over M (in the sense of $\mathcal{L}$) subgroup of G of bounded index which contains μ.

Note that we need the $\mathcal{L}'$-structure to make sense of μ, and G^{00}_{top}, etc., although $G^{00}_{\mathrm{def,top}}$ is nevertheless still type-definable over M in $\mathcal{L}$.

It turns out that $G^{00}_{\mathrm{def,top}}$ is a normal subgroup of G and the map $G(M) \to G/G^{00}_{\mathrm{def,top}}$ is the (unique up to isomorphism) universal compactification of $G(M)$ among definable (in the sense of $\mathcal{L}$), continuous group compactifications of $G(M)$.

Note that the definitions $G^{00}_{\mathrm{def,top}} := \mu \cdot G^{00}_M$ and $G^{000}_{\mathrm{def,top}} := \langle \mu^G \rangle \cdot G^{000}_M$ make sense even in the wider context when $\mathcal{L}'$ is any extension of $\mathcal{L}$ such that all members of some basis of neighborhoods of the identity in $G(M)$ are definable in $\mathcal{L}'$ (with parameters from M); the difference is that now more monster models are allowed, because we do not require $\mathcal{L}'$ to contain predicates for all open subsets of G. By a standard argument, we get that the quotients $G/G^{00}_{\mathrm{def,top}}$ and $G/G^{000}_{\mathrm{def,top}}$ do not depend on the choice of both the language $\mathcal{L}'$ and the monster model in which they are computed. The property $G^{00}_{\mathrm{def,top}} = G^{000}_{\mathrm{def,top}}$ is also independent of the choice of $\mathcal{L}'$ and the monster model, which follows directly from definitions.

Remark 1.5. i) If $G(M)$ is discrete, then $G^{000}_{\mathrm{def,top}} = G^{000}_M \geq G^{000}_{\mathrm{top}}$ and $G^{00}_{\mathrm{def,top}} = G^{00}_M \geq G^{00}_{\mathrm{top}}$.
ii) If all open subsets of $G(M)$ are definable in M (in the language $\mathcal{L}$), then $G^{000}_{\mathrm{def,top}} = G^{000}_{\mathrm{top}} \geq G^{000}_M$ and $G^{00}_{\mathrm{def,top}} = G^{00}_{\mathrm{top}} \geq G^{00}_M$.

Recall that a group $G(M)$ definable in M is *definably amenable* if and only if there is a left-invariant, Borel probability measure on $S_G(M)$. In order to give a suitable generalization of this notion in the "definable-topological category", one needs to assume that all members of some basis of (not necessarily open) neighborhoods of the identity in $G(M)$ are definable in M (in the original language $\mathcal{L}$). In [27], we assumed more, namely, that there is such a basis consisting of open neighborhoods of the identity, but in the more general context everything works in the same way. In particular, $S^{\mu}_G(M)$ defined as above is still a $G(M)$-ambit. The following definition was proposed in [27, Section 3].

Definition 1.6. Assume that all members of some basis of neighborhoods of the identity in the topological group $G(M)$ are definable in M (in $\mathcal{L}$). We say that $G(M)$ is *definably topologically amenable* if there exists a left-invariant, Borel probability measure on the $G(M)$-ambit $S^{\mu}_G(M)$.

Conjecture 0.1 recalled in the introduction is the main conjecture of [27]. As was recalled in the introduction, one of the main results of [27] was [27, Theorem 0.5] saying that Conjecture 0.1 is true if $G(M)$ has a basis of open neighborhoods of the identity consisting of definable, open subgroups. This implies Conjecture 1.3 for groups possessing a basis of open neighborhoods of the identity consisting open subgroups. In Subsection 2.6 of this paper (see Corollary 2.36), we prove Conjecture 0.1 (and so also Conjecture 1.3) in its full generality.

The definition of amenability of a topological group is by saying that there is a well-behaved measure on the universal topological ambit. The definitions of definable amenability or definable topological amenability are by saying that there is a well-behaved measure on the $G(M)$-ambits $S_G(M)$ or $S_G^{\mu}(M)$, respectively. But these ambits are not universal in any of the categories of ambits considered in [27]. So based on [23], we proposed in [27] more general notions of amenability, which we recall now.

As was pointed out in [23], there is a unique closed equivalence relation E on $S_G(M)$ such that $S_G(M)/E$ is the universal definable $G(M)$-ambit; a description of E can be found in Section 3 of [23]. In [27, Subsection 2.2], we described a closed equivalence relation E_1 on $S_G(M)$ such that $S_G(M)/E_1$ is the universal definable topological $G(M)$-ambit (where $G(M)$ is a topological group definable in M).

Definition 1.7. 1) We say that $G(M)$ is *weakly definably amenable* if there exists a left-invariant, Borel probability measure on the universal definable G-ambit, i.e. on $S_G(M)/E$.
2) We say that G is *weakly definably topologically amenable* if there exists a left-invariant, Borel probability measure on the universal definable topological G-ambit, i.e. on $S_G(M)/E_1$.

Conjecture 0.2 from the introduction is the most general conjecture of [27]. In Section 3, we will show that it is false, even in the case when $G(M)$ is discrete (i.e. working in the definable category). In Subsection 3.4 of [27], a weaker form of this conjecture was proposed. Namely, let $G^{000+}_{\mathrm{def,top}}$ be the normal subgroup generated by all products ab^{-1} for $(a,b) \in E_1'$, where $aE_1'b \iff \mathrm{tp}(a/M)E_1\,\mathrm{tp}(b/M)$. It is M-invariant, and by Proposition 3.10 of [23], we easily get that $G^{000}_{\mathrm{def,top}} \leq G^{000+}_{\mathrm{def,top}} \leq G^{00}_{\mathrm{def,top}}$.

Conjecture 1.8. *Let $G(M)$ be a topological group definable in an arbitrary structure M. If G is weakly definably topologically amenable, then $G^{00}_{\mathrm{def,top}} = G^{000+}_{\mathrm{def,top}}$.*

At first glance it seems that this conjecture should be reachable by the methods of Section 2, but we do not quite see how to prove it.

2. Means and connected components

The main goal of this section is to prove the equality of various connected components under the existence of a suitable measure or mean. In particular, we will prove Conjecture 0.1. As mentioned in the introduction, this conjecture was proved in [27] but under the stronger assumption that there is a basis of open neighborhoods of the identity consisting of definable open subgroups. Similarly to [27], our proofs are based on the idea of the proof of Massicot-Wagner version of the stabilizer lemma. Our key tricks to deal with the general case will be using means instead of measures (so something like measures but defined only on certain lattices of subsets), positively $\bigvee$-definable sets, and a notion of largeness. As to the Massicot-Wagner result, we will prove a variant of it (see Proposition 2.10 and Corollary 2.11) which is applicable to various situations. The main results of this section are contained in Subsection 2.6. In Subsection 2.7, we study groups equipped with $\bigvee$-definable group topologies, also proving that existence of a mean on the appropriate lattice of subsets implies equality of the closures of the appropriate connected components.

2.1. **$\bigvee$-definable sets.** Let T be any (complete) theory, $M \models T$, and $\mathfrak{C}$ be a monster model of T. By a [type-]definable set we usually mean a set which is [type-]definable with parameters in $\mathfrak{C}$. We can identify it with the corresponding formula [or sets of formulas]. We will be often talking about sets which are A-type-definable, so using parameters from a set A. One can often incorporate parameters into the language and work over $\emptyset$, e.g. in this and in the next subsection we work with $\emptyset$-definability, but sometimes parameters are essential (e.g. in Proposition 2.10 and the applications to Theorem 2.34 and Proposition 2.51).

By the category of *$\bigvee$-positively definable sets*, we mean the category whose objects are expressions of the form $\bigvee_{i\in\omega} D_i$, where $D_0 \subseteq D_1 \subseteq \dots$ are positively definable sets, where two such expressions are considered to be equal if they agree in any model of T (equivalently, in the monster model; so working in the monster model, any object can be identified with the corresponding subset of the model). A morphism $F\colon \bigvee_{i\in\omega} D_i \to \bigvee_{i\in\omega} E_i$ is a collection of definable functions $F_i\colon D_i \to E_{j_i}$, where i ranges over ω and j_i is some index in ω, such that $\bigcup F_i$ is a well-defined function, and two such collections of functions are identified if they yield the same function from $\bigvee_{i\in\omega} D_i(M)$ to $\bigvee_{i\in\omega} E_i(M)$ for every model M, equivalently for the monster model. We write $\bigvee_{i\in\omega} D_i \subseteq \bigvee_{i\in\omega} E_i$ if this holds in every model (equivalently, in $\mathfrak{C}$); this is equivalent to saying that for every i there is j_i such that $D_i \subseteq E_{j_i}$. Whenever $\bigvee_{i\in\omega} D_i(\bar{x})$ is $\bigvee$-positively definable and $\bar{a}$ is a tuple of parameters, we say that $\bigvee_{i\in\omega} D_i(\bar{a})$ *holds* if there is i such that $M \models D_i(\bar{a})$ for some [any] model M containing $\bar{a}$.

In fact, we can consider any $\bigvee_{i\in I} D_i$ for a countable set I and positively definable sets D_i, as then one can replace I by ω and the D_i's by the unions of initial sets D_i, $i < n$. We will be doing this freely without mentioning. Also, one could extend the context to uncountable sets I, but countable families are sufficient for the purpose of our main theorems.

Recall that a subset D of a group G is said to be *(left) generic* if finitely many left translates of it cover G; D is said to be *thick* if there is n such that for every $g_1, \dots, g_n \in G$ there is $i < j$ such that $g_j^{-1} g_i \in D$. It is clear that each thick subset of G is generic. As to the converse, if $D \subseteq G$ is generic, then $D^{-1}D$ is thick.

Let G be a group definable in T. For a positively definable set $D(x,\bar{y}) \vdash G(x)$, by $(\exists^{\mathrm{gen}} x)D(x,\bar{y})$ we mean the $\bigvee$-positively definable set $\bigvee_{l\in\omega}(\exists^{l\text{-gen}} x)D(x,\bar{y})$, where

$$(\exists^{l\text{-gen}} x)D(x,\bar{y}) \ := \ (\exists x_1,\dots,x_l)(\forall z)\bigvee_{i=1}^{l} D(x_i z,\bar{y}).$$

(Formally, the quantifiers in the last formula are restricted to G; if $(G,\cdot)$ is a sort, then this formula is clearly positive, so $(\exists^{\mathrm{gen}} x)D(x,\bar{y})$ is $\bigvee$-positively definable. Abusing terminology by allowing in positive formulas both the group operation on G and quantification over G, we can say that $(\exists^{\mathrm{gen}} x)D(x,\bar{y})$ is $\bigvee$-positively definable also for any definable G.)

In particular, for any parameters $\bar{b}$, $(\exists^{\mathrm{gen}} x)D(x,\bar{b})$ holds iff $D(M,\bar{b}) := \{a \in G(M) : M \models D(a,\bar{b})\}$ is generic in $G(M)$ for some [any] model M containing $\bar{b}$. For a $\bigvee$-positively definable set $D(x,\bar{y}) = \bigvee_{i\in\omega} D_i(x,\bar{y})$ such that $D(x,\bar{y}) \vdash G(x)$ by $(\exists^{\mathrm{gen}} x)D(x,\bar{y})$ we mean the $\bigvee$-positively definable set $\bigvee_{i\in\omega}(\exists^{\mathrm{gen}} x)D_i(x,\bar{y})$. In particular, for any parameters $\bar{b}$, $(\exists^{\mathrm{gen}} x)\bigvee_{i\in\omega} D_i(x,\bar{b})$ holds iff for some i the set $D_i(M,\bar{b}) := \{a \in G(M) : M \models D_i(a,\bar{b})\}$ is generic in $G(M)$ for some [any] model M containing $\bar{b}$. Working in the monster model, this is equivalent to saying that $D(\mathfrak{C},\bar{b}) := \{a \in G(\mathfrak{C}) : \mathfrak{C} \models D(a,\bar{b})\} = \bigcup_{i\in\omega} D_i(\mathfrak{C},\bar{b})$ is generic in $G(\mathfrak{C})$ (but this is not true in a non $\aleph_0$-saturated model).

Analogous definitions apply when we replace "generic" by "thick". The only difference is, of course, that the displayed formula above is now the following

$$(\exists^{l\text{-thick}} x)D(x,\bar{y}) \ := \ (\forall x_1,\dots,x_l)\bigvee_{i<j} D(x_j^{-1}x_i,\bar{y}).$$

2.2. **A largeness notion.** Throughout, G is a group acting on X. We work in the language of group actions, $(G,\cdot,X,\cdot,\dots)$. ($\cdot$ refers both to the group operation and the action, and $\dots$ to possible additional structure.) In the particular case when G acts on itself via left translations, the results which we will obtain for $(G,\cdot,X,\cdot,\dots)$ transfer automatically to the corresponding statements in the language of groups $(G,\cdot,\dots)$ (i.e. without the extra sort for X), just identifying X with G.

We define a largeness notion $\mathcal{L}_k$ for subsets of X, resembling "rank $\geq k$" for certain model-theoretic ranks. In fact, we define two largeness notions $\mathcal{L}_k^{\mathrm{gen}}$ and $\mathcal{L}_k^{\mathrm{thick}}$. The stronger notion $\mathcal{L}_k^{\mathrm{thick}}$ corresponds to non-forking in stable theories (see Remark 2.4). For our purposes, both notions work in the same way, so later we will just write $\mathcal{L}_k$. It would be interesting to further investigate $\mathcal{L}_k^{\mathrm{gen}}$ and $\mathcal{L}_k^{\mathrm{thick}}$ (and variants) for unstable theories.

In what follows, we deal with $\mathcal{L}_k^{\mathrm{gen}}$, but everything works also for the analogously defined $\mathcal{L}_k^{\mathrm{thick}}$.

Definition 2.1. Let $Y(x,\bar{y}) \subseteq X(x)$ be a $\bigvee$-positively definable set $\bigvee_i Y_i(x,\bar{y})$.

(1) $\mathcal{L}_0^{\mathrm{gen}}(Y(x,\bar{y}))$ is the $\bigvee$-positively definable set $\bigvee_i(\exists x)Y_i(x,\bar{y})$.

(2) For $k>0$, $\mathcal{L}_k^{\mathrm{gen}}(Y(x,\bar{y}))$ is the $\bigvee$-positively definable set in variables $\bar{y}$
$$(\exists^{\mathrm{gen}} z)\mathcal{L}_{k-1}^{\mathrm{gen}}(Y(x,\bar{y})\cap Y(z^{-1}x,\bar{y})).$$

In particular, using terminology from Subsection 2.1, for a $\bigvee$-positively definable set $Y=Y(x)\subseteq X(x)$ we have a well-defined meaning of "$\mathcal{L}_k^{\mathrm{gen}}(Y)$ holds". Namely, $\mathcal{L}_0^{\mathrm{gen}}(Y)$ holds iff $Y\neq\emptyset$, and $\mathcal{L}_k^{\mathrm{gen}}(Y)$ holds iff $\{g\in G:\mathcal{L}_{k-1}^{\mathrm{gen}}(Y\cap gY)\ \text{holds}\}$ is generic. The word "hold" will be often skipped from now on.

Remark 2.2. $\mathcal{L}_k^{\mathrm{gen}}(Y(x,\bar{y}))$ can be expressed by a disjunction of positive, translation invariant formulas $\psi_j(\bar{y})$ of the language $(G,\cdot,X,\cdot,Y_i)_i$, where $Y=\bigvee Y_i$. (Here, by a translation invariant formula we mean a formula $\psi_{(Y_i:i<\omega)}(\bar{y})$ depending on the Y_i's (and with variables $\bar{y}$ appearing only in the $Y_i(x,\bar{y})$'s) such that $\psi_{(Y_i:i<\omega)}(\bar{y})$ is equivalent to $\psi_{(gY_i:i<\omega)}(\bar{y})$ for any $g\in G$.)

Proof. The proof is by induction on k. Clearly $\mathcal{L}_0^{\mathrm{gen}}(Y(x,\bar{y}))$ can by expressed as $\bigvee_i(\exists x)Y_i(x,\bar{y})$ which does the job. Now, suppose that $\mathcal{L}_k^{\mathrm{gen}}(Y(x,\bar{y}))$ can be expressed as $\bigvee_{j\in\omega}\psi_{j,(Y_i:i<\omega)}(\bar{y})$, where each $\psi_{j,(Y_i:i<\omega)}(\bar{y})$ is positive and translation invariant. Then $\mathcal{L}_{k+1}^{\mathrm{gen}}(Y(x,\bar{y}))$ can be expressed as $\bigvee_{j,l\in\omega}(\exists^{l\text{-}\mathrm{gen}}z)\psi_{j,(Y_i\cap zY_i:i<\omega)}(\bar{y},z)$. By induction hypothesis, it is clear that each $(\exists^{l\text{-}\mathrm{gen}}z)\psi_{j,(Y_i\cap zY_i:i<\omega)}(\bar{y},z)$ is positive and translation invariant. $\square$

It is also easy to express the $\mathcal{L}_k^{\mathrm{gen}}$ directly, e.g. $\mathcal{L}_2^{\mathrm{gen}}(Y)\equiv\bigvee_{l,l'}\mathcal{L}_2^{\mathrm{gen},l,l'}(Y)$, where
$$\mathcal{L}_2^{\mathrm{gen},l,l'}(Y)\equiv(\exists^{l\text{-}\mathrm{gen}}z)(\exists^{l'\text{-}\mathrm{gen}}z')\mathcal{L}_0^{\mathrm{gen}}(Y\cap zY\cap z'Y\cap zz'Y).$$

Let $Y=Y(x)\subseteq X(x)$ and $Y=\bigvee Y_i$. Since translation by h on X and conjugation by h on G gives an isomorphism $(G,\cdot,X,\cdot,Y_i)_i\to(G,\cdot,X,\cdot,,hY_i)_i$, it follows directly that $\mathcal{L}_k^{\mathrm{gen}}$ is translation invariant, i.e. $\mathcal{L}_k^{\mathrm{gen}}(Y)\iff\mathcal{L}_k^{\mathrm{gen}}(hY)$.

Define
$$\mathrm{St}_{\mathcal{L}_k^{\mathrm{gen}}}(Y):=\{g:\mathcal{L}_k^{\mathrm{gen}}(gY\cap Y)\}.$$
This is an operator from the class of $\bigvee$-positively definable sets to itself. Note that $\mathcal{L}_{k+1}^{\mathrm{gen}}(Y)$ holds iff $\mathrm{St}_{\mathcal{L}_k^{\mathrm{gen}}}(Y)$ is generic as a $\bigvee$-positive definable set (which remember means that writing $\mathrm{St}_{\mathcal{L}_k^{\mathrm{gen}}}(Y)$ as a suitable countable increasing union of positively definable sets U_n say, one of the U_n's is generic). By Remark 2.2, we get

Remark 2.3. $S:=\mathrm{St}_{\mathcal{L}_k^{\mathrm{gen}}}(Y)$ satisfies $S=S^{-1}$. If additionally $\mathcal{L}_k^{\mathrm{gen}}(Y)$, then $1\in S$, so S is symmetric. Even more: S can be expressed by a disjunction of positive formulas which are closed under inversion; if additionally $\mathcal{L}_k^{\mathrm{gen}}(Y)$, then these formulas can be chosen to contain 1, so they are symmetric.

As already mentioned, the above definitions and facts have obvious counterparts with "generic" replaced by "thick". In the rest of the paper, we can work with any of these two versions, so we will be writing $\mathcal{L}$ in place of $\mathcal{L}^{\mathrm{gen}}$ or $\mathcal{L}^{\mathrm{thick}}$. An exception is the next remark which holds for $\mathcal{L}^{\mathrm{thick}}$.

Remark 2.4. When $G = \mathrm{Aut}(\mathfrak{C})$ is the automorphism group of a monster model of a stable theory T, and Y is definable (over $\mathfrak{C}$), then $\mathcal{L}_k^{\mathrm{thick}}(Y)$ holds for all $k \in \omega$ if and only if Y does not fork over $\emptyset$. (Here, $\mathcal{L}_k^{\mathrm{thick}}(Y)$ is computed in $(G, \cdot, \mathfrak{C}, \cdot)$ with $\mathfrak{C}$ equipped with its original stable structure.)

Proof. Let $T = \mathrm{Th}(\mathfrak{C})$. The structure in which we will be working is $(G, \cdot, \mathfrak{C}, \cdot)$, with $\mathfrak{C}$ equipped with its original stable structure.
$(\leftarrow)$. It is enough to show this implication working in a monster model $(G^*, \cdot, \mathfrak{C}^*, \cdot) \succ (G, \cdot, \mathfrak{C}, \cdot)$ (as neither the definition of $\mathcal{L}_k^{\mathrm{thick}}(Y)$ nor non-forking changes under passing to an elementary extension). We argue by induction on k.

If Y does not fork over $\emptyset$, then $Y \neq \emptyset$, so $\mathcal{L}_0^{\mathrm{thick}}(Y)$. For the induction step, consider any Y which does not fork over $\emptyset$. By inductive hypothesis, it is enough to show that

$$S := \{g \in G^* : gY \cap Y \text{ does not fork over } \emptyset\}$$

is thick. Take $p^* \in S(\mathfrak{C}^*)$ which does not fork over $\emptyset$ and contains Y. By stability, we know that the orbit $G^* \cdot p^*$ is bounded (of cardinality at most $2^{|T|}$), so $\mathrm{Stab}_{G^*}(p^*)$ is a bounded index subgroup of G^*. Write explicitly $Y(x) = \varphi(x, \bar{a})$. Then $\mathrm{Stab}_{G^*}(p^*)$ is contained in

$$S' := \{g \in G^* : gY \in p^*\} = \{g \in G^* : \varphi(x, g\bar{a}) \in p^*\} = \{g \in G^* : \mathfrak{C}^* \models d_{p^*}\varphi(g\bar{a})\}.$$

By stability, S' is a definable subset of G^* (in the sense of the structure $(G^*, \cdot, \mathfrak{C}^*, \cdot)$). All of this implies that S' is thick, as otherwise, by the sufficient saturation of $(G^*, \cdot, \mathfrak{C}^*, \cdot)$, we would get a sequence $(g_i)_{i<(2^{|T|})^+}$ of elements of G^* such that $g_j^{-1}g_i \notin S'$ for all $i < j < (2^{|T|})^+$, which contradicts the fact that $[G^* : \mathrm{Stab}_{G^*}(p^*)] < (2^{|T|})^+$. On the other hand, S' is clearly contained in S.
$(\rightarrow)$. Suppose Y forks over $\emptyset$. Then, by stability, Y k-divides over $\emptyset$ for some k. Then one can easily check that $\mathcal{L}_{k-1}^{\mathrm{thick}}(Y)$ does not hold. We will check it for $k = 2$ and $k = 3$, leaving the general case for the reader.

Suppose Y 2-divides over $\emptyset$. Then, by the strong $\aleph_0$-homogeneity of $\mathfrak{C}$, there are $g_0, g_1, \dots \in G$ such that for all $i < j$, $g_iY \cap g_jY = \emptyset$. If $\mathcal{L}_1^{\mathrm{thick}}(Y)$ holds, then $\{g : gY \cap Y \neq \emptyset\}$ is thick, so there are $i < j$ such that $g_j^{-1}g_iY \cap Y \neq \emptyset$, a contradiction. (Note that this argument does not work for "generic" in place of "thick".)

Suppose Y 3-divides over $\emptyset$. Then there are $g_0, g_1, \dots \in G$ such that for all $i < j < k$, $g_iY \cap g_jY \cap g_kY = \emptyset$, and for all i and j, $g_ig_j = g_{i+j}$. Suppose for a contradiction that $\mathcal{L}_2^{\mathrm{thick}}(Y)$ holds. Then there are $i < j$ such that $\mathcal{L}_1^{\mathrm{thick}}(g_i^{-1}g_jY \cap Y)$ holds. Hence, we can find $k < l$ such that $(g_i^{-1}g_jY \cap Y) \cap (g_k^{-1}g_lg_i^{-1}g_jY \cap g_k^{-1}g_lY) \neq \emptyset$. In particular, $g_jY \cap g_iY \cap g_{l-k+j}Y \neq \emptyset$, a contradiction as $i < j < l - k + j$. □

Let us finish with the following easy remark.

Remark 2.5. (1) Let $Y(x,\bar{y}) \subseteq Y'(x,\bar{y}) \subseteq X(x)$ be $\bigvee$-positively definable sets. Then $\mathcal{L}_k(Y(x,\bar{y})) \subseteq \mathcal{L}_k(Y'(x,\bar{y}))$ (as $\bigvee$-definable sets in variables $\bar{y}$). In particular, if the tuple $\bar{y}$ is empty, then "$\mathcal{L}_k(Y)$ holds" implies "$\mathcal{L}_k(Y')$ holds".
(2) Let $Y(x,\bar{y}) \subseteq X(x)$ be a $\bigvee$-positively definable set. Then for every $k \in \omega$, $\mathcal{L}_k^{\text{thick}}(Y(x,\bar{y})) \subseteq \mathcal{L}_k^{\text{gen}}(Y(x,\bar{y}))$. In particular, if the tuple $\bar{y}$ is empty, then "$\mathcal{L}_k^{\text{thick}}(Y)$ holds" implies "$\mathcal{L}_k^{\text{gen}}(Y)$ holds".

2.3. **Means and stabilizers.** Let X be a G-set. By a *G-lattice* we mean a family of subsets of X including $\emptyset$ and X, which is closed under G-translations, and intersections and unions of pairs.

Definition 2.6. Let G be a group acting on X, $\mathcal{D}$ a G-lattice of subsets of X. A *mean* is a monotone, (non-negative), translation-invariant function $m\colon \mathcal{D} \to \mathbb{R}$ satisfying $m(\emptyset) = 0$, and for $Y, Z \in \mathcal{D}$

$$m(Y \cup Z) = m(Y) + m(Z) - m(Y \cap Z).$$

The mean m is *normalized*, if $m(X) = 1$.

Given a mean m and $\epsilon \in \mathbb{R}$, the *ϵ-stabilizer* of a set $Y \subseteq X$ is defined to be

$$\mathrm{St}_\epsilon(Y) := \{g \in G : m(gY \cap Y) > (1-\epsilon)m(Y)\}.$$

Lemma 2.7. *Let X be a G-set and $\mathcal{D}$ a G-lattice. Let m be a mean on $\mathcal{D}$ (so $m(X) < \infty$), and let $W \in \mathcal{D}$ satisfy $m(W) > 0$. Then:*

(1) $\mathrm{St}_1(W) = \{g \in G : m(gW \cap W) > 0\}$ *is thick (so generic).*
(2) We have $\mathcal{L}_k(W)$ for all k (working in $(G,\cdot,X,\cdot,W,\dots)$).

Proof. (1) For some $n \in \mathbb{N}$ we have $n \cdot m(W) > m(X)$. Suppose $\mathrm{St}_1(W)$ is not n-thick. Then one can find $g_i \in G$, $i = 1,\dots,n$, satisfying $g_j^{-1}g_i \notin \mathrm{St}_1(W)$ for all $i < j$. Therefore, $m(g_iW \cap g_jW) = m(g_j^{-1}g_iW \cap W) = 0$ for all $i < j$. Hence, $n \cdot m(W) \leq m(X)$, a contradiction.
(2) Let us work with $\mathcal{L} = \mathcal{L}^{\text{thick}}$ which clearly implies the version with $\mathcal{L} = \mathcal{L}^{\text{gen}}$. Without loss, we can work in a monster model $(G^*,\cdot,X^*,\cdot,W^*,\dots) \succ (G,\cdot,X,\cdot,W,\dots)$. To see this, apply a standard construction with incorporating m to the language (as the collection of functions $m_{\varphi(x,\bar{y})}$, where $m_{\varphi(x,\bar{y})}(\bar{b}) := m(\varphi(x,\bar{b}))$ when it is defined, and say symbol ∞ otherwise), extending to the monster model, and taking the standard part; this yields a mean (which we still denote by m) on a certain G^*-lattice of subsets of X^*, including W^*, and such that $m(X^*) = m(X) < \infty$ and $m(W^*) = m(W) > 0$. So without loss $(G,\cdot,X,\cdot,W,\dots)$ is a monster model.

We argue by induction on k. For $k = 0$, $m(W) > 0$ ensures $\mathcal{L}_0(W)$. For higher k, we know by induction that $\mathcal{L}_{k-1}(gW \cap W)$ holds whenever $m(gW \cap W) > 0$. Thus, $\{g \in G : \mathcal{L}_{k-1}(gW \cap W)\}$ is thick by (1), so $\mathcal{L}_k(W)$ holds by the sufficient saturation of the model and the definition of $\mathcal{L}_k$. (Note that $\mathcal{L}_{k-1}(gW \cap W)$ is a $\bigvee$-positively definable set $\bigvee_i D_i(g)$, so saturation is needed to deduce that $\{g \in G : D_i(g)\}$ is thick for some i.) $\square$

Remark 2.8. In fact, the ideal $\mathcal{I}_m = \{Y : m(Y) = 0\}$ is an S1-ideal, i.e. $\mathcal{I}_m$ is a G-invariant ideal on the lattice $\mathcal{D}$ such that whenever $W \in \mathcal{D}$ and there are arbitrary long finite sequences (g_i) of elements of G such that $g_iW \cap g_jW \in I$, then $W \in I$. The stabilizer St_1 can be defined for any S1-ideal $\mathcal{I}$ as $\{g : gW \cap W \notin \mathcal{I}\}$, and Lemma 2.7 continues to hold for $W \notin \mathcal{I}$. The assumption on m' in Proposition 2.10 can be replaced by: $\mathcal{D}'$ carries an S1-ideal.

Lemma 2.9. *Let X be a G-set and $\mathcal{D}$ a G-lattice. Let m be a mean on $\mathcal{D}$. Then, for any $Z \in \mathcal{D}$ and $\epsilon_1, \epsilon_2 \in \mathbb{R}$, $\mathrm{St}_{\epsilon_1}(Z)\,\mathrm{St}_{\epsilon_2}(Z) \subseteq \mathrm{St}_{\epsilon_1+\epsilon_2}(Z)$.*

Proof. The natural argument uses symmetric differences of sets, but here our lattice is not closed under set-theoretic difference, so we will mimic means of symmetric differences. (In fact, using Proposition 2.20, we could work with the Boolean algebra generated by $\mathcal{D}$ and use symmetric differences, but we do not do it here to keep this argument self-contained and completely elementary.)

Note that, by the invariance of m, for any ϵ we have

$$(\dagger) \quad g \in \mathrm{St}_\epsilon(Z) \iff m(gZ) + m(Z) - 2m(gZ \cap Z) < 2\epsilon m(Z).$$

Consider any $g_i \in \mathrm{St}_{\epsilon_i}(Z)$ for $i = 1, 2$. Then, $m(g_iZ) + m(Z) - 2m(g_iZ \cap Z) < 2\epsilon_i m(Z)$ for $i = 1, 2$. Hence, by invariance, we easily get

$$m(g_1g_2Z) + 2m(g_1Z) + m(Z) - 2m(g_1g_2Z \cap g_1Z) - 2m(g_1Z \cap Z) < 2(\epsilon_1 + \epsilon_2)m(Z).$$

By $(\dagger)$, it is enough to show that the left hand side of the above inequality is greater than or equal to $m(g_1g_2Z) + m(Z) - 2m(g_1g_2Z \cap Z)$. By the modularity of m, this is easily seen to be equivalent to $m(g_1Z \cup (g_1g_2Z \cap Z)) \geq m(g_1Z \cap (g_1g_2Z \cup Z))$ which is true by the monotonicity of m. □

The following proposition is our strong version of the Massicot-Wagner elaboration of the stabilizer theorem of the first author. It will be the engine for most of our main results. We will actually need it only in case $X = G$, but the more general statement clarifies some aspects of the proof. Note that when $X = G$, the conclusion $Y^N \subseteq \mathrm{St}_1(BA)$ implies that $Y^N \subseteq BAA^{-1}B^{-1}$. A suitable version also holds for approximate groups (yielding information on amenable approximate groups as in Massicot-Wagner), but we will stick with the global assumptions.

Proposition 2.10. *Let $A \subseteq X$, $B \subseteq G$, $N \in \mathbb{N}$. Let $\mathcal{D}'$ be the set of finite intersections of translates gB. Let $\mathcal{D}$ be a G-lattice including A and $B'A$ for $B' \in \mathcal{D}'$. Let m be an invariant mean on $\mathcal{D}$, $m(A) > 0$, and m' an invariant mean on the lattice generated by $\mathcal{D}'$, with $m'(B) > 0$. Then there exists a generic, symmetric set $Y \subseteq G$ that is positively definable in $(G, \cdot, B)$ over parameters from G, and such that $Y^N \subseteq \mathrm{St}_1(BA)$.*

Proof. We use the mean m' only for the largeness of B. Namely, by Lemma 2.7, we have $\mathcal{L}_k(B)$ for all $k \in \omega$. We will show:

(**) for some k and $B' \in \mathcal{D}'$ with $B' \subseteq B$ and $\mathcal{L}_{k+1}(B')$, the set $Y := \mathrm{St}_{\mathcal{L}_k}(B')$ is generic as a $\bigvee$-definable set, and $Y^N \subseteq \mathrm{St}_1(B'A)$.

This means that if we present (using Remark 2.3 and the comments preceding it) Y as $\bigvee_n Y_n$ with the Y_n's increasing, symmetric and positively definable over G, then some Y_n is generic, and, of course, $Y_n^N \subseteq Y^N$. So (**) will suffice.

Let $\epsilon = 1/N$. Let $f(k)$ be the infimum of $m(B'A)$ over all $B' \in \mathcal{L}_k \cap \mathcal{D}'$ with $B' \subseteq B$. So $0 < m(A) \leq f(k) \leq m(X)$. Thus, we cannot have $f(l) \geq \sqrt{1+\epsilon} f(l-1)$ for all $l > 0$. Fix $l > 0$ with $f(l) < \sqrt{1+\epsilon} f(l-1)$. Let $\lambda = f(l)\sqrt{1+\epsilon}$. Let $B' \in \mathcal{D}'$ satisfy

(***) $\mathcal{L}_l(B')$ and $m(B'A) < \lambda$.

We will show that any such B' satisfies (**) (with $k = l-1$.) Let $Y = \mathrm{St}_{\mathcal{L}_{l-1}}(B')$. Since $B' \in \mathcal{L}_l$, Y is generic as a $\bigvee$-definable set. For $g \in Y$ we have $\mathcal{L}_{l-1}(gB' \cap B')$, so

$$m(gB'A \cap B'A) \geq m((gB' \cap B')A) \geq f(l-1) > f(l)/\sqrt{1+\epsilon} > m(B'A)/(1+\epsilon).$$

Hence, $g \in \mathrm{St}_\epsilon(B'A)$. So $Y \subseteq \mathrm{St}_\epsilon(B'A)$. By Lemma 2.9, for any Z, $\mathrm{St}_\epsilon(Z)^N \subseteq \mathrm{St}_{N\epsilon}(Z)$. Thus, we conclude that $Y^N \subseteq \mathrm{St}_1(B'A)$, i.e. $m(gB'A \cap B'A) > 0$ for $g \in Y^N$. This proves (**). □

We will also need the following corollary of the proof of Proposition 2.10.

Corollary 2.11. *Let $A \subseteq X = G$, $\mathcal{B} \subseteq \mathcal{P}(G)$, $N \in \mathbb{N}$. Put $\mathcal{D}' = \{g_1 B \cap \cdots \cap g_n B : B \in \mathcal{B}, g_1, \dots, g_n \in G\}$. Let $\mathcal{D}$ be a G-lattice containing $\mathcal{D}'$ and including A and $B'A$ for $B' \in \mathcal{D}'$. Let m be an invariant mean on $\mathcal{D}$ with $m(A) > 0$ and $m(B) > 0$ for $B \in \mathcal{B}$. Then there exist $l \in \mathbb{N}_{>0}$, $\lambda \in \mathbb{R}$, $B \in \mathcal{B}$ and $g_1, \dots, g_n \in G$ such that for $B' := B \cap g_1 B \cap \cdots \cap g_n B$ we have*

$$\mathcal{L}_l(B') \text{ and } m(B'A) < \lambda,$$

and whenever $E \in \mathcal{B}$ and $h_1, \dots, h_m \in G$ are chosen so that for $E' := E \cap h_1 E \cap \cdots \cap h_m E$ one has $\mathcal{L}_l(E')$ and $m(E'A) < \lambda$, then $S := \mathrm{St}_{\mathcal{L}_{l-1}}(E')$ is generic (as a set $\bigvee$-definable in $(G, \cdot, E)$), symmetric, and $S^N \subseteq E'A(E'A)^{-1} \subseteq EA(EA)^{-1}$.

The above corollary will be used later for $N = 8$ and for $N = 16$.

2.4. **From pre-mean to mean.** We show how to extend a pre-mean to a mean canonically; if the pre-mean is G-invariant, the resulting mean will therefore be G-invariant, too. This will be essential in the proofs of the main results of Section 2.

Definition 2.12. A *normalized mean* on a lattice $(L, \cup, \cap)$ is a monotone function $\rho : L \to [0, 1]$, satisfying:

$$\rho(Y \cup Y') = \rho(Y) + \rho(Y') - \rho(Y \cap Y'),$$

and $\rho(\emptyset) = 0$, $\rho(L) = 1$.

Whenever we present a type-definable set Z as an intersection $\bigcap_i Z_i$, we mean that the Z_i's are definable, i ranges over a directed set $(I, <)$, and $Z_j \subseteq Z_i$ for $i < j$.

Let $E = \bigcap_{i \in I} R_i$ be a type-definable equivalence relation on a definable set X, where without loss each R_i is reflexive and symmetric.

Working in the monster model, we write Y/E for the image of $Y \subseteq X$ in X/E, and YE for the pullback of Y/E in X. For a binary relation R on X, and $Y \subseteq X$, by $R \circ Y$ we mean $\{x \in X : (\exists y \in Y) R(y, x)\}$. In particular, $YE = E \circ Y$.

The following definition and lemma can be read over any base set of parameters.

Definition 2.13. A *pre-mean* for X/E is a monotone function m from definable subsets of X into $[0, 1]$, with $m(\emptyset) = 0$, $m(X) = 1$, and $m(Y \cup Y') \leq m(Y) + m(Y')$, such that equality holds whenever $(R_i \circ Y) \cap Y' = \emptyset$ for some i.

By compactness, the condition "$(R_i \circ Y) \cap Y' = \emptyset$ for some i" is equivalent to "$(E \circ Y) \cap Y' = \emptyset$".

Lemma 2.14. *Let m be a pre-mean for X/E. Then m induces a normalized mean ν on the lattice of sets Y/E, with Y type-definable, or equivalently on the lattice of type-definable sets Y with $YE = Y$, in the following way*

$$\nu(Y) := \inf\{m(D) : D \text{ definable, } Y \subseteq D\}.$$

Proof. Let L be the lattice of all $\bigwedge$-definable sets Y with $YE = Y$. For $Y \in L$, define

$$\nu(Y) = \inf\{m(D) : D \text{ definable, } Y \subseteq D\}.$$

Clearly $\nu(\emptyset) = 0$, $\nu(X) = 1$, ν is monotone, and $\nu(Y \cup Y') \leq \nu(Y) + \nu(Y')$. If $Y, Y' \in L$ are disjoint, then $\bigwedge_i R_i(y, y') \wedge y \in Y \wedge y' \in Y'$ is inconsistent. By compactness, for some i and some definable $D \supseteq Y$ and $D' \supseteq Y'$, we have $(R_i \circ D) \cap D' = \emptyset$. As m is a pre-mean, we have $m(D \cup D') = m(D) + m(D')$, and likewise for any definable subsets of D, D'. Hence, in this case, $\nu(Y \cup Y') = \nu(Y) + \nu(Y')$.

Now, L is not complemented, but we do have:

Claim 1: Let $Y \subseteq Z$ be both in L. For any $\epsilon > 0$ there exists $Y' \subseteq Z$, $Y' \in L$, Y' disjoint from Y, and with $\nu(Y) + \nu(Y') \geq \nu(Z) - \epsilon$.

Proof. Write $Y = \bigcap_{k \in K} Y_k$ with definable Y_k such that $R_{i(k)} \circ Y_{j(k)} \subseteq Y_k$ (here $i(k) \in I$ and $j(k) \in K$ are some functions of k). Similarly write $Z = \bigcap_l Z_l$. Find k such that $\nu(Y) \geq m(Y_k) - \epsilon$. We have

$$Y_{j(k)} \cap R_{i(k)} \circ (X \setminus Y_k) = \emptyset.$$

Let

$$Y' = E \circ (Z \setminus Y_k) = E \circ \left(\bigcap_l Z_l \setminus Y_k \right) = \bigcap_{i,l} R_i \circ (Z_l \setminus Y_k).$$

Then $Y' \in L$, and $Y' \subseteq E \circ Z = Z$. Also, $Y' \subseteq E \circ (X \setminus Y_k) \subseteq R_{i(k)} \circ (X \setminus Y_k)$, so $Y \cap Y' = \emptyset$. Finally, $\nu(Y') = \inf_{i,l} m(R_i \circ (Z_l \setminus Y_k)) \geq \inf_l m(Z_l \setminus Y_k)$, so

$$\nu(Y') + m(Y_k) \geq \inf_l m(Z_l \setminus Y_k) + m(Y_k) \geq \inf_l m(Z_l) = \nu(Z)$$

As $m(Y_k) \leq \nu(Y) + \epsilon$, we obtain $\nu(Y') + \nu(Y) + \epsilon \geq \nu(Z)$ as required. □(claim)

From this, the equality $\nu(Y \cup Z) = \nu(Y) + \nu(Z) - \nu(Y \cap Z)$ can be shown as follows. Take any $\epsilon > 0$. Find $Y' \in L$ such that $Y' \subseteq Y$, Y' disjoint from $Y \cap Z$, and $\delta_1 := \nu(Y) - \nu(Y') - \nu(Y \cap Z) \leq \frac{1}{2}\epsilon$. Similarly, find $Z' \in L$ such that $Z' \subseteq Z$, Z' disjoint from $Y \cap Z$, and $\delta_2 := \nu(Z) - \nu(Z') - \nu(Y \cap Z) \leq \frac{1}{2}\epsilon$. Then $Y \cap Z, Y', Z'$ are pairwise disjoint subsets of $Y \cup Z$. Finally, find $T \in L$ such that $T \subseteq Y \cup Z$, T disjoint from $Y \cap Z$, and $\delta := \nu(Y \cup Z) - \nu(T) - \nu(Y \cap Z) \leq \epsilon$. Put $Y'' = Y' \cup (T \cap Y) \in L$ and $Z'' = Z' \cup (T \cap Z) \in L$. Then Y'', Z'' and $Y \cap Z$ are pairwise disjoint subsets of $Y \cup Z$. Put $T'' = Y'' \cup Z'' \in L$. We see that $\delta'' := \nu(Y \cup Z) - \nu(T'') - \nu(Y \cap Z) \leq \delta \leq \epsilon$ and T'' is disjoint from $Y \cap Z$. Moreover, $\delta_1'' := \nu(Y) - \nu(Y'') - \nu(Y \cap Z) \leq \delta_1 \leq \frac{1}{2}\epsilon$ and $\delta_2'' := \nu(Z) - \nu(Z'') - \nu(Y \cap Z) \leq \delta_2 \leq \frac{1}{2}\epsilon$. Note also that $\delta_1'', \delta_2'', \delta'' \geq 0$.

We get $|\nu(Y \cup Z) - \nu(Y) - \nu(Z) + \nu(Y \cap Z)| = |\delta'' + \nu(T'') - \nu(Y) - \nu(Z) + 2\nu(Y \cap Z)| = |\delta'' - (\nu(Y) - \nu(Y'') - \nu(Y \cap Z)) - (\nu(Z) - \nu(Z'') - \nu(Y \cap Z))| = |\delta'' - \delta_1'' - \delta_2''|$. Since $\delta'' \in [0, \epsilon]$ and $\delta_1'', \delta_2'' \in [0, \frac{1}{2}\epsilon]$, we see that $|\delta'' - \delta_1'' - \delta_2''| \leq \epsilon$. Letting $\epsilon \to 0$, we obtain the desired equality. □

Lemma 2.14 will be sufficient to deal with Case 1 in Subsection 2.6, i.e. to prove Theorem 2.34. In order to deal with Case 2 and prove Theorem 2.35, we will need some variant of this lemma. Namely, suppose that the type-definable equivalence relation E is on a definable group G.

Definition 2.15. A *G-pre-mean* for G/E is a pre-mean for G/E such that $m(Y \cup Y') = m(Y) + m(Y')$ whenever $((g_1 R_i \cap \cdots \cap g_n R_i) \circ Y) \cap Y' = \emptyset$ for some $g_1, \ldots, g_n \in G$ and some $i \in I$.

The following variant of Lemma 2.14 follows from Lemma 2.14.

Corollary 2.16. *Let m be a G-pre-mean for X/E. Then m induces a normalized mean ν on the lattice of type-definable sets Y with $Y(g_1 E \cap \cdots \cap g_n E) = Y$ for some $g_1, \ldots, g_n \in G$, in the following way*

$$\nu(Y) := \inf\{m(D) : D \text{ definable, } Y \subseteq D\}.$$

2.5. **Means and measures.** In this subsection, we will prove that, in a certain general context, the existence of an invariant mean is equivalent to the existence of an invariant measure on an appropriate space. This is interesting in its own right, but also yields model-theoretic absoluteness of various notions of "amenability", i.e. the existence of invariant measures on appropriate spaces computed for a given model M does not depend on the choice of M.

Let us recall some definitions from measure theory.

Definition 2.17. Let R be a ring of subsets of a given set X, namely closed under finite unions and differences (for example given by a Boolean algebra of subsets of X).
1) A *content* on R is a function $m\colon R \to [0, +\infty]$ which is finitely additive and satisfies $m(\emptyset) = 0$.
2) A *pre-measure* on R is a content which is σ-additive, namely if $(A_n)_{n<\omega}$ is a sequence of pairwise disjoint members of R whose union A is also in R, then $m(A) = \sum_n m(A_n)$.
3) A *measure* is a pre-measure on a σ-algebra of subsets of a given set.

A content m on a ring R of subsets of X is called *σ-finite* if X is the union of an increasing sequence $(X_n)_{n<\omega}$ of elements of R with $m(X_n) < \infty$.

Fact 2.18 (Carathéodory extension theorem)**.** *Let ν be a σ-finite pre-measure on a ring R of subsets of X. Then there is a unique extension of ν to a measure on the σ-algebra $\sigma(R)$ generated by R.*

From the proof, or from a more precise statement which says that the extended measure (restricted to $\sigma(R)$) is just the outer measure induced by ν, it follows that if R is a G-ring (for an action of a group G on X) and ν is G-invariant, then so is the extended measure. It is clear that the converse of the above theorem is also true, i.e. if a content ν on R extends to a measure on $\sigma(R)$, then ν is a pre-measure.

When $(X_n)_{n<\omega}$ is a descending sequence of sets whose intersection is empty, we will write $X_n \downarrow \emptyset$; when $(X_n)_{n<\omega}$ is an ascending sequence of sets whose union is X, we will write $X_n \uparrow X$.

Remark 2.19. Let ν be a content on a ring R of subsets of X taking only finite values. Then ν is a pre-measure if and only if for every sequence $(X_n)_{n<\omega}$ of sets from R such that $X_n \downarrow \emptyset$ one has $\lim_n \nu(X_n) = 0$ (in this case we say that ν is continuous at 0). If R is a Boolean algebra, these conditions are also equivalent to the condition that for every sequence $(X_n)_{n<\omega}$ of sets from R such that $X_n \uparrow X$ one has $\lim_n \nu(X_n) = \nu(X)$.

Proposition 2.20. *If ρ is a normalized mean on a lattice $(L, \cap, \cup)$ of subsets of a set X, then it extends uniquely to a content ν on the Boolean algebra $B(L)$ generated by L. If L is a G-lattice and ρ is G-invariant, then so is ν.*

Proof. **Case 1:** L is finite, say equal to $\{A_0, \dots, A_n\}$.

It is clear that there is a unique possible candidate for ν, namely ν is determined by the formulas

$$\nu\left(A_0^{\epsilon(0)} \cap \dots \cap A_n^{\epsilon(n)}\right) = \rho\left(\bigcap_{i\in\Delta_\epsilon^+} A_i\right) - \rho\left((\bigcap_{i\in\Delta_\epsilon^+} A_i) \cap (\bigcup_{i\in\Delta_\epsilon^-} A_i)\right),$$

for any $\epsilon \in \{0,1\}^{n+1}$, where $\Delta_\epsilon^+ := \{i \leq n : \epsilon(i) = 1\}$, $\Delta_\epsilon^- := \{i \leq n : \epsilon(i) = 0\}$, and $A_i^0 := X \setminus A_i$, $A_i^1 := A_i$. This follows by finite additivity of ρ and the fact that each element of $B(L)$ can be (uniquely) written as a (disjoint) union of sets of the form $A_0^{\epsilon(0)} \cap \cdots \cap A_n^{\epsilon(n)}$.

Conversely, it is clear that when we define ν be the above formulas on the atoms of $B(L)$ and then extend additively, then we get a content. It is also clear that if ρ is G-invariant, so is ν. The remaining thing to check is that ν extends ρ, i.e. $\nu(A_k) = \rho(A_k)$ for all $k \leq n$.

We argue by induction on n, where the base induction step for $n = 0$ is clear. Assume the conclusion holds for numbers less then a given $n > 0$. It is enough to show that $\nu(A_n) = \rho(A_n)$.

$$\nu(A_n) = \sum_{\epsilon \in 2^n} \rho\left(A_n \cap \bigcap_{i \in \Delta_\epsilon^+} A_i\right) - \rho\left(A_n \cap (\bigcap_{i \in \Delta_\epsilon^+} A_i) \cap (\bigcup_{i \in \Delta_\epsilon^-} A_i)\right) = S_1 + S_2,$$

where

$$S_1 = \sum_{\epsilon \in 2^{n-1}} \rho\left(A_n \cap A_{n-1} \cap \bigcap_{i \in \Delta_\epsilon^+} A_i\right) - \rho\left(A_n \cap A_{n-1} \cap (\bigcap_{i \in \Delta_\epsilon^+} A_i) \cap (\bigcup_{i \in \Delta_\epsilon^-} A_i)\right),$$

$$S_2 = \sum_{\epsilon \in 2^{n-1}} \rho\left(A_n \cap \bigcap_{i \in \Delta_\epsilon^+} A_i\right) - \rho\left(A_n \cap (\bigcap_{i \in \Delta_\epsilon^+} A_i) \cap (A_{n-1} \cup \bigcup_{i \in \Delta_\epsilon^-} A_i)\right).$$

By the modularity of ρ,

$$\begin{aligned} S_2 \;=\; & \textstyle\sum_{\epsilon \in 2^{n-1}} \rho\left(A_n \cap \bigcap_{i \in \Delta_\epsilon^+} A_i\right) - \rho\left(A_n \cap (\bigcap_{i \in \Delta_\epsilon^+} A_i) \cap (\bigcup_{i \in \Delta_\epsilon^-} A_i)\right) \\ & -\rho\left(A_n \cap A_{n-1} \cap \textstyle\bigcap_{i \in \Delta_\epsilon^+} A_i\right) + \rho\left(A_n \cap A_{n-1} \cap (\bigcap_{i \in \Delta_\epsilon^+} A_i) \cap (\bigcup_{i \in \Delta_\epsilon^-} A_i)\right). \end{aligned}$$

Thus, $S_1 + S_2 = \sum_{\epsilon \in 2^{n-1}} \rho\left(A_n \cap \bigcap_{i \in \Delta_\epsilon^+} A_i\right) - \rho\left(A_n \cap (\bigcap_{i \in \Delta_\epsilon^+} A_i) \cap (\bigcup_{i \in \Delta_\epsilon^-} A_i)\right)$, which is equal to $\rho(A_n)$ by induction hypothesis. Thus, the induction step has been completed.

Case 2: L is arbitrary.

For uniqueness notice that any content on $B(L)$ extending ρ is determined by its restrictions to all Boolean algebras generated by finite sublattices of L and that these restrictions are unique by Case 1. To show existence, for any finite sublattice $L_0 \subseteq L$ let ν_{L_0} be the unique content on $B(L_0)$ extending $\rho|_{L_0}$, which exists by Case 1. Then note that by uniqueness in Case 1, $\bigcup_{L_0} \nu_{L_0}$ is the desired content. □

The next easy example shows that it may happen that a mean ρ is continuous at 0, but the unique extension ν to a content on the generated Boolean algebra is not continuous at 0, i.e. ν is not a pre-measure and so it cannot be extended to a measure.

Example 2.21. Take any infinite set X and present it as an increasing union of sets X_n. Let the lattice L consist of $\emptyset, X_0, X_1, \dots, X$. Define a mean on L by: $\rho(\emptyset) = 0$, $\rho(X) = 1$, $\rho(X_n) = \frac{1}{2}$. Then, if $A_n \downarrow \emptyset$, where $A_n \in L$, then eventually $A_n = \emptyset$, so ρ is continuous at 0. Let ν be the unique extension of ρ to a content on $B(L)$. Then $X \setminus X_n \downarrow \emptyset$, but $\lim \nu(X \setminus X_n) = \frac{1}{2} \neq 0$, so ν is not a pre-measure.

The means that we are interested in come from pre-means, and we will see that this rules out obstacles as in the above example.

From now on, we work in models of a given theory T. As is well-known, a definable family of definable sets is given by a formula $\varphi(\bar{x}, \bar{y})$, in the sense that the family is precisely the collection of sets defined by the formulas $\varphi(\bar{x}, \bar{b})$ as $\bar{b}$ varies (over a given model, or over the monster model). We generalize this to the notion of a $\bigvee$-definable family (of definable sets), given now by a collection $\{\varphi_i(\bar{x}, \bar{z}_i) : i \in I\}$ of formulas. Namely, the family is the collection of sets definable by formulas $\varphi_i(\bar{x}, \bar{b}_i)$ for $i \in I$ and varying parameters $\bar{b}_i$.

Definition 2.22. We will say that a $\bigvee$-definable family $\mathcal{E} := \{\varphi_i(x, y, \bar{z}_i) : i \in I, \bar{z}_i \text{ belongs to any model}\}$ *defines* an equivalence relation if for every model M, $E_M := \bigcap \mathcal{E}_M$ is an (M-type-definable) equivalence relation, where $\mathcal{E}_M := \{\varphi_i(x, y, \bar{b}_i) : i \in I, \bar{b}_i \subseteq M\}$.

By a standard trick, we can and do assume that I is a directed set and for every $i < j$, $(\forall \bar{z}_i)(\exists \bar{z}_j)(\varphi_j(x, y, \bar{z}_j) \to \varphi_i(x, y, \bar{z}_i))$.

The above definition is introduced in order to capture for example the following situations. A $\emptyset$-type-definable equivalence relation $E = \bigcap_{i \in I} R_i(x, y)$ is defined by the $\bigvee$-definable family $\{R_i(x, y) : i \in I\}$ (so here there are no parametric variables $\bar{z}_i$). In particular, the relation of lying in the same left [resp. right] coset of a $\emptyset$-type-definable subgroup H of a $\emptyset$-definable group G is defined by a $\bigvee$-definable family (without parametric variables). To get another important example, consider any $\emptyset$-type-definable subgroup $H = \bigcap_{i \in I} X_i$ (where I is directed and $X_j \subseteq X_i$ whenever $i < j$). Put $\mathcal{E} = \{G(x) \wedge G(y) \wedge z^{-1}(yx^{-1})z \in X_i : i \in I, z\}$ (where for $z \notin G$ and $a \in G$ we put $z^{-1}az := a$). Then, for any model M, E_M is the M-type-definable equivalence relation of lying in the same right coset of $\bigcap_{g \in G(M)} H^g$. More generally, when G is equipped with a $\bigvee$-definable group topology as in Subsection 2.7, then the relation of lying in the same left [resp. right] coset of the infinitesimals (i.e. μ_M from Definition 2.38) is also naturally defined by a $\bigvee$-definable family.

From now on, let G be a $\emptyset$-definable group and let E be an equivalence relation on G defined by a $\bigvee$-definable family $\mathcal{E} := \{\varphi_i(x, y, \bar{z}_i) : i \in I, \bar{z}_i\}$; we assume that each $\varphi_i(x, y, \bar{z}_i)$ implies that $x, y \in G$. Work in a monster model M^*; so $G = G(M^*)$.

Definition 2.23. By a *G-pre-mean* for $\mathcal{E}_M$ we mean a G-pre-mean for G/E_M (see Definition 2.15), i.e. a monotone function m on definable (with parameters) subsets of G into $[0, 1]$, with $m(\emptyset) = 0, m(G) = 1$, and $m(Y \cup Y') \leq m(Y) + m(Y')$, such that equality holds whenever $((g_1\varphi_i(x,y,\bar{b}_1)\cap\cdots\cap g_n\varphi_i(x,y,\bar{b}_n))\circ Y)\cap Y'=\emptyset$ for some $g_1,\dots,g_n\in G$, $i\in I$, and $\bar{b}_1,\dots,\bar{b}_n$ from M.

By a standard construction (incorporating the mean into the language, as was recalled in the proof of Lemma 2.7(2)), we have the following remark.

Remark 2.24. A G-pre-mean for $\mathcal{E}_M$, but defined only on M-definable sets and satisfying the "equality criterion" only for $g_1,\dots,g_n\in G(M)$, extends to a G-pre-mean for $\mathcal{E}_M$ (defined on all definable sets). In fact, it extends to a G-pre-mean for $\mathcal{E}_{M^*}$, which is clearly also a G-pre-mean for $\mathcal{E}_N$ for any $N\prec M^*$. If the initial G-pre-mean is $G(M)$-invariant, then the G-pre-mean for $\mathcal{E}_{M^*}$ is $G(M^*)$-invariant, so it is also a G-invariant G-pre-mean for $\mathcal{E}_N$ for any $N\prec M^*$.

For a model M, let $\mathcal{D}_{\mathcal{E}_M}$ be the G-lattice of type-definable (with parameters) subsets D of $G(M^*)$ such that $(g_1E_M\cap\cdots\cap g_nE_M)\circ D=D$ for some $g_1,\dots,g_n\in G$.

From Corollary 2.16 and Remark 2.24, we get:

Corollary 2.25. *Let m be a G-pre-mean for $\mathcal{E}_M$. Then m induces a normalized mean ρ on the lattice $\mathcal{D}_{\mathcal{E}_M}$ via $\rho(Y):=\inf\{m(D): D$ definable, $Y\subseteq D\}$. If m is $G(M)$-invariant, then we can replace it by a G-invariant pre-mean, and then the induced ρ is G-invariant as well.*

The converse is easy is check.

Remark 2.26. A normalized mean ρ on the lattice $\mathcal{D}_{\mathcal{E}_M}$ induces a G-pre-mean m for $\mathcal{E}_M$ via $m(Y):=\inf\{\rho((g_1E_M\cap\cdots\cap g_nE_M)\circ Y): g_1,\dots,g_n\in G\}$. If ρ is $G(M)$-invariant [resp. G-invariant], so is m.

Corollary 2.27. *If $\mathcal{D}_{\mathcal{E}_M}$ carries a [$G(M)$-invariant], normalized mean, then it carries such a mean ρ which is induced from a [G-invariant] G-pre-mean m for $\mathcal{E}_M$ via $\rho(Y):=\inf\{m(D): D$ definable, $Y\subseteq D\}$.*

Corollary 2.28. *The existence of a G-invariant normalized mean on $\mathcal{D}_{\mathcal{E}_M}$ does not depend either on the choice of M or the monster model M^* in which the lattice is computed.*

The next proposition is the main observation of this subsection.

Proposition 2.29. *Assume E_M is $G(M)$-invariant. The following conditions are equivalent.*

1. *$S_{G/E_M}(M)$ carries a $G(M)$-invariant, Borel probability measure.*
2. *There is a $G(M)$-invariant [G-invariant] G-pre-mean for $\mathcal{E}_M$.*
3. *The lattice $\mathcal{D}_{\mathcal{E}_M}$ carries a $G(M)$-invariant [G-invariant], normalized mean.*

Proof. First note that E_M being $G(M)$-invariant guarantees that $G(M)$ acts naturally on G/E_M, which induces an action of $G(M)$ on $S_{G/E_M}(M)$.
(1) $\rightarrow$ (2). Let μ witnesses (1). For an M-definable subset D of G define $m(D):=\mu(\bar{D})$, where $\bar{D}$ is the set of complete types (over M) of elements of

D/E_M. Since E_M is $G(M)$-invariant, we easily see that m is a $G(M)$-invariant G-pre-mean for $\mathcal{E}_M$, but defined only on M-definable sets and satisfying the "equality criterion" only for $g_1, \dots, g_n \in G(M)$. By Remark 2.24, it extends to an actual G-invariant G-pre-mean (defined on all definable sets) for $\mathcal{E}_M$.
(2) $\to$ (3). This follows from Corollary 2.25.
(3) $\to$ (1). Take a $G(M)$-invariant, normalized mean on $\mathcal{D}_{\mathcal{E}_M}$. By Corollary 2.27, there exists a G-invariant, normalized mean ρ induced from a G-invariant G-pre-mean m for $\mathcal{E}_M$ via $\rho(Y) := \inf\{m(D) : D \text{ definable}, Y \subseteq D\}$. By Proposition 2.20, let ν be the unique extension of ρ to a G-invariant content on the Boolean algebra $B(\mathcal{D}_{\mathcal{E}_M})$. We will show that ν is a pre-measure, which by the Carathédory theorem can be further extended to a G-invariant measure $\bar{\nu}$ on the generated σ-algebra $\sigma(B(\mathcal{D}_{\mathcal{E}_M}))$. Then $\bar{\mu}$ induces a $G(M)$-invariant, Borel probability measure on $S_{G/E_M}(M)$ via $\mu(P) := \bar{\mu}(\{a \in M^* : \mathrm{tp}((a/E_M)/M) \in P\})$ for any Borel subset P of $S_{G/E_M}(M)$, and the proof will be complete. So it remains to show

Claim 1: ν is a pre-measure.

Proof. Put $R := B(\mathcal{D}_{\mathcal{E}_M})$. By Remark 2.19, it is enough to show that for every sequence $(X_n)_{n<\omega}$ of sets from R such that $X_n \uparrow G$ one has $\lim_n \nu(X_n) = 1$. Take any $\epsilon > 0$. We need to show that $\nu(X_n) > 1 - \epsilon$ for some n.

One can find sets $Z_k \subseteq Y_k$ (for $k \in \omega$) from $\mathcal{D}_{\mathcal{E}_M}$ and natural numbers $n_0 < n_1 < \dots$ such that
$$X_i = (Y_0 \setminus Z_0) \sqcup \dots \sqcup (Y_{n_i} \setminus Z_{n_i})$$
for every $i < \omega$ (where $\sqcup$ stands for disjoint union). Then
$$\nu(X_i) = \sum_{k=0}^{n_i} \rho(Y_k) - \rho(Z_k).$$

For each k we can choose a definable set $D_k \supseteq Y_k$ such that
$$\sum_{k=0}^{\infty} m(D_k) - \rho(Y_k) < \epsilon.$$

For each k let $\mathcal{F}_k$ be the family of all sets F definable over the set of parameters over which Z_k is defined and such that $Z_k \subseteq F \subseteq D_k$. Then $\bigcap \mathcal{F}_k = Z_k$ for every k. Therefore,
$$G = \bigcup_i X_i = \bigcup_k Y_k \setminus Z_k \subseteq \bigcup_k D_k \setminus Z_k = \bigcup_k \left(D_k \setminus \bigcap \mathcal{F}_k\right) = \bigcup_k \bigcup_{F \in \mathcal{F}_k} D_k \setminus F,$$
so by the saturation of M^*, there are $k_1 < \dots < k_n < \omega$ and $F_{k_1} \in \mathcal{F}_{k_1}, \dots, F_{k_n} \in \mathcal{F}_{k_n}$ such that
$$G = (D_{k_1} \setminus F_{k_1}) \cup \dots \cup (D_{k_n} \setminus F_{k_n}).$$
(Note that this is not necessarily a disjoint union.)

We also have $Z_{k_j} \subseteq F_{k_j}$. Since $Z_{k_j} \in \mathcal{D}_{\mathcal{E}_M}$, by compactness, it is easy to see that there are definable sets $F'_{k_j} \in \mathcal{F}_{k_j}$ contained in F_{k_j} such that $((g_1\varphi_i(x, y, \bar{b}_1) \cap$

$\dots \cap g_n\varphi_i(x, y, \bar{b}_n)) \circ F'_{k_j}) \cap (D_{k_j} \setminus F_{k_j}) = \emptyset$ for some $g_1, \dots, g_n \in G$, $i \in I$, and $\bar{b}_1, \dots, \bar{b}_n$ from M (all depending on j of course). Hence, $m((D_{k_j} \setminus F_{k_j}) \cup F'_{k_j}) = m(D_{k_j} \setminus F_{k_j}) + m(F'_{k_j})$, which implies that $m(D_{k_j} \setminus F_{k_j}) \leq m(D_{k_j}) - m(F'_{k_j})$.

From all these observations, we get

$$1 = m(G) = m(\bigcup_{j=1}^n D_{k_j} \setminus F_{k_j}) \leq \sum_{j=1}^n m(D_{k_j} \setminus F_{k_j}) \leq \sum_{j=1}^n m(D_{k_j}) - m(F'_{k_j}) < (\sum_{j=1}^n \rho(Y_{k_j}) - \rho(Z_{k_j})) + \epsilon.$$

Hence, $\nu(X_{k_n}) \geq \nu((Y_{k_1} \setminus Z_{k_1}) \sqcup \dots \sqcup (Y_{k_n} \setminus Z_{k_n})) = \sum_{j=1}^n \rho(Y_{k_j}) - \rho(Z_{k_j}) > 1 - \epsilon$. □(claim)

The proof of the proposition is complete . □

Remark 2.30. In Proposition 2.29, one can add one more equivalent condition:

(4) The $G(M)$-lattice $\mathcal{D}_{E_M}$ of type-definable subsets D of $G(M^*)$ such that $E_M \circ D = D$ carries a $G(M)$-invariant, normalized mean.

Proof. The implication (3) $\to$ (4) is trivial, while (4) $\to$ (1) follows similarly to (3) $\to$ (1) (note that, in the proof of (3) $\to$ (1), it is enough to work with $G(M)$-invariant pre-means, means and contents in order to get that μ is $G(M)$-invariant). □

The reason why we work with the more complicated lattice $\mathcal{D}_{\mathcal{E}_M}$ instead of $\mathcal{D}_{E_M}$ is that the former is a G-lattice which is needed in Case 2 in Subsection 2.6.

From Corollary 2.28 and Proposition 2.29, we get

Corollary 2.31. *Assume E_M is $G(M)$-invariant for every model M. Then, the existence of a $G(M)$-invariant, Borel probability measure on $S_{G/E_M}(M)$ does not depend on the choice of M.*

For the type-definable equivalence relation $E(x, y)$ given by $x^{-1}y \in H$, where H is a $\emptyset$-type-definable subgroup of G, Corollary 2.31 specializes to

Corollary 2.32. *The existence of a $G(M)$-invariant, Borel probability measure on $S_{G/H}(M)$ does not depend on the choice of M.*

Corollary 2.31 specializes to more absolutness results in the context of $\bigvee$-definable group topologies, which will be discussed in Subsection 2.7 (see Corollary 2.45).

Remark 2.33. If E_M is not $G(M)$-invariant, then there is no natural (left) action of $G(M)$ on $S_{G/E_M}(M)$. But we can always replace the family $\mathcal{E}$, by $\mathcal{E}' := \{\varphi_i(t_i x, t_i y, \bar{z}_i) : i \in I, t_i, \bar{z}_i\}$ (where for $a \notin G$ or $b \notin G$ we put $ab := b$). Then, for any model M, the induced equivalence relation E'_M will be the intersection of all gE_M for g ranging over $G(M)$. And, by Corollary 2.31, the existence of a $G(M)$-invariant, Borel probability measure on $S_{G/E'_M}(M)$ does not depend on the choice of M.

2.6. **Measures, means, and connected components.** Now, consider a structure M, a $\emptyset$-definable group G, and an M-type-definable subgroup H of G. Usually G will stand for the interpretation of G in a monster model M^* (i.e. $G = G^* = G(M^*)$); by $G(M)$ we denote the interpretation of G in M.

We will be interested in the following two cases.

Case 1: The type space $S_{G/H}(M)$ (i.e. the space of complete types over M of left cosets modulo H) carries a $G(M)$-invariant, Borel probability measure.

The discussion below repeats some arguments from the previous subsection in a special case, but since this will be the context of the main results of Section 2, we prefer to write it explicitly.

Let $\bar{m}$ be a $G(M)$-invariant, Borel probability measure on $S_{G/H}(M)$. We define a $G(M)$-invariant pre-mean (see Definition 2.13, where the equivalence relation is $xH = yH$) m' on M-definable subsets of G, by $m'(Y) := \bar{m}(\bar{Y})$, where $\bar{Y}$ is the set of complete types over M of elements of Y/H.

As in the proof of Lemma 2.7, the standard construction allows us to extend m' to a G-invariant pre-mean on M^*-definable subsets of $G = G(M^*)$. Note that this extended pre-mean is definable over $\emptyset$ in some expansion of the language (meaning that for any closed interval I and for any formula $\varphi(x, \bar{y})$ of the original language the set $\{\bar{b} : m'(\varphi(x, \bar{b})) \in I\}$ is $\emptyset$-type-definable in this expansion of the language), and M^* can be chosen so that $M \prec M^*$ in the expanded language. Next, using Lemma 2.14, we obtain a normalized, G-invariant mean m on $\mathcal{D}_H$ – the G-lattice of M^*-type-definable subsets Y of G satisfying $YH = Y$ – which satisfies $m(Y) = \inf\{m'(D) : D \text{ definable}, Y \subseteq D\}$.

Case 2: H is normalized by $G(M)$, and $S_{H\backslash G}(M)$ carries a $G(M)$-invariant, Borel probability measure, where the action of $G(M)$ on $S_{H\backslash G}(M)$ is induced by the action on $H\backslash G$ given by $g * (Ha) := gHa = H(ga)$.

Let $\bar{m}$ be a $G(M)$-invariant, Borel probability measure on $S_{H\backslash G}(M)$. As in Case 1, we obtain a $G(M)$-invariant pre-mean (for the equivalence relation $Hx = Hy$) m' on M-definable subsets of G. The standard construction allows us to extend it to a definable over $\emptyset$ (in some expansion of the language), G-invariant pre-mean m' on M^*-definable subsets of $G = G(M^*)$, for some monster model M^* such that $M^* \succ M$ also in the expand language. Moreover, since H is normalized by $G(M)$, the standard construction gives us the following additional property of m': For any Y and Z definable subsets of G, M-definable superset D of H, and $g_1, \dots, g_n \in G$, if $(D^{g_1} \cap \dots \cap D^{g_n})Y \cap Z = \emptyset$, then $m'(Y \cup Z) = m'(Y) + m'(Z)$, i.e. m' is a G-pre-mean for $H\backslash G$, using the terminology from Definition 2.15. By Corollary 2.16, we obtain a normalized, G-invariant mean m on $\mathcal{D}'_H$ – the G-lattice of M^*-type-definable subsets Y of G satisfying $(H^{g_1} \cap \dots \cap H^{g_n})Y = Y$ for some $g_1, \dots, g_n \in G$ – which satisfies $m(Y) = \inf\{m'(D) : D \text{ definable}, Y \subseteq D\}$.

We are ready to prove the main results of this section. They concern situations from the above Cases 1 and 2, respectively. We will give a detailed proof of the first theorem and only explain how to modify it to get the second one.

In the rest of this section, we will write Z^4 to mean $ZZZ^{-1}Z^{-1}$; Z^8 denotes Z^4Z^4.

Theorem 2.34. *Let H be a $\emptyset$-type-definable subgroup of G, normalized by $G(M)$. Let N be the normal subgroup generated by H. Then* (1) $\leftrightarrow$ (2) $\leftrightarrow$ (3) $\leftrightarrow$ (4) $\rightarrow$ (5)*:*

(1) $S_{G/H}(M)$ carries a $G(M)$-invariant, Borel probability measure.
(2) There is a $G(M)$-invariant pre-mean for G/H on M-definable subsets of G.
(3) There is a G-invariant pre-mean for G/H which is definable over $\emptyset$ in some expansion of the language in which $M \prec M^*$ (enlarging M^* if necessary).
(4) The lattice $\mathcal{D}_H$ carries a normalized, G-invariant mean.
(5) $G^{00}_M \leq NG^{000}_M$.

Proof. The equivalence of conditions (1)-(4) essentially follows from Proposition 2.29 applied to $\mathcal{E} := \{G(x) \wedge G(y) \wedge x^{-1}y \in X_i : i \in I\}$, where $H = \bigcap_{i\in I} X_i$ (with I directed and $X_j \subseteq X_i$ whenever $i < j$). For that notice that the relation E_M in this special case is just lying in the same left coset of H, so it is G-invariant, and the lattice $\mathcal{D}_{\mathcal{E}_M}$ coincides with $\mathcal{D}_H$.

However, for the reader's convenience, we explain some of these equivalences more explicitly. By the above discussion of Case 1, any $G(M)$-invariant, Borel probability measure on $S_{G/H}(M)$ induces a $G(M)$-invariant pre-mean on M-definable subsets of G, which then can be extended to a G-invariant pre-mean m' on definable subsets of G which is definable over $\emptyset$ in some expansion of the language in which $M \prec M^*$ (enlarging M^* if necessary). This in turn induces a normalized, G-invariant mean m on the lattice $\mathcal{D}_H$, which satisfies $m(Y) = \inf\{m'(D) : D \text{ definable}, Y \subseteq D\}$. So (1) $\rightarrow$ (2) $\leftrightarrow$ (3) $\rightarrow$ (4). The implication (4) $\rightarrow$ (1) follows from the implication (3) $\rightarrow$ (1) in Proposition 2.29.

It remains to prove (4) $\rightarrow$ (5). So assume (4). By (4) $\rightarrow$ (2) and the above discussion, we have a G-invariant mean m on $\mathcal{D}_H$ given by $m(Y) = \inf\{m'(D) : D \text{ definable}, Y \subseteq D\}$ for some pre-mean m' satisfying (3).

Let $p \in S_G(M)$ be a wide type of G, in the sense that $m(DH) > 0$ for any $D \in p$. In order to finish the proof, it is enough to show that $(HpH)^4$ contains G^{00}_M. Indeed, then, since $pp^{-1} \subseteq G^{000}_M$ implies $ppp^{-1}p^{-1} \subseteq G^{000}_M$, and so $(HpH)^4 \subseteq NG^{000}_M$, we get $G^{00}_M \leq NG^{000}_M$ which is the desired conclusion.

As HpH is an intersection of partial types P over M satisfying $HPH = P$ and $m(P) > 0$ (namely the appropriate HDH with D M-definable), it suffices to show that for each such P, P^4 contains G^{00}_M. For this, it suffices to find for any M-definable set P' containing P a generic, M-type-definable set $Q = HQH$ with $Q^8 \subseteq P'^4$, for then $m(Q) > 0$ and we can find an M-definable set Q' containing Q such that $Q'^8 \subseteq P'^4$, and we can iterate: find a generic, M-type-definable

$R = HRH$ with $R^8 \subseteq Q'^4$ and an M-definable R' containing R and satisfying $R'^8 \subseteq Q'^4$, etc., and at the limit take the intersection $P'^4 \cap Q'^4 \cap R'^4 \cap \dots$ – an M-type-definable, bounded index, subgroup contained in P'^4, which clearly contains G_M^{00}. Since this is true for any M-definable P' containing P, we get $G_M^{00} \subseteq P^4$.

So consider a partial type P over M satisfying $HPH = P$ and $m(P) > 0$. Consider any M-definable P' containing P. We will apply Corollary 2.11 to: $X := G$, $A := P$, and the family

$$\mathcal{B} := \{HQH : P \subseteq HQH \subseteq P' \text{ and } Q \text{ is } M\text{-definable}\}$$

of subsets of G. Recall that $\mathcal{D}'$ is the collection of all intersections $g_1B \cap \dots \cap g_nB$, where $B \in \mathcal{B}$ and $g_1, \dots, g_n \in G$, and as $\mathcal{D}$ take the lattice generated by: $\mathcal{D}'$, the set A, and all sets $B'A$ for $B' \in \mathcal{D}'$. Note that $\mathcal{D} \subseteq \mathcal{D}_H$, so our mean m is defined on $\mathcal{D}$. By Corollary 2.11, we find $l \in \mathbb{N}$, $\lambda \in \mathbb{R}$, $B \in \mathcal{B}$ and $g_1, \dots, g_n \in G$ such that for $B' := B \cap g_1B \cap \dots \cap g_nB$ we have

(***) $\mathcal{L}_l(B')$ and $m(B'A) < \lambda$,

and whenever $E \in \mathcal{B}$ and $h_1, \dots, h_m \in G$ are chosen so that for $E' := E \cap h_1E \cap \dots \cap h_mE$ one has $\mathcal{L}_l(E')$ and $m(E'A) < \lambda$, then $\mathrm{St}_{\mathcal{L}_{l-1}}(E')$ is generic (as a set $\bigvee$-definable in $(G, \cdot, E)$), symmetric and has 8th power contained in $E'P(E'P)^{-1} \subseteq E^4$.

We can find M-definable sets C and D such that $B \subseteq C \subseteq P'$, $P \subseteq D$ and for $C' := C \cap g_1C \cap \dots \cap g_nC$, we have $m'(C'D) < \lambda$ (where m' is the pre-mean on definable subsets of G chosen at the beginning of the proof of (4) $\to$ (5)). Now, choose any M-definable set Q such that $B \subseteq Q \subseteq HQH \subseteq C$. Let $Q' = Q \cap g_1Q \cap \dots \cap g_nQ$. Then $B' \subseteq Q'$, so, by (***), we get $\mathcal{L}_l(Q')$. Since $\mathcal{L}_l(Q')$ is a $\bigvee$-definable (over $\emptyset$) condition on $g_1, \dots, g_n$ in the structure $(G, \cdot, Q)$ and Q is M-definable in the original theory, we see that $\mathcal{L}_l(Q')$ is an M-$\bigvee$-definable condition on $g_1, \dots, g_n$ in the original theory. On the other hand, $m'(C'D) < \lambda$ is a $\bigvee$-definable (over $\emptyset$) condition on $g_1, \dots, g_n$ in the expanded language (in which m' is definable over $\emptyset$). Since $M \prec M^*$ also in this expanded language, we can find $g_1, \dots, g_n \in G(M)$ such that $\mathcal{L}_l(Q')$ and $m'(C'D) < \lambda$ still holds for the corresponding Q' and C'. Finally, take $E := HQH$ and $E' := E \cap g_1E \cap \dots \cap g_nE$. We see that $E \in \mathcal{B}$, $\mathcal{L}_l(E')$ and $m(E'A) < \lambda$.

Define $Y := \mathrm{St}_{\mathcal{L}_{l-1}}(E')$. By the the choice of l and λ, we have that $Y = \bigvee_\nu Y_\nu$ is generic and $Y^8 \subseteq E^4 \subseteq P'^4$.

As H is normalized by $g_1, \dots, g_n$, we have $HE'H = E'$. Since $HE' = E'$, we have $HYH = Y$, and moreover Y is a disjunction of sets Y_ν positively M-definable in $(G, \cdot, E)$ and satisfying $HY_\nu H = Y_\nu$. Indeed, let $R(x, \bar{y})$ be a new predicate. By the approximations to $\mathcal{L}_l$ mentioned in and after Remark 2.2, we have that for any $s \in \omega$ there are increasing sets $P_{\nu,s}(R)(\bar{y})$, $\nu \in \omega$, positively $\emptyset$-definable in $(G, \cdot, X, \cdot, R)$ such that $P_{\nu,s}(R(gx, \bar{y}))(\bar{y}) \iff P_{\nu,s}(R(x, \bar{y}))(\bar{y})$ for all $g \in G$, and $\mathcal{L}_s(R(x, \bar{y}))$ can be presented as the $\bigvee$-positively definable set

$\bigvee_\nu P_{\nu,s}(R(x,\bar{y}))(\bar{y})$. In particular, if $R(x,\bar{y})$ is positively definable in $(G,\cdot,X,\cdot,E)$, then the $P_{\nu,s}(R)(\bar{y})$ are positively definable in $(G,\cdot,X,\cdot,E)$ over the same parameters over which $R(x,\bar{y})$ is defined. Applying this to our situation for $s := l-1$ and $R(x,y) := (x \in (yE' \cap E'))$, we get that $Y = \{y : \mathcal{L}_s(yE' \cap E')\}$ can be presented as $\bigvee_\nu P_{\nu,s}(R(x,y))(y)$. Putting $Y_\nu(y) = P_{\nu,s}(R)(y)$, we have that Y_ν is positively M-definable in $(G,\cdot,E)$, and, since $HE' = E'$, we get that for any $h_1, h_2 \in H$: $Y_\nu(h_1yh_2) \iff P_{\nu,s}(R(x,h_1yh_2))(y) \iff P_{\nu,s}(x \in h_1yh_2E' \cap E')(y) \iff P_{\nu,s}(x \in h_1yE' \cap E')(y) \iff P_{\nu,s}(x \in yE' \cap h_1^{-1}E')(y) \iff P_{\nu,s}(x \in yE' \cap E')(y) \iff Y_\nu(y)$. So $HY_\nu H = Y_\nu$ as was claimed at the beginning of this paragraph.

Since the $Y_\nu \subseteq G$ are positively M-definable in $(G,\cdot,E)$ and E is M-type-definable in the original theory, we easily get that the Y_ν are M-type-definable in the original theory. Moreover, some Y_ν will be generic, and $HY_\nu H = Y_\nu$, and $Y_\nu^8 \subset P'^4$. □

In the situation of Case 2, we have

Theorem 2.35. *Let H be a $\emptyset$-type-definable subgroup of G, normalized by $G(M)$. Let N be the normal subgroup generated by H. Then $(1) \leftrightarrow (2) \leftrightarrow (3) \leftrightarrow (4) \leftrightarrow (5) \to (6)$:*

(1) $S_{H\backslash G}(M)$ carries a $G(M)$-invariant, Borel probability measure.
(2) There is a $G(M)$-invariant pre-mean for $H\backslash G$ on M-definable subsets of G.
(3) There is a G-invariant G-pre-mean m' for $H\backslash G$ (i.e. a G-invariant pre-mean m' for $H\backslash G$ such that $m'(Z \cup Z') = m'(Z) + m'(Z')$ whenever $D'Z \cap Z' = \emptyset$ for some M-definable superset D of H and $D' = D^{g_1} \cap \dots \cap D^{g_n}$ for some $g_1, \dots, g_n \in G$).
(4) There is a G-invariant G-pre-mean for $H\backslash G$ which is definable over $\emptyset$ in some expansion of the language in which $M \prec M^*$ (enlarging M^* if necessary).
(5) The lattice $\mathcal{D}'_H$ carries a normalized, G-invariant mean.
(6) $G^{00}_M \leq NG^{000}_M$.

Proof. The equivalence of conditions (1)-(5) essentially follows from Proposition 2.29 applied to $\mathcal{E} := \{G(x) \wedge G(y) \wedge yx^{-1} \in X_i : i \in I\}$, where $H = \bigcap_{i \in I} X_i$ (with I directed and $X_j \subseteq X_i$ whenever $i < j$). For that notice that the relation E_M in this special case is just lying in the same right coset of H, so it is $G(M)$-invariant by the assumption that H is normalized by $G(M)$, and the lattice $\mathcal{D}_{\mathcal{E}_M}$ coincides with $\mathcal{D}'_H$. One should also use the above discussion of Case 2.

It remains to justify $(5) \to (6)$.

So assume (5). By $(5) \to (2)$ and the discussion of Case 2, we have a G-invariant mean m on $\mathcal{D}'_H$ given by $m(Y) = \inf\{m'(D) : D \text{ definable}, Y \subseteq D\}$ for some pre-mean m' satisfying (4). We follow the lines of the proof of $(4) \to (5)$ in Theorem 2.34, but now it is enough to work with right cosets modulo $H^{g_1} \cap \dots \cap H^{g_n}$ for some $g_1, \dots, g_n \in G(M)$ (in place of two-sided cosets of H), e.g. P is a partial type over M satisfying $(H^{g_1} \cap \dots \cap H^{g_n})P = P$ (for some $g_1, \dots, g_n \in G(M)$) and $m(P) > 0$. The way how $\mathcal{D}'_H$ was defined is essential to ensure that $\mathcal{D} \subseteq \mathcal{D}'_H$ (and so m is defined on $\mathcal{D}$). □

Conjecture 0.1 follows immediately from Theorem 2.35, taking $H := \mu$:

Corollary 2.36. *1) Let $G(M)$ be a topological group and assume that the members of a basis of neighborhoods of the identity are definable in M. If G is definably topologically amenable, then $G^{00}_{\mathrm{def,top}} = G^{000}_{\mathrm{def,top}}$.*
2) Let $G(M)$ be a topological group. If $G(M)$ is amenable, then $G^{00}_{\mathrm{top}} = G^{000}_{\mathrm{top}}$.

2.7. **$\bigvee$-definable group topologies.** In Section 1, we recalled two contexts to deal with topological groups model-theoretically: one with all open subsets being definable, and a more general one with a basis of open neighborhoods at the identity consisting of definable sets. Notice, however, that in each of these contexts we do not get a natural group topology when passing to elementary extensions. In order to get a group topology in an arbitrary elementary extension, one usually considers a more special context with a uniformly definable basis of open neighborhoods at the identity (in other words, when a basis of open sets at the identity is a definable family).

As usual, let G be a $\emptyset$-definable group, and M or N denotes a model. Here, we extend the last context, for example to cover topologies induced on $G(M)$ by type-definable subgroups of G normalized by $G(M)$. Note that any $\emptyset$-type-definable subgroup $H = \bigcap_{i \in I} X_i$ (with the definable sets X_i, where without loss I is a directed set such that $X_j \subseteq X_i$ for $i < j$), normalized by $G(M)$, can be viewed as topologizing $G(M)$ in the sense that the family $\{X_i : i \in I\}$ is a basis of (not necessarily open!) neighborhoods at the identity; but on a bigger model it will not in general give a topology. It is thus natural to consider a slightly stronger condition.

We first elaborate on some terminology introduced briefly in Subsection 2.5. By a *$\bigvee$-definable family of definable subsets of* G, we mean a class $\mathcal{T} = \{\varphi_i(x, \bar{y}_i) : i \in I,\ \bar{y}_i$ belongs to any model$\}$, where $\varphi_i(x, \bar{y}_i)$ are some formulas implying $G(x)$. In any model M,

$$\mathcal{T}(M) := \{\varphi_i(M, \bar{y}_i) : i \in I, \bar{y}_i \in M\}$$

is an actual collection of subsets of $G(M)$; also, put

$$\mathcal{T}_M := \{\varphi_i(x, \bar{y}_i) : \bar{y}_i \in M\}.$$

By a standard trick, we can, and will from now on, assume that I is a directed set, and for every $i < j$ we have $(\forall \bar{y}_i)(\exists \bar{y}_j)(\varphi_j(x, \bar{y}_j) \to \varphi_i(x, \bar{y}_i))$; the last condition is equivalent to the property that for every model M and $i < j$, each member of the definable family $\{\varphi_i(M, \bar{y}_i) : \bar{y}_i \in M\}$ contains a member of the family $\{\varphi_j(M, \bar{y}_j) : \bar{y}_j \in M\}$. (In fact, by the aforementioned standard trick, we could even replace the word "contains" by "equals", but we will not need it.)

Definition 2.37. A $\bigvee$-*definable group topology* on G is a $\bigvee$-definable family $\mathcal{T} = \{\varphi_i(x, \bar{y}_i) : i \in I, \bar{y}_i\}$ of definable subsets of G containing 1 such that in any model M, $\mathcal{T}(M)$ forms a basis of (not necessarily open) neighborhoods of the identity for a topology on $G(M)$, making the group operations continuous. Equivalently, for any model M, $\mathcal{T}(M)$ consists of subsets of $G(M)$ containing 1, such that each of the following sets

(1) the intersection of any two members of $\mathcal{T}(M)$,
(2) the inversion of any member of $\mathcal{T}(M)$,
(3) the conjugate of any member of $\mathcal{T}(M)$ by an element of $G(M)$

contains a member of $\mathcal{T}(M)$, and, additionally, if $A \in \mathcal{T}(M)$, then there exists $B \in \mathcal{T}(M)$ with $B^2 \subseteq A$.

It easy to check that in the above definition, it is enough to take a sufficiently saturated model M.

By compactness, a $\bigvee$-definable group topology on G is a $\bigvee$-definable family $\mathcal{T} = \{\varphi_i(x, \bar{y}_i) : i \in I, \bar{y}_i\}$ of definable subsets of G containing 1 such that:

(1) For every $i, j \in I$ there is $k \in I$ such that $(\forall \bar{y}_i)(\forall \bar{y}_j)(\exists \bar{y}_k)(\varphi_k(x, \bar{y}_k) \rightarrow (\varphi_i(x, \bar{y}_i) \wedge \varphi_j(x, \bar{y}_j)))$.
(2) For every $i \in I$ there is $j \in I$ such that $(\forall \bar{y}_i)(\exists \bar{y}_j)(\varphi_j(x^{-1}, \bar{y}_j) \rightarrow \varphi_i(x, \bar{y}_i))$.
(3) For every $i \in I$ there is $j \in I$ such that $(\forall \bar{y}_i)(\forall z)(\exists \bar{y}_j)(\varphi_j(z^{-1}xz, \bar{y}_j) \rightarrow \varphi_i(x, \bar{y}_i))$.
(4) For every $i \in I$ there is $j \in I$ such that $(\forall \bar{y}_i)(\exists \bar{y}_j)((\exists x_1, x_2)(\varphi_j(x_1, \bar{y}_j) \wedge \varphi_j(x_2, \bar{y}_j) \wedge x = x_1 \cdot x_2) \rightarrow \varphi_i(x, \bar{y}_i))$.

Let $\mathcal{T} = \{\varphi(x, \bar{z}_i) : i \in I, \bar{z}_i\}$ be a $\bigvee$-definable group topology on G. Work in a fixed monster model M^* (so $M \prec M^*$ by convention).

Definition 2.38. We let $\mu_M^{\mathcal{T}}$ be the M-type-definable subgroup $\bigcap_{D \in \mathcal{T}_M} D$. When the identity of $\mathcal{T}$ is clear, we write μ_M.

It is clear that $\mu_M^{\mathcal{T}}$ is normalized by $G(M)$.

Remark 2.39. For any A-definable set D, there exists an A-type-definable set $\mathrm{cl}(D)$ such that for any model M, the closure of $D(M)$ is $\mathrm{cl}(D)(M)$. Namely, $a \in \mathrm{cl}(D)$ iff $\bigwedge_i (\forall \bar{y}_i)(a \cdot \varphi_i(x, \bar{y}_i) \cap D(x) \neq \emptyset)$ (more formally, $\bigwedge_i (\forall \bar{y}_i)(\exists x)(\varphi_i(a^{-1}x, \bar{y}_i) \wedge D(x))$.

For a type-definable set $D = \bigcap_i D_i$ (where $D_j \subseteq D_i$ for $i < j$), let $\mathrm{cl}(D) = \bigcap_i \mathrm{cl}(D_i)$. For any sufficiently saturated model M, the closure of $D(M)$ is $\mathrm{cl}(D)(M)$.

Remark 2.40. For P M-type-definable, $\mathrm{cl}(P) \subseteq P\mu_M^{\mathcal{T}}$. Indeed, $\mathrm{cl}(P) = \bigcap\{PH : H \in \mathcal{T}\}$, which formally means that $\mathrm{cl}(P)(z)$ is the type $\bigwedge_i (\forall \bar{y}_i)(\exists x_1, x_2)(P(x_1) \wedge \varphi_i(x_2, \bar{y}_i) \wedge z = x_1 \cdot x_2)$. In particular, $\mu_M^{\mathcal{T}}$ is closed. Similarly, $\mathrm{cl}(P) \subseteq \mu_M^{\mathcal{T}} P$. In fact, $\mathrm{cl}(P)$ is contained in both $P((\mu_M^{\mathcal{T}})^{g_1} \cap \cdots \cap (\mu_M^{\mathcal{T}})^{g_n})$ and $((\mu_M^{\mathcal{T}})^{g_1} \cap \cdots \cap (\mu_M^{\mathcal{T}})^{g_n})P$ for any $g_1, \ldots, g_n \in G$.

By $\mathcal{C}^{\mathcal{T}}$ (or just $\mathcal{C}$) we will denote cl(1). Then $\mathcal{C} = \bigcap \mathcal{T}$, so it is $\emptyset$-type-definable, and it is a normal subgroup of G. It is clear that $\mathcal{C} \leq \mu_M$ for any M. Note that formally $\mathcal{C}$ coincides with μ_{M^*} which happens to be $\emptyset$-type-definable in the monster model M^*.

We say that $\mathcal{T}$ is *strongly Hausdorff* if $G(M)$ is Hausdorff in every model M; equivalently $\mathcal{C} = \{1\}$; equivalently, $\bigcap \mathcal{F} = \{1\}$ for some definable family $\mathcal{F} \subseteq \mathcal{T}$. Note that, in contrast with definable families, $\mathcal{T}(M)$ may be Hausdorff for one M, without $\mathcal{T}$ being strongly Hausdorff. This occurs when $\mu_M^{\mathcal{T}}(M) = \{1\}$, see Example 2.44. Note also that given a Hausdorff topological group, we can expand it to a first order structure in which there is an even definable $\mathcal{T}$ which is strongly Hausdorff and induces the given topology on the group we started from.

Define the following G-lattices of subsets of G (a $\emptyset$-definable group equipped with a $\bigvee$-definable group topology $\mathcal{T}$).

(1) $\mathcal{D}_{\mathcal{T}}^{\mu_M}$ – the G-lattice of sets D type-definable in M^* over arbitrary parameters, such that $D\mu_M^{\mathcal{T}} = D$.
(2) $\mathcal{D}_{\mathcal{T}}^{\mu_M{}'}$ – the G-lattice of sets D type-definable in M^* over arbitrary parameters, such that $((\mu_M^{\mathcal{T}})^{g_1} \cap \cdots \cap (\mu_M^{\mathcal{T}})^{g_n})D = D$ for some $g_1, \dots . g_n \in G$.
(3) $\mathcal{D}_{\mathcal{T}}$ – the G-lattice of closed sets D type-definable in M^* over arbitrary parameters.
(4) $\mathcal{D}_{\mathcal{C}}$ – the G-lattice of sets D type-definable in M^* over arbitrary parameters, such that $D\mathcal{C} = D$.

By Remark 2.40, it is clear that $\mathcal{D}_{\mathcal{T}}^{\mu_M}$ and $\mathcal{D}_{\mathcal{T}}^{\mu_M{}'}$ are both contained in $\mathcal{D}_{\mathcal{T}} \subseteq \mathcal{D}_{\mathcal{C}}$.

Now, we give an example showing that type-definable subgroups lead, in a natural way, to $\bigvee$-definable group topologies.

Example 2.41. Let $H = \bigcap_{i \in I} X_i$ be any $\emptyset$-type-definable subgroup of G (and without loss I is directed, and $X_j \subseteq X_i$ whenever $i < j$). Let $\mathcal{T}$ be the union of all families $\mathcal{T}_{i,m}$, where $\mathcal{T}_{i,m}$ is the class of m-fold intersections of conjugates of X_i, for instance $\mathcal{T}_{i,1} = \{g^{-1}X_i g : g \in G\}$. It is clear that with the order $(i, m) < (j, n) \iff i < j \wedge m < n$, $\mathcal{T}$ can be treated as a $\bigvee$-definable family of definable subsets of G containing 1. Clearly, for any model M, $\bigcap \mathcal{T}_M$ is a type-definable subgroup of G normalized by $G(M)$; it follows that $\mathcal{T}(M)$ defines a group topology on $G(M)$.

In case when H is invariant under conjugation by elements of $G(M)$, we can recover H as the intersection of all M-definable neighborhoods of the identity.

All of this works also for H type-definable over M (allowing formulas with parameters from M in the definition of $\bigvee$-definable group topology).

In case H is a normal subgroup of G, the family $\mathcal{T}$ yields the same topology as the family $\{X_i : i \in I\}$ (where $X_i(x)$ are definable sets which do not depend on any parameters $\bar{y}_i$), $\mu_M = \mathcal{C} = H$ does not depend on M, and $\mathrm{cl}(P) = PH$ for any type-definable set P. In particular, $\mathcal{D}_{\mathcal{T}}^{\mu_M} = \mathcal{D}_{\mathcal{T}}^{\mu_M{}'} = \mathcal{D}_{\mathcal{T}} = \mathcal{D}_{\mathcal{C}}$ for every M.

Remark 2.42. Example 2.41 shows a connection between $\bigvee$-definable group topologies and $G(M)$-normal, M-type-definable subgroups:

- each $\bigvee$-definable group topology $\mathcal{T}$ yields the $G(M)$-normal, M-type-definable subgroup $\mu_M^{\mathcal{T}}$;
- each $G(M)$-normal, type-definable over $\emptyset$ [or over M] subgroup H yields the $\bigvee$-definable group topology $\mathcal{T}$ on G [defined over M, if G is defined over M] described in Example 2.41, such that $\mu_M^{\mathcal{T}} = H$.

However, the former notion, namely that of a $\bigvee$-definable group topology is more precise, as it is a priori given without reference to the particular small model M. Also, the map from $\bigvee$-definable group topologies to $G(M)$-normal, M-type-definable subgroups (or topologies on $G(M)$), given by $\mathcal{T} \mapsto \mu_M^{\mathcal{T}}$, is not injective; Example 2.41 provides the smallest $\bigvee$-definable group topology specializing to a given topology on $G(M)$, but there can certainly be others, e.g. in the Abelian case, non-discrete, strongly Hausdorff topologies are never deduced from a single model in this way (see also Example 2.44).

Remark 2.43. Let us change the notation only for the purpose of this remark. Let G be an arbitrary topological group. Choose a basis $\{X_i : i \in I\}$ of open sets at the identity, with $X_j \subseteq X_i$ whenever $i < j$. Expand the pure group language with predicates for all X_i's, and denote the resulting structure by M and the resulting language by $\mathcal{L}$. Let M^* be a monster model, $G^* = G(M^*)$ and $X_i^* = X_i(M^*)$. Then $H := \bigcap X_i^*$ is a $\emptyset$-type-definable group which is normalized by $G = G(M)$. So Example 2.41 yields a $\bigvee$-definable group topology $\mathcal{T}$ which specializes to the original topology on G. This is the smallest (in a strong sense) $\bigvee$-definable group topology which specializes to the original topology on G, namely, any other such topology $\mathcal{T}'$ which is $\bigvee$-definable in an expansion of the pure group structure on G whose language is denoted by $\mathcal{L}'$, and for any model $N \succ M$ in the sense of $\mathcal{L} \cup \mathcal{L}'$, the topology on $G(N)$ given by $\mathcal{T}$ is weaker than the one given by $\mathcal{T}'$. This shows that Example 2.41 allows us to extend the given group topology on G to the canonical (i.e. smallest among topologies $\bigvee$-definable in arbitrary expansions) group topology on elementary extensions.

Let us look at a concrete example illustrating some of the above discussions.

Example 2.44. Take $M := (\mathbb{Z}, +, \cdot)$ and $G(M) := (\mathbb{Z}, +)$. Take the $\emptyset$-type-definable subgroup $H := \bigcap_{n \in \mathbb{N}} n!G$. The family $\mathcal{T}$ from Example 2.41 coincides with the $\bigvee$-definable family $\{n!G : n \in \mathbb{Z}\}$. So $\mathcal{T}(M)$ is Hausdorff, but $\mathcal{T}$ is not strongly Hausdorff. Now, consider the definable family $\mathcal{F} = \{g \cdot G : g \in G\}$ of definable subsets of G containing 0. It is clear that the family $\mathcal{T}'$ of finite intersections of members of $\mathcal{F}$ is a strongly Hausdorff $\bigvee$-definable group topology on G, and $\mathcal{T}(M)$ and $\mathcal{T}'(M)$ induce on $G(M)$ the same topology. But for every $\aleph_0$-saturated model M, the topologies $\mathcal{T}(M)$ and $\mathcal{T}'(M)$ on $G(M)$ are different (as the later is Hausdorff, but the former is not).

We return to the general context where $\mathcal{T} = \{\varphi(x, \bar{z}_i) : i \in I, \bar{z}_i\}$ is a $\bigvee$-definable group topology on G. Recall that the group $G(M)$, with the topology induced by $\mathcal{T}(M)$, is said to be definably topologically amenable if there is a (left) $G(M)$-invariant, Borel probability measure on $S_{\mu_M\backslash G}(M)$ (equivalently, on $S_G^{\mu_M}(M)$). A natural question arises, whether the definable topological amenability of $(G(M), \mathcal{T}(M))$ is independent of the choice of M. The positive answer follows from Corollary 2.31 applied to the family $\mathcal{E} := \{G(x) \wedge G(y) \wedge \varphi_i(yx^{-1}, \bar{z}_i) : i \in I, \bar{z}_i\}$; similarly, applying Corollary 2.31 to the family $\mathcal{E} := \{G(x) \wedge G(y) \wedge \varphi_i(x^{-1}y, \bar{z}_i) : i \in I, \bar{z}_i\}$, we get item (2) of the following corollary.

Corollary 2.45. *Let $\mathcal{T}$ be a $\bigvee$-definable group topology on G. Then:*

(1) the definable topological amenability of $(G(M), \mathcal{T}(M))$ does not depend on the choice of M,

(2) the existence of a $G(M)$-invariant, Borel probability measure on $S_{G/\mu_M}(M)$ does not depend on the choice of M.

But the following question remains open.

Question 2.46. *1) Let $\mathcal{T}$ be a $\bigvee$-definable group topology on G. Does amenability of $(G(M), \mathcal{T}(M))$ as a topological group depend on the choice of M?*
2) Let G be an arbitrary topological group. Let $\mathcal{T}$ be the smallest topology among topologies $\bigvee$-definable in expansions of $(G, \cdot)$ which specialize to the given topology on G (see Remark 2.43). Does amenability of G (as a topological group) imply amenability of $(G(N), \mathcal{T}(N))$ for $N \succ M$ (where $\mathcal{T}$ is defined in $\mathrm{Th}(M)$).

It is clear that the positive answer to (1) implies the positive answer to (2). Question (2) is interesting, as it asks whether there is any chance to transfer (topological) amenability to elementary extensions. (Note that whenever we have two group topologies $\mathcal{T}_1 \subseteq \mathcal{T}_2$ on a given group G, then amenability of $(G, \mathcal{T}_2)$ implies amenability of $(G, \mathcal{T}_1)$.)

Still, $\mathcal{T} = \{\varphi(x, \bar{z}_i) : i \in I, \bar{z}_i\}$ is a $\bigvee$-definable group topology on G.

Definition 2.47. 1) A *right pre-mean* for $\mathcal{T}_M$ is a G-pre-mean for $\mathcal{E}_M$ (in the sense of Definition 2.23) with $\mathcal{E} := \{G(x) \wedge G(y) \wedge \varphi_i(x^{-1}y, \bar{z}_i) : i \in I, \bar{z}_i\}$. Explicitly, it is a monotone function m on definable subsets of G into $[0, 1]$, with $m(\emptyset) = 0, m(G) = 1$, and $m(Y \cup Y') \leq m(Y) + m(Y')$, such that equality holds whenever $YD \cap Y' = \emptyset$ for some $D \in \mathcal{T}_M$.
2) A *left G-pre-mean* for $\mathcal{T}_M$ is a G-pre-mean for $\mathcal{E}_M$ (in the sense of Definition 2.23) with $\mathcal{E} := \{G(x) \wedge G(y) \wedge \varphi_i(yx^{-1}, \bar{z}_i) : i \in I, \bar{z}_i\}$. Explicitly, it is a monotone function m on definable subsets of G into $[0, 1]$, with $m(\emptyset) = 0, m(G) = 1$, and $m(Y \cup Y') \leq m(Y) + m(Y')$, such that equality holds whenever $(D^{g_1} \cap \cdots \cap D^{g_n})Y \cap Y' = \emptyset$ for some $D \in \mathcal{T}_M$ and $g_1, \ldots, g_n \in G$.

Then Proposition 2.29 specializes to the following two statements.

Corollary 2.48. *The following conditions are equivalent.*

(1) *$S_{G/\mu_M}(M)$ carries a $G(M)$-invariant, Borel probability measure.*
(2) *There is a $G(M)$-invariant [G-invariant] right pre-mean for $\mathcal{T}_M$.*
(3) *The lattice $\mathcal{D}_{\mathcal{T}}^{\mu_M}$ carries a $G(M)$-invariant [G-invariant], normalized mean.*

Corollary 2.49. *The following conditions are equivalent.*

(1) *$(G(M), \mathcal{T}(M))$ is definably topologically amenable (i.e. $S_{\mu_M\backslash G}(M)$ carries a $G(M)$-invariant, Borel probability measure).*
(2) *There is a $G(M)$-invariant [G-invariant] left G-pre-mean for $\mathcal{T}_M$.*
(3) *The lattice $\mathcal{D}_{\mathcal{T}}^{\mu_M}{}'$ carries a $G(M)$-invariant [G-invariant], normalized mean.*

By Corollary 2.28, we get that the existence of a left-invariant mean on $\mathcal{D}_{\mathcal{T}}^{\mu_M}$ [or on $\mathcal{D}_{\mathcal{T}}^{\mu_M}{}'$, respectively] is independent of the choice of both M and M^*. Similarly, the existence of an invariant mean on $\mathcal{D}_{\mathcal{C}}$ is independent of the choice of M^*. A question is whether the existence of an invariant mean on $\mathcal{D}_{\mathcal{T}}$ depends on the choice of M^*.

Along with Remark 2.40, Corollaries 2.48 and 2.49 seem to suggest that one can get (from amenability) a G-invariant, normalized mean on the lattice $\mathcal{D}_{\mathcal{T}}$ of closed, type-definable sets; but we do not quite see this. It is certainly not true about $\mathcal{D}_{\mathcal{C}}$: to see this, it is enough to take an amenable (as a topological group) but not definably amenable group $G(M)$ such that G is strongly Hausdorff (as then $\mathcal{D}_{\mathcal{C}}$ consists of all type-definable subsets of $G = G(M^*)$, so the restriction of an invariant mean defined on $\mathcal{D}_{\mathcal{C}}$ to the algebra of all definable subsets would be a left-invariant Keisler measure, contradicting the failure of definable amenability).

The following is a corollary of the proofs of Theorems 2.34 and 2.35 applied for $H := \mu_M$; the set $\hat{D}$ from the conclusion below will be $p^4 := ppp^{-1}p^{-1}$ for a type $p \in S_G(M)$ which is wide in the sense that $m(D\mu_M) > 0$ [resp. $m(\mu_M D) > 0$] for every $D \in p$.

Corollary 2.50. *Let $\mathcal{T}$ be a $\bigvee$-definable group topology. Assume $\mathcal{D}_{\mathcal{T}}^{\mu_M}$ [or $\mathcal{D}_{\mathcal{T}}^{\mu_M}{}'$, respectively] carries a G-invariant mean m. Then $G_M^{00} \leq G_M^{000}\langle \mu_M^G \rangle$. More precisely, there exists an M-type-definable set $\widehat{D} \subseteq G_M^{000}$, with $\widehat{D}\langle \mu_M^G \rangle \supseteq G_M^{00}$. In fact, for any wide, M-type-definable set $P = \mu_M P \mu_M$ we have $P^4 := PPP^{-1}P^{-1} \supseteq G_M^{00}$.*

The main result of this subsection is the the following

Proposition 2.51. *Let $\mathcal{T}$ be a $\bigvee$-definable group topology such that for all $n \in \mathbb{N}$ the projections (to all subproducts) of every type-definable, closed set in G^n are closed. Assume $\mathcal{D}_{\mathcal{T}}$ carries a G-invariant mean m. Then $\mathrm{cl}(G_M^{00}) = \mathrm{cl}(G_M^{000})$. More precisely, there exists an M-type-definable set $\widehat{D} \subseteq G_M^{000}$, with $\mathrm{cl}(\widehat{D}) = \mathrm{cl}(G_M^{00})$. In fact, for any closed, wide (i.e. of positive mean), M-type-definable set P we have $P^4 := PPP^{-1}P^{-1} \supseteq G_M^{00}$.*

Proof. We start from

Claim 1: i) The product of any two closed, type-definable sets is always closed (and clearly type-definable).

ii) For all type-definable sets P and Q, $\mathrm{cl}(\mathrm{cl}(P) \cdot \mathrm{cl}(Q)) = \mathrm{cl}(P \cdot Q)$.
iii) For all type-definable sets P and Q, $\mathrm{cl}(P) \cdot \mathrm{cl}(Q) = \mathrm{cl}(P \cdot Q)$.
iv) For every type-definable set $P = \bigcap P_i$ (where $P_j \subseteq P_i$ whenever $i < j$), $\mathrm{cl}(P) = \bigcap_i \mathrm{cl}(P_i)$.

Proof. i) This would follow immediately from the assumption that projections of closed, type-definable sets are closed if the topology induced by $\mathcal{T}$ on $G = G(M^*)$ was Hausdorff. But if it is not Hausdorff, we can always pass to the Hausdorff quotient $G/\mathcal{C}$, where $\mathcal{C} = \mathrm{cl}(1)$. Working with $G/\mathcal{C}$ in place of G, we still have that projections of closed, type-definable sets are closed, so the product of any two closed, type-definable subsets of $G/\mathcal{C}$ is closed. Now, take any two closed, type-definable subsets P and Q of G. Then $P = P\mathcal{C}$ and $Q = Q\mathcal{C}$. So $P/\mathcal{C}$ and $Q/\mathcal{C}$ are closed, type-definable subsets of $G/\mathcal{C}$, and so $PQ/\mathcal{C} = P/\mathcal{C} \cdot Q/\mathcal{C}$ is closed in $G/\mathcal{C}$, hence PQ is closed in G.

ii) is a general property of all topological groups.

iii) Using (ii), we immediately see that (iii) is equivalent to (i).

iv) follows from Remark 2.39. □(claim)

Claim 2: For any closed, M-type-definable set P with $m(P) > 0$, there exists a generic, closed set Q type-definable over some parameters and such that $Q^8 \subseteq P^4$.

Proof. We use Proposition 2.10, with $G = X$ the present $G/\mathcal{C}$ (where $\mathcal{C} = \mathrm{cl}(1)$), $A = B = P/\mathcal{C}$, $N = 8$, $\mathcal{D}$ being the lattice of closed, type-definable subsets of $G/\mathcal{C}$, and $m = m'$ being the pushforward of the mean m from the statement of Proposition 2.51. (Item (i) of the first claim is used to ensure that the assumptions of Proposition 2.10 hold.) So there exists a generic, symmetric $\bar{Q} \subseteq G/\mathcal{C}$ positively definable in $(G/C, \cdot, P/\mathcal{C})$, and with $\bar{Q}^8 \subseteq (P/\mathcal{C})^4$. By the assumption that projections of closed, type-definable sets are closed (and the fact that $G/\mathcal{C}$ is Hausdorff), it follows that $\bar{Q}$ is closed and type-definable in the original structure M^*. So the pullback Q of $\bar{Q}$ by the quotient map $G \to G/\mathcal{C}$ is also generic, closed, type-definable, and $Q^8 \subseteq P^4$. □(claim)

Since we are going to deal with G^{00}_M, we need to be more careful about parameters, and force Q to be defined over M.

First, we will prove the last statement of Proposition 2.51, and then we will quickly explain how to deduce the previous one.

So take any closed, wide, M-type-definable set P (where wide means that $m(P) > 0$). Consider any M-definable set P' containing P^4.

By the first and last item of the first claim, we can find an M-definable set P'' such that $P^4 \subseteq P'' \subseteq \mathrm{cl}(P'') \subseteq P'$. Let Q be a set provided by the second claim. We can find an M-definable, generic set Q_0 such that $Q_0^8 \subseteq P''$, and so, by item (iii) of the first claim, $\mathrm{cl}(Q_0)^8 = \mathrm{cl}(Q_0^8) \subseteq \mathrm{cl}(P'') \subseteq P'$. By the last item of the first claim, we can find an M-definable set Q_1 such that $\mathrm{cl}(Q_0) \subseteq Q_1$ and $\mathrm{cl}(Q_1)^8 \subseteq P'$.

Put

$$C_1 := \mathrm{cl}(Q_1)^4.$$

Now, apply the above argument to $\mathrm{cl}(Q_0)$ (which is M-type-definable by Remark 2.39) in place of P, and Q_1^4 in place of P'. As a result, we obtain M-definable, generic sets R_0 and R_1 such that $\mathrm{cl}(R_0) \subseteq R_1$ and $\mathrm{cl}(R_1)^8 \subseteq Q_1^4$. Put

$$C_2 := \mathrm{cl}(R_1)^4.$$

Continuing in this way, we obtain a sequence $C_1, C_2, \dots$ of M-type-definable, generic and symmetric subsets of P' such that $C_{i+1}^2 \subseteq C_i$ for all i. Then $\bigcap_i C_i$ is a bounded index, M-type-definable subgroup contained in P'. Therefore, $G_M^{00} \subseteq P'$. Since P' was an arbitrary M-definable set containing P^4, we conclude that $G_M^{00} \subseteq P^4$, which is the desired conclusion.

Let us prove now the existence of $\widehat{D}$. Let p be a wide type of G over M, in the sense that $m(\mathrm{cl}(D)) > 0$ for any $D \in p$. Let $P = \mathrm{cl}(p)$. By the last item of the first claim, and by what we have just proved, we get that P^4 contains G_M^{00}. Put $\widehat{D} := p^4$. It is clearly contained in G_M^{000}. On the other hand, by item (iii) of the first claim, $\mathrm{cl}(\widehat{D}) = \mathrm{cl}(p^4) = \mathrm{cl}(p)^4 = P^4 \supseteq G_M^{00}$. □

Remark 2.52. The assumption in Proposition 2.51 that the projections of closed, type-definable sets are closed may seem a bit artificial, perhaps it can be changed. At any rate, it holds in each of the following two situations.

(1) The situation from the last paragraph of Example 2.41, namely: $H = \bigcap_{i \in I} X_i$ is a normal, type-definable subgroup of G (and without loss $X_j \subseteq X_i$ when $i < j$), and $\mathcal{T} := \{X_i : i \in I\}$.
(2) $\mathcal{T}$ is a definable family and $(G(M), \mathcal{T}(M))$ is compact, Hausdorff for some model M.

Proof. (1) follows from the observation that $F \subseteq G^n$ is closed if and only if $F = F \cdot \mathcal{C}^n$, where $\mathcal{C} = \mathrm{cl}(1)$.
(2) By the compactness and Hausdorffness of $G(M)$, the projections of any closed subset of $G(M)^n$ are closed. Thus, since $\mathcal{T} = \{\varphi(x, \bar{y}) : \bar{y}\}$ is a definable family, we easily get that the projections of any closed and definable subset F of G^n are closed. On the other hand, for any type-definable, closed set $F = \bigcap F_i \subseteq G^n$ (where $F_j \subseteq F_i$ whenever $i < j$), using the last item of the first claim of the proof of Proposition 2.51, we get that $F = \bigcap_i \mathrm{cl}(F_i)$ and each $\mathrm{cl}(F_i)$ is definable (by the definability of $\mathcal{T}$), and, by compactness, any projection of F is the intersection of the projections of the $\mathrm{cl}(F_i)$'s. So the conclusion follows. □

By virtue of Remark 2.52(1), the following obvious corollary of Theorem 2.34 also follows from Proposition 2.51.

Corollary 2.53. *Let N be any normal, $\emptyset$-type-definable subgroup of G. Assume the lattice $\mathcal{D}_N$ (of type-definable subsets Y of G such that $YN = N$) carries a G-invariant mean. Then $G_M^{00} \leq NG_M^{000}$.*

3. Definable actions, weakly almost periodic actions, and stability

One aim of this section is to give a negative answer to Conjecture 0.3 of [27] about *definable actions* of definable groups on compact spaces: see Corollary 3.3 below. But we go rather beyond this, discussing the relationships between the notions in the title of the section. Weakly almost periodic actions (or flows) of a (topological) group G on a compact space X are important in topological dynamics. Weak almost periodicity (for functions on a topological group) was introduced in [9], and discussed later in [14]. We will be referring to [10] where weak almost periodicity of G-flows is defined and studied. The connection of weak almost periodicity with stability is by now fairly well-known, although much of what is in print or published, such as [4] and [18], deals with the case where the relevant group G is the (topological) automorphism group of a countable ω-categorical structure. In contrast, we are here concerned with an action of a group $G(M)$ definable in a structure M on a compact space X where $G(M)$ is viewed as a discrete group, but where the action on X is assumed to factor through the action of $G(M)$ on its space $S_G(M)$ of types over M.

We will give some background below on both continuous logic (in an appropriate form) and weak almost periodicity. The connection between stability in continuous logic and weak almost periodicity goes through results of Grothendieck [14] in functional analysis, which have been commented on in several expository papers such as [5] and later [34]. However, it is relative stability, namely stability of a formula *in a model* M which is relevant, and only equivalent to stability when the model is saturated enough.

One of our main structural results is Theorem 3.16 below characterizing when the action of $G(M)$ on X is weakly almost periodic in terms of stable in M formulas. When M is ω_1-saturated, another equivalent condition is that the action of $G(M)$ on X is *definable*, which will yield the desired conclusions (Theorem 3.2 and Corollary 3.3).

Although this is a model theory paper, it is convenient for us to quote heavily from the topological dynamics literature, especially for results which have not yet been developed in the parallel model-theoretic environment.

We will generally be assuming any ambient theory T to be countable.

The notion of a definable action of a definable group on a compact space was given in [12] and explored in some degree of generality in [23]. We repeat the definition.

Definition 3.1. (i) Let X be a set definable over M. A function f from $X(M)$ to a compact space C is said to be *definable* if if for every pair C_1, C_2 of closed disjoint subsets of C, there is a definable (in M) set $Z \subseteq X(M)$ such that $f^{-1}(C_1) \subseteq Z$, and $f^{-1}(C_2) \subseteq G(M) \setminus Z$.
(ii) Suppose G is a group definable over M. A group action by $G(M)$ on a compact

space X by homeomorphisms is said to be *definable* if for every $x \in X$ the map from G to X taking g to $g \cdot x$ is definable.

When all types over M are definable, then the natural action of $G(M)$ on $S_G(M)$ is a definable action and is moreover the universal definable $G(M)$-ambit (see [12]). This is interesting for structures M such as the reals or p-adics where all types over M are definable, although the complete theories are unstable. However, in general, definability of an action of $G(M)$ on a compact space X is a rather restrictive condition. In [23], it was shown that there is always a universal definable $G(M)$-ambit (which will of course factor through $S_G(M)$). As in [27], $G(M)$ is said to be *weakly definably amenable* iff whenever $G(M)$ acts definably on the compact space X, then X supports a $G(M)$-invariant Borel probability measure, equivalently the universal definable $G(M)$ ambit supports a $G(M)$-invariant Borel probability measure. In [27], it was conjectured that if $G(M)$ is weakly definably amenable, then $G^{00}_M = G^{000}_M$. We will show here that this fails drastically, by proving that when M is sufficiently saturated, then $G(M)$ is *always* weakly definably amenable.

Theorem 3.2. *Suppose M is ω_1-saturated. Then $G(M)$ is weakly definably amenable: for any any definable action of $G(M)$ on a compact space X, X supports a $G(M)$-invariant Borel probability measure.*

We deduce a negative answer to Conjecture 0.3 of [27]:

Corollary 3.3. *There is a model M, and a group $G(M)$ definable in M such that $G(M)$ is weakly definably amenable but $G^{00}_M \neq G^{000}_M$.*

Proof. In fact, whenever G is a group definable in a NIP theory T and $G^{00} \neq G^{000}$, then choosing an ω_1-saturated model M of T, we see from Theorem 3.2 that $G(M)$ is weakly definably amenable. Moreover $G^{00}_M = G^{00} \neq G^{000} = G^{000}_M$. There are many such examples, such as from [8]: T is the theory of the 2-sorted structure M with sorts $(\mathbb{R}, +, \times)$ and $(\mathbb{Z}, +)$ and no additional structure. As pointed out there, the universal cover of $\mathrm{SL}(2, \mathbb{R})$ is naturally definable in M. T is NIP, and if G is the interpretation of this group in a saturated model, then $G^{00} \neq G^{000}$. □

3.1. **Continuous logic.** Continuous logic is about real-valued relations and formulas, or, more generally, formulas with values in compact spaces, and, as such, is present in a lot of recent work which does not explicitly mention continuous logic (even in Definition 3.1 above).

There have been various approaches to continuous logic, starting with [7]. An attractive formalism was developed in [2] and [3], and our set up will be a special case. In this section, we will give relatively self-contained proofs, for reasons explained below. In Subsection 4.3, we will again make use of continuous logic, but be more precise about the relations between our set-up and [3].

T will be a complete first order theory in the usual (non-continuous) sense, which is countable (for convenience) and we work as earlier in a big saturated (or monster) model $\mathfrak{C}$. We fix a sort X (which will be a definable group G in the applications). As usual $M, N, \ldots$ denote small elementary submodels of $\mathfrak{C}$, and $A, B, \ldots$ small subsets of this monster model. There is no harm assuming that $T = T^{eq}$.

Definition 3.4. (i) By a *continuous logic (CL) formula on X over A*, we mean a continuous function $\phi\colon S_X(A) \to \mathbb{R}$.
(ii) If ϕ is such a CL-formula, then for any $b \in X$ (in the monster model) by $\phi(b)$ we mean $\phi(\mathrm{tp}(b/A))$. Hence, we have a map $\phi\colon X(N) \to \mathbb{R}$ for all models N, in particular a map $\phi\colon X = X(\mathfrak{C}) \to \mathbb{R}$. As the notation suggests, we are identifying a CL-formula on X over A with the latter map, and so may write it as $\phi(x)$ where x is a variable of sort X.
(iii) We consider two such CL-formulas on X, ϕ, ψ, over sets A, B, respectively to be *equivalent* if they agree in the sense of (ii), namely if for all $a \in X$, $\phi(a) = \psi(a)$.

Remark 3.5. (i) The range of any CL-formula is a compact subset of $\mathbb{R}$.
(ii) A CL-formula ϕ (over some A) is equivalent to a CL-formula over B if ϕ is invariant under automorphisms of the monster model which fix B pointwise.
(iii) The maps $\phi\colon X(M) \to \mathbb{R}$ given by CL-formulas ϕ over M are precisely the *definable* maps from $X(M)$ to $\mathbb{R}$ in the sense of Definition 3.1.
(iv) Any CL-formula over a set A is (equivalent to a CL-formula) over a countable subset of A.

Definition 3.6. (i) Let M be a model, and $\phi(x, y)$ a CL-formula over M, where x, y are variables of sorts X, Y, respectively. Let $a \in X$. Then $\mathrm{tp}_\phi(a/M)$ is the function taking $b \in Y(M)$ to $\phi(a, b)$, and is called a *complete $\phi(x, y)$-type over M*.
(ii) In the context of (i), $\mathrm{tp}_\phi(a/M)$ is *definable* if it is definable in the sense of Definition 3.1, equivalently, by Remark 3.5(iii), given by or rather induced by a CL-formula on Y over M.

Remark 3.7. Suppose M is ω_1-saturated, $\phi(x, y)$ is a CL-formula over M, and a is in the big model. Then $\mathrm{tp}_\phi(a/M)$ is definable if and only if for each closed subset C of $\mathbb{R}$, $\{b \in Y(M) : \phi(a, b) \in C\}$ is type-definable over some countable subset of M.

Proof. This follows from Remark 3.5(iv). $\square$

Definition 3.8. Let $\phi(x, y)$ be a CL-formula over M.
(i) We say that $\phi(x, y)$ is *stable* (for the theory T) if for all $\epsilon > 0$ there do not exist a_i, b_i for $i < \omega$ (in the monster model) such that for all $i < j$, $|\phi(a_i, b_j) - \phi(a_j, b_i)| \geq \epsilon$.
(ii) We say that $\phi(x, y)$ is *stable in M* if for all $\epsilon > 0$ there do not exist a_i, b_i for $i < \omega$ *in M* such that for all $i < j$, $|\phi(a_i, b_j) - \phi(a_j, b_i)| \geq \epsilon$

Remark 3.9. (i) Routine methods show that $\phi(x, y)$ is stable (for T) iff whenever $(a_i, b_i)_{i<\omega}$ is indiscernible (over M), then $\phi(a_i, b_j) = \phi(a_j, b_i)$ for $i < j$.

(ii) On the other hand, stability of $\phi(x, y)$ in M is easily seen to be equivalent to Grothendieck's double limit condition: given a_i, b_i in M for $i < \omega$ we have that $\lim_i \lim_j \phi(a_i, b_j) = \lim_j \lim_i \phi(a_i, b_j)$ if both double limits exist.
(iii) A CL-formula $\phi(x, y)$ is stable [in M] iff $\phi^{op}(x, y) := \phi(y, x)$ is stable [in M].

The following is due to Grothendieck (modulo a routine translation), and we give an explanation below.

Proposition 3.10. *Let $\phi(x, y)$ be a CL-formula over M. Then the following are equivalent.*

(i) $\phi(x, y)$ is stable in M.
(ii) Whenever $M \prec M^*$, and $\operatorname{tp}(a/M^*) \in S_x(M^*)$ is finitely satisfiable in M, then $\operatorname{tp}_\phi(a/M^*)$ is definable over M, namely the function taking $b \in M^*$ to $\phi(a, b)$ is given by a CL-formula $\psi(y)$ over M.

Proof. Consider the (compact) space $Z = S_y(M)$ of complete types over M in variable y, and let $C(Z)$ be the real Banach space of continuous real valued functions on Z. Let A denote the subset of $C(Z)$ consisting of the functions $\phi(a, y)$ for $a \in M$. Note that A is bounded. Let Z_0 be the set of realized types, namely those $\operatorname{tp}(b/M)$ for $b \in M$, a dense subset of Z. With this notation, Grothendieck's Theorem 6 in [14] says that the following are equivalent.

(i)' If $f_i \in A$ and $q_i \in Z_0$ for $i < \omega$, then $\lim_i \lim_j f_i(q_j) = \lim_j \lim_i f_i(q_j)$ if both double limits exist.
(ii)' The closure of A in the pointwise convergence topology on $C(Z)$ is compact.

Now, if f_i is $\phi(a_i, y)$ and $q_i = \operatorname{tp}(b_i/M)$, then (i)' says precisely that $\lim_i \lim_j \phi(a_i, b_j) = \lim_j \lim_i \phi(a_i.b_j)$ for all sequences $a_i, b_i \in M$ with $i < \omega$ for which both double limits exist, which by Remark 3.9(ii) says that $\phi(x, y)$ is stable in M, namely condition (i) in the proposition.

On the other hand (ii)' implies that the closure of A in $C(Z)$ (in the pointwise topology) is a compact, so closed, subset of the space $\mathbb{R}^Z$ of *all* functions from Z to $\mathbb{R}$ (equipped with the pointwise, equivalently Tychonoff topology). So every function in the closure of A in $\mathbb{R}^Z$ is already in $C(Z)$, so is continuous. So it is clear that (ii)' is equivalent to

(ii)" whenever $f \in \mathbb{R}^Z$ is in the closure of A, then f is continuous.

It is now easy to see that if $f \in \mathbb{R}^Z$ is in the closure of $\{\phi(a, y) : a \in M\}$, then f is of the form $\phi(a^*, y)$, where M^* is a saturated model containing M, and $\operatorname{tp}(a^*/M^*)$ is finitely satisfiable in M. So for $q \in Z = S_y(M)$, $f(q) = \phi(a^*, b)$ for some (any) realization b of q in M^*. The continuity of f means that it is given by a CL-formula $\psi(y)$ over M, which precisely means that $\psi(y)$ is a definition over M of $\operatorname{tp}_\phi(a^*/M^*)$. So we get that (ii) implies (ii)", and it is again easy to see that they are equivalent. ☐

Remark 3.11. (a) Actually the original statement of (ii)' in [14] is that the closure of A in the *weak* topology on $C(Z)$ is compact. The weak topology

on $C(Z)$ is the one whose basic open neighbourhoods of a point f_0 are of the form $\{f \in C(Z) : |g_1(f - f_0)| < \epsilon, \ldots, |g_r(f - f_0)| < \epsilon\}$, where $g_1, \ldots, g_r$ are in $L(C(Z), \mathbb{R})$ — the space of bounded linear functions on $C(Z)$. This so-called weak topology is stronger than the pointwise convergence topology on $C(Z)$ whose basic open neighbourhoods of a point f_0 are as above but where g_i is evaluation at some point $x_i \in Z$. It is pointed out in [14] that relative compactness of a bounded subset A of $C(Z)$ in the weak topology is equivalent to relative compactness of A in the pointwise convergence topology, yielding the statement (ii)' in the proof of Proposition 3.10.
(b) In [5], which seems to be the first model theory paper to recognize Grothendieck's contribution, only the implication "$\phi(x, y)$ stable in M implies that all ϕ-types over M are definable" is deduced from Grothendieck's theorem, rather than the stronger equivalence in Proposition 3.10.
(c) Grothendieck's proof in [14] is basically a model theory proof. See [34] for the case of classical ($\{0, 1\}$-valued) formulas.

Proposition 3.12. *The CL-formula $\phi(x, y)$ is stable (for T) if and only if every complete $\phi(x, y)$-type over any model over which ϕ is defined is definable.*

Proof. In the more general metric structures formalism, this appears in [3] (Proposition 7.7 there) and adapts to our context. However, we give a relatively self contained account. Left implies right is given by Proposition 3.10. The other direction is the easy one and can be seen as follows. Assume $\phi(x, y)$ to be unstable (for a contradiction). By (or as in) Remark 3.9, we can find $a_i, b_i \in \mathfrak{C}$ for $i \in \mathbb{Q}$, and real numbers $r < s$ that $\phi(a_i, b_j) \leq r$ for $i < j$ and $\phi(a_i, b_j) \geq s$ for $i > j$. Let M be a countable model containing the b_i for $i \in \mathbb{Q}$. By compactness, for each cut C in $\mathbb{Q}$ there is some $a_C \in \mathfrak{C}$ such that $\phi(a_C, b_j) \geq s$ for $j < C$ and $\phi(a_C, b_j) \leq s$ for $j > C$. Now, by assumption, each $\mathrm{tp}_\phi(a_C/M)$ is definable, so for each C there is some (ordinary) formula $\psi_C(y)$ over M such that for any $b \in M$, $\phi(a_C, b) \leq r$ implies $\psi_C(b)$, and $\phi(a_C, b) \geq s$ implies $\neg\psi_C(b)$. This is a contradiction, as there are continuum many distinct C's but only countably many (ordinary) formulas over the countable model M. $\square$

Proposition 3.13. *Suppose M is ω_1-saturated, $\phi(x, y)$ is a CL-formula over M, and every complete $\phi(x, y)$-type over M is definable. Then every complete $\phi(x, y)$-type over any model N (over which ϕ is defined) is definable, and hence, by Proposition 3.10, $\phi(x, y)$ is stable.*

Proof. Let $A \subset M$ be countable such that $\phi(x, y)$ is over A. It suffices to prove that every complete ϕ-type over a countable model containing A is definable. As M is ω_1-saturated, it is enough to prove that every complete ϕ-type over any countable submodel M_0 of M which contains A is definable. So let $p(x)$ and M_0 be such. Let p' be a coheir of p over M, namely $p' = \mathrm{tp}_\phi(a/M)$, $p = p'|M_0$, and $\mathrm{tp}(a/M)$ is finitely satisfiable in M_0. By our assumptions, p' is definable. So to prove that p is definable it suffices to prove:

Claim 1: p' is definable over M_0.

Proof. Let C be a compact subset of $\mathbb{R}$, and let $\Psi(y, b)$ be a partial type over a countable sequence b from M such that for all $c \in M$, $\phi(a, c) \in C$ iff $M \models \Psi(c, b)$. We will show that in fact $\Psi(y, b)$ is equivalent to a partial type over M_0. For this it is enough to show that if b' realizes $\mathrm{tp}(b/M_0)$ in M, then $\Psi(y, b')$ is equivalent to $\Psi(y, b)$.

Suppose $c' \in M$ and suppose $M \models \Psi(c', b')$. Let $c \in M$ be such that $\mathrm{tp}(c, b/M_0) = \mathrm{tp}(c', b'/M_0)$. As $\mathrm{tp}(a/M)$ is finitely satisfiable in M_0, $\phi(a, c) = \phi(a, c')$. As $M \models \Psi(c, b)$, we have that $\phi(a, c) \in C$. Hence, $\phi(a, c') \in C$, whereby $M \models \Psi(c', b)$. Hence, $\Psi(y, b')$ is equivalent to $\Psi(y, b)$, as required. This finishes the proof of the claim. □(claim)

Hence, the proof of the proposition is also finished. □

3.2. **Weakly almost periodic actions.** The context here is a G-flow (X, G), where X is a compact space and G a topological group. For f a continuous function from X to $\mathbb{R}$ and $g \in G$, gf denotes the (continuous) function taking $x \in X$ to $f(gx)$. We will take our definition of a weakly almost periodic G-flow from Theorem II.1 of [10].

Definition 3.14. (i) A continuous function $f : X \to \mathbb{R}$ is *weakly almost periodic* (or wap) if whenever $h \in \mathbb{R}^X$ is in the closure of $\{gf : g \in G\}$ (in the pointwise convergence topology) then h is continuous.
(ii) The G-flow (X, G) is *weakly almost periodic* (or wap), if every continuous function $f : X \to \mathbb{R}$ is weakly almost periodic.

Fact 3.15. *Suppose that (X, G) is wap. Then there is a G-invariant Borel probability measure on X.*

Proof. This is well-known within topological dynamics, but we nevertheless give an account with some references. We may assume that (X, G) is minimal (by passing to a minimal subflow). By Proposition II.8 of [10], the flow (X, G) is almost periodic (also known as equicontinuous). The minimal equicontinuous flows have been classified in [1] for example (see [1, Chapter 3, Theorem 6]), as homogeneous spaces for compact groups (on which G acts as subgroups of the compact groups in question), whereby the Haar measure induces the required G-invariant measure on X. □

We now pass to the model-theoretic context, which here means that we consider actions of a definable group $G(M)$ on a compact space X which factor through $S_G(M)$.

Theorem 3.16. *Let M be a structure, $G(M)$ a group definable in M, and let a $G(M)$-flow $(X, G(M))$ be given, which factors through the action of $G(M)$ on*

$S_G(M)$ via a continuous surjective ($G(M)$-equivariant) map $\pi\colon S_G(M) \to X$. Consider the following three conditions:

(i) $(X, G(M))$ is wap,
(ii) for each continuous function $F\colon S_G(M) \to \mathbb{R}$ of the form $f \circ \pi$ for $f\colon X \to \mathbb{R}$ continuous, the CL-formula $F(yx)$ is stable in M,
(iii) the action of $G(M)$ on X is definable.

Then:

(a) (i) and (ii) are equivalent, and imply (iii),
(b) if M is ω_1-saturated, then (i), (ii), (iii) are equivalent, and, moreover, in (ii) we have that $F(yx)$ is stable for T (not just in M).

Proof. (a) Suppose $(X, G(M))$ is wap. Let $F = f \circ \pi$ for some $f \in C(X)$. Let $h\colon S_G(M) \to \mathbb{R}$ be in the pointwise closure of $\{gF : g \in G(M)\}$. Then clearly for any $p \in S_G(M)$, $h(p)$ depends only on $\pi(p)$, so $h = h_1 \circ \pi$ for a unique $h_1\colon X \to \mathbb{R}$. But h_1 is in the pointwise closure of $\{gf : g \in G(M)\}$, so, by assumption, h_1 is continuous. Hence, h is continuous. By the proof of Proposition 3.10, or, more precisely, the equivalence of (i) and (ii)" in there, the CL-formula $F(xy)$ is stable in M, and so is $F(yx)$ by Remark 3.9(iii).

The converse goes the same way: Let $f \in C(X)$, and $h\colon X \to \mathbb{R}$ be in the closure, again in the pointwise topology, of $\{gf : g \in G(M)\}$. Let $F = f \circ \pi \in C(S_G(M))$. Let $h_1 = h \circ \pi$. Then clearly h_1 is in the closure of $\{gF : g \in G(M)\}$. As $F(yx)$ is assumed to be stable in M, by Proposition 3.10 (or rather its proof), h_1 is continuous, so clearly h is continuous.

So far we have shown (i) if and only if (ii). We now show that either of these equivalent conditions imply that the action of $G(M)$ on X is definable. Let $x_0 \in X$.

Claim 1: For any continuous function $f\colon X \to \mathbb{R}$, the function from $G(M) \to \mathbb{R}$ taking g to $f(gx_0)$ is definable (over M).

Proof. Let $p \in S_G(M)$ be such that $\pi(p) = x_0$. Consider the lift F of f to $S_G(M)$ via π. We use x, y to denote variables of sort G. By (ii), the formula $F(yx)$ (in variables x, y) is stable in M, so, by Proposition 3.10, the function taking $g \in G(M)$ to $F(gp)$ is definable over M, namely induced by a CL-formula $\psi(y)$ over M. But $F(gp) = f(gx_0)$. Hence, the claim is proved. □(claim)

Definability of the action of $G(M)$ on X now follows from the claim and Urysohn's lemma: Let X_0, X_1 be disjoint closed subsets of X. By Urysohn, there is a continuous function $f \in C(X)$ such that f is 0 on X_0 and 1 on X_1. By the claim, there is some definable (in M) subset Z of $G(M)$, such that for all $g \in G(M)$, if $f(gx_0) = 0$ then $g \in Z$, and if $f(gx_0) = 1$ then $g \notin Z$. But this implies that if $gx_0 \in X_0$ then $g \in Z$, and if $gx_0 \in X_1$ then $g \notin Z$. As $x_0 \in X$ was arbitrary, this shows that the action of $G(M)$ on X is definable.

(b) We assume now that M is ω_1-saturated. All we have to do is to prove that (iii) implies the stronger version of (ii) (with stability for T). Now, exactly as in the previous paragraph, definability of the action of $G(M)$ on X means precisely that whenever $F\colon S_G(M) \to \mathbb{R}$ lifts some continuous function f on X, then every complete $F(yx)$-type over M is definable. By Proposition 3.13, each such $F(yx)$ is stable (for T). □

Proof of Theorem 3.2. We may assume that X is a (definable) $G(M)$-ambit, in which case by [12] or [23, Remark 3.2], the action factors through the action of $G(M)$ on $S_G(M)$. By Theorem 3.16(b), and ω_1-saturation of M, the action of G on X is wap, so by Fact 3.15, X has a $G(M)$-invariant Borel probability measure. □

3.3. **On universal ambits and minimal flows.** We give a description of the universal definable wap ambit and universal minimal definable wap flow for a group $G(M)$ definable in a structure M. As seen by the material above, this is closely connected to stable group theory in a continuous logic sense, but unless M is saturated enough it will be stability in M. Actually even in the classical case, stable group theory relative to a model M (i.e. where relevant formulas $\phi(x, y)$ are stable in M) has not been written down, so it is not surprising if we happen to rely on the topological dynamical literature. By G we mean $G(M^*)$ for a suitably saturated elementary extension M^* of M.

M will be an arbitrary structure and $G(M)$ a group definable in M. Following on from notation in the previous section, if $F(x)$ is a CL-formula on G (i.e. where the variable x ranges over G) over M, then we will call F *stable in M* if the CL-formula $F(yx)$ (in variables x, y) is stable in M. Let $\mathcal{A}$ be the collection (in fact algebra) of such stable in M, CL-formulas $F(x)$ on G. For $p(x) \in S_G(M)$ by $p|\mathcal{A}$ we mean the map which for each $F \in \mathcal{A}$ gives $F(p)$. The collection $\{p|\mathcal{A} : p \in S_G(M)\}$ is clearly a quotient of $S_G(M)$ by a closed equivalence relation, hence is naturally a compact space which we call the type space over M of the stable in M, CL-formulas over M, and which we denote here by $\mathcal{S}$. Let $\pi_0\colon S_G(M) \to \mathcal{S}$ be the canonical surjective continuous map. Note that $G(M)$ acts on $\mathcal{S}$, and that π_0 is a map of $G(M)$-flows (in fact ambits, where $\pi_0(e)$ is taken as the distinguished point of $\mathcal{S}$). With the above notation we have:

Proposition 3.17. *(i) $(\mathcal{S}, G(M))$ is the universal definable wap ambit of $G(M)$. (ii) The universal minimal definable wap flow of $G(M)$ is G/G^{00}_M.*

Proof. Let us first note:

Claim 1: With above notation, a continuous function $F\colon S_G(M) \to \mathbb{R}$ is stable in M if and only if it is induced, via π_0, by a continuous function from $\mathcal{S}$ to $\mathbb{R}$.

Proof. This follows from the Stone-Weierstrass theorem and the easy fact that $\mathcal{A}$ is a closed subalgebra of the Banach algebra $C(S_G(M))$ of all real valued continuous functions on $S_G(M)$ (where $C(S_G(M))$ is equipped with the uniform convergence topology). $\square$(claim)

(i) follows easily from the claim and previous results. First, by Theorem 3.16 and the claim, $(\mathcal{S}, G(M))$ (with distinguished point $s_0 = \pi_0(e)$) is definable and wap. Secondly, suppose $(X, G(M))$ is a definable wap ambit with distinguished point x_0, and corresponding $\pi\colon S_G(M) \to X$ (taking e to x_0). By Theorem 3.16 again, for every continuous function f on X, $F = f \circ \pi$ is stable, hence is in the algebra $\mathcal{A}$. This easily induces a surjective continuous $G(M)$-equivariant map from $\mathcal{S}$ to X taking s_0 to x_0.
(ii) The action of $G(M)$ on G/G^{00}_M is induced by multiplication on the left. Clearly every orbit is dense, in particular the image of $G(M)$ in G/G^{00}_M under the canonical homomorphism ι taking g to g/G^{00}_M is dense. The action factors through the type space. Why is it wap? Let f be a continuous function from G/G^{00}_M to $\mathbb{R}$, and $F\colon G \to \mathbb{R}$ be $f \circ \pi$ where $\pi\colon G \to G/G^{00}_M$ is the canonical homomorphism. So F is a CL-formula on G over M such that $F(g)$ depends only on the coset of g modulo G^{00}_M. We claim that the CL-formula $F(yx)$ in variables x, y (so on $G \times G$) is stable for the theory, in particular F is stable in M. If not, we can find a large indiscernible sequence $((g_i, h_i) : i \in I)$ such that for $i < j$, $F(g_i h_j) \neq F(g_j h_i)$ which is clearly a contradiction. So we have shown that the action of $G(M)$ on G/G^{00}_M is a minimal wap flow which factors through $S_G(M)$ so is also definable by Theorem 3.16.

To see that it is universal such, we will appeal again to the topological dynamics literature. So let $(X, G(M))$ be a minimal definable wap flow. As already remarked, we deduce from II.8 of [10] that the flow $(X, G(M))$ is equicontinuous. By Theorem 3.3 from Chapter I of [13], the Ellis semigroup $E(X)$ is a compact topological group acting by homeomorphisms on X, and, moreover, $(X, G(M))$ is isomorphic to $E(X)/H$ for a suitable closed subgroup H of $E(X)$ (with the action of $G(M)$ on $E(X)/H$ given by $g(\eta H) = (g\eta)H$). So it remains to show that the natural homomorphism $h\colon G(M) \to E(X)$ is definable, because in that case $E(X)$ will be a definable group compactification of $G(M)$ (in the sense of [12]) and we know from Proposition 3.4 of [12] that G/G^{00}_M is the universal such definable group compactification.

The fact that $h\colon G(M) \to E(X)$ is definable follows from the fact that $E(X)$ is a subflow of the product $G(M)$-flow X^X which is definable (because a product of definable flows is always definable [26, Remark 1.12]). $\square$

The above proposition together with Theorem 3.16 yields

Corollary 3.18. *When M is ω_1-saturated, the universal definable wap ambit coincides with the universal definable ambit and can be described as the type space of the collection (algebra) of CL-formulas F on G over M which are stable (for T).*

We can also give a description of the universal definable $G(M)$-ambit for an arbitrary (not necessarily ω_1-saturated) M. For this recall that in the proof of

Theorem 3.16 (Claim 1 and the paragraph afterwards; see also the proof of (b)) we showed that definability of the action of $G(M)$ on X means precisely that whenever $F\colon S_G(M) \to \mathbb{R}$ lifts some continuous function f on X, then every complete $F(yx)$-type over M is definable. Thus, applying Stone-Weierstrass as in the proof of Claim 1 in the proof of Proposition 3.17 and following the lines of the easy proof of item (i) of this proposition, we get

Corollary 3.19. *Let $\mathcal{D}$ be the quotient of $S_G(M)$ corresponding to the algebra $\mathcal{B}$ of all CL-formulas $F(x)$ on G over M for which every complete $F(yx)$-type over M is definable. Then $G(M)$ acts naturally on $\mathcal{D}$, and $(\mathcal{D}, G(M))$ is the universal definable ambit.*

Finally, as promised in the introduction, we give a negative answer to Problem 4.11 (1) from [23], concerning whether the natural map from $S_G(M)$ to G/G^{000}_M factors through the universal definable ambit. When M is sufficiently saturated, Corollary 3.18 says that the universal definable ambit is precisely the universal definable wap ambit. So we consider, as in the proof of Corollary 3.3, a group G definable in a countable NIP theory T such that $G^{000} \neq G^{00}$. Let M be an ω_1-saturated model over which G is defined. Then $\mathcal{S}$, as defined above, is, by Proposition 3.17, the universal definable wap ambit of $G(M)$, and likewise the universal minimal definable wap flow of $G(M)$ is G/G^{00}_M which is a, in fact the unique, minimal subflow of $\mathcal{S}$. The natural map from $S_G(M)$ to G/G^{000}_M referred to above takes $\mathrm{tp}(g/M)$ to g/G^{000}_M. If I is a minimal subflow of $S_G(M)$ then the same map takes I onto G/G^{000}_M. Following earlier notation, let π_0 be the canonical map from $S_G(M)$ to $\mathcal{S}$. Then $\pi_0[I]$ is a (the unique) minimal subflow of $\mathcal{S}$, G/G^{00}_M. So if $S_G(M) \to G/G^{000}_M$ factored through π_0, we would obtain a (natural) homomorphism from G/G^{00}_M to G/G^{000}_M, implying that $G^{00}_M = G^{000}_M$, a contradiction.

4. Amenability and G-compactness

In this section, we introduce the notions of amenable types and theories, and study their consequences. In Subsection 4.2, we analyze various basic properties and provide equivalent definitions. In Subsections 4.3 and 4.4, we prove the main result of this section that amenability implies G-compactness: in Subsection 4.3, we give a relatively simpler proof assuming the existence of definable measures, and in Subsection 4.4, we give a proof in full generality, using in particular Corollary 2.11 and arguments in the spirit of Section 2 (although in a different context).

4.1. **Preliminaries on G-compactness.** We only recall a few basic definitions and facts about Lascar strong types and Galois groups. For more details the reader is referred to [29], [6] or [37].

Let $\mathfrak{C}$ be a monster model of a complete theory T.

Definition 4.1.

i) *The group of Lascar strong automorphisms*, which is denoted by $\mathrm{Autf}_L(\mathfrak{C})$, is the subgroup of $\mathrm{Aut}(\mathfrak{C})$ which is generated by all automorphisms fixing a small submodel of $\mathfrak{C}$ pointwise, i.e. $\mathrm{Autf}_L(\mathfrak{C}) = \langle \sigma : \sigma \in \mathrm{Aut}(\mathfrak{C}/M)$ for a small $M \prec \mathfrak{C}\rangle$.
ii) *The Lascar Galois group of* T, which is denoted by $\mathrm{Gal}_L(T)$, is the quotient group $\mathrm{Aut}(\mathfrak{C})/\mathrm{Autf}_L(\mathfrak{C})$ (which makes sense, as $\mathrm{Autf}_L(\mathfrak{C})$ is a normal subgroup of $\mathrm{Aut}(\mathfrak{C})$). It turns out that $\mathrm{Gal}_L(T)$ does not depend on the choice of $\mathfrak{C}$.

The orbit equivalence relation of $\mathrm{Autf}_L(\mathfrak{C})$ acting on any given product S of boundedly (i.e. less than the degree of saturation of $\mathfrak{C}$) many sorts of $\mathfrak{C}$ is usually denoted by E_L. It turns out that this is the finest bounded (i.e. with boundedly many classes), invariant equivalence relation on S; and the same is true after the restriction to the set of realizations of any type in $S(\emptyset)$. The classes of E_L are called *Lascar strong types*. It turns out that $\mathrm{Autf}_L(\mathfrak{C})$ coincides with the the group of all automorphisms fixing setwise all E_L-classes on all (possibly infinite) products of sorts.

For any small $M \prec \mathfrak{C}$ enumerated as $\bar{m}$, we have a natural surjection from $S_{\bar{m}}(M) := \{p \in S(M) : \mathrm{tp}(\bar{m}/M) \subseteq p\}$ to $\mathrm{Gal}_L(T)$ given by $\mathrm{tp}(\sigma(\bar{m})/M) \mapsto \sigma/\mathrm{Autf}_L(\mathfrak{C})$ for $\sigma \in \mathrm{Aut}(\mathfrak{C})$. We can equip $\mathrm{Gal}_L(T)$ with the quotient topology induced by this surjecion, and it is easy to check that this topology does not depend on the choice of M. In this way, $\mathrm{Gal}_L(T)$ becomes a topological (but not necessarily Hausdorff) group (see [37] for a detailed exposition).

Definition 4.2.
i) By $\mathrm{Gal}_0(T)$ we denote the closure of the identity in $\mathrm{Gal}_L(T)$.
ii) *The group of Kim-Pillay strong automorphisms*, which is denoted by $\mathrm{Autf}_{KP}(\mathfrak{C})$, is the preimage of $\mathrm{Gal}_0(T)$ under the quotient homomorphism $\mathrm{Aut}(\mathfrak{C}) \to \mathrm{Gal}_L(T)$.
iii) *The Kim-Pillay Galois group of* T, which is denoted by $\mathrm{Gal}_{KP}(T)$, is the quotient group $\mathrm{Gal}_L(T)/\mathrm{Gal}_0(T) \cong \mathrm{Aut}(\mathfrak{C})/\mathrm{Autf}_{KP}(\mathfrak{C})$ equipped with the quotient topology. It is a compact, Hausdorff topological group.

The orbit equivalence relation of $\mathrm{Autf}_{KP}(\mathfrak{C})$ acting on any given product S of (boundedly many) sorts of $\mathfrak{C}$ is usually denoted by E_{KP}. It turns out that this is the finest bounded (i.e. with boundedly many classes), type-definable over $\emptyset$ equivalence relation on S; and the same is true after the restriction to the set of realizations of any type in $S(\emptyset)$. The classes of E_{KP} are called *Kim-Pillay strong types*. It turns out that $\mathrm{Autf}_{KP}(\mathfrak{C})$ coincides with the the group of all automorphisms fixing setwise all E_{KP}-classes on all (possibly infinite) products of sorts.

The theory T is said to be *G-compact* if the following *equivalent* conditions hold.
(1) $\mathrm{Autf}_L(\mathfrak{C}) = \mathrm{Autf}_{KP}(\mathfrak{C})$.
(2) $\mathrm{Gal}_L(T)$ is Hausdorff.

(3) Lascar strong types coincide with Kim-Pillay strong types on any (possibly infinite) products of sorts.

By the definition of E_L, we see that $\bar{\alpha}\, E_L\, \bar{\beta}$ if and only if there are $\bar{\alpha}_0 = \bar{\alpha}, \bar{\alpha}_1, \dots, \bar{\alpha}_n = \bar{\beta}$ and models $M_0, \dots, M_{n-1}$ such that

$$\bar{\alpha}_0 \equiv_{M_0} \bar{\alpha}_1 \equiv_{M_1} \dots \bar{\alpha}_{n-1} \equiv_{M_{n-1}} \bar{\alpha}_n.$$

In this paper, by the *Lascar distance* from $\bar{\alpha}$ to $\bar{\beta}$ (denoted by $d_L(\bar{\alpha}, \bar{\beta})$) we mean the smallest natural number n as above. By the *Lascar diameter* of a Lascar strong type $[\bar{\alpha}]_{E_L}$ we mean the supremum of $d_L(\bar{\alpha}, \bar{\beta})$ with $\bar{\beta}$ ranging over $[\bar{\alpha}]_{E_L}$. It is well known (proved in [33]) that $[\bar{\alpha}]_{E_L} = [\bar{\alpha}]_{E_{KP}}$ if and only if the Lascar diameter of $[\bar{\alpha}]_{E_L}$ is finite.

4.2. **Amenable theories.** As usual, $\mathfrak{C}$ is a monster model of an arbitrary complete theory T. Let $\bar{c}$ be an enumeration of $\mathfrak{C}$ and let $S_{\bar{c}}(\mathfrak{C}) = \{\mathrm{tp}(\bar{a}/\mathfrak{C}) \in S(\mathfrak{C}) : \bar{a} \equiv \bar{c}\}$. More generally, for a partial type $\pi(\bar{x})$ over $\emptyset$, put $S_\pi(\mathfrak{C}) = \{q(\bar{x}) \in S(\mathfrak{C}) : \pi \subseteq q\}$. In particular, if $p(\bar{x}) \in S(\emptyset)$ and $\bar{\alpha} \models p$, then $S_p(\mathfrak{C}) = S_{\bar{\alpha}}(\mathfrak{C}) = \{q(\bar{x}) \in S(\mathfrak{C}) : p \subseteq q\}$. (Note that we allow here tuples $\bar{x}$ of unbounded length (i.e. greater than the degree of saturation of $\mathfrak{C}$)). Each $S_\pi(\mathfrak{C})$ is naturally an $\mathrm{Aut}(\mathfrak{C})$-flow.

Let us start from the local version of amenability.

Definition 4.3. A partial type $\pi(\bar{x})$ over $\emptyset$ is *amenable* if there is an $\mathrm{Aut}(\mathfrak{C})$-invariant, Borel (regular) probability measure on $S_\pi(\mathfrak{C})$.

Remark 4.4. The following conditions are equivalent for a type $\pi(\bar{x})$ over $\emptyset$.

(1) $\pi(\bar{x})$ is amenable.
(2) There is an $\mathrm{Aut}(\mathfrak{C})$-invariant, Borel (regular) probability measure μ on $S_{\bar{x}}(\mathfrak{C})$ concentrated on $S_\pi(\mathfrak{C})$, i.e. for any formula $\varphi(\bar{x}, \bar{a})$ inconsistent with $\pi(\bar{x})$, $\mu([\varphi(\bar{x}, \bar{a})]) = 0$ (where $[\varphi(\bar{x}, \bar{a})]$ is the subset of $S_{\bar{x}}(\mathfrak{C})$ consisting of all types containing $\varphi(\bar{x}, \bar{a})$).
(3) There is an $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive probability measure on relatively $\mathfrak{C}$-definable subsets of $\pi(\bar{x})$.
(4) There is an $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive probability measure on $\mathfrak{C}$-definable sets in variables $\bar{x}$, concentrated on $\pi(\bar{x})$ (i.e. for any formula $\varphi(\bar{x}, \bar{a})$ inconsistent with $\pi(\bar{x})$, $\mu(\varphi(\bar{x}, \bar{a})) = 0$).

Proof. Follows easily using the fact (see [36, Chapter 7.1]) that whenever G acts by homeomorphisms on a compact, Hausdorff, 0-dimensional space X, then each G-invariant, finitely additive probability measure on the Boolean algebra of clopen subsets of X extends to a G-invariant, Borel (regular) probability measure on X. □

Thus, by a *global* $\mathrm{Aut}(\mathfrak{C})$*-invariant Keisler measure extending* $\pi(\bar{x})$ we mean a measure from any of the items of Remark 4.4. And similarly working over any model M in place of $\mathfrak{C}$.

Proposition 4.5. *Amenability of a given type $\pi(\bar{x})$ (over $\emptyset$) is absolute in the sense that it does not depend on the choice of the monster model $\mathfrak{C}$. It is also equivalent to the amenability of $\pi(\bar{x})$ computed with respect to an $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model M in place of $\mathfrak{C}$.*

Proof. Let M and M' be two $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous models. Assume that there is an $\mathrm{Aut}(M)$-invariant, Borel (regular) probability measure μ on $S_\pi(M)$. We want to find such $\mathrm{Aut}(M')$-invariant measure μ' on $S_\pi(M')$.

Consider any formula $\varphi(\bar{x}, \bar{a}')$ with $\bar{a}' \in M'$. Choose (using the $\aleph_0$-saturation of M) any $\bar{a} \in M$ such that $\bar{a}' \equiv \bar{a}$, and define

$$\mu'([\varphi(\bar{x}, \bar{a}')] \cap S_\pi(M')) := \mu([\varphi(\bar{x}, \bar{a})] \cap S_\pi(M)).$$

By the strong $\aleph_0$-homogeneity of M and $\mathrm{Aut}(M)$-invariance of μ, we see that μ' is well-defined and $\mathrm{Aut}(M')$-invariant. It is also clear that $\mu'(S_\pi(M')) = 1$. It remains to check μ' is finitely additive on clopen subsets (as then μ' extends to the desired Borel measure). Take $\varphi(\bar{x}, \bar{a}')$ and $\psi(\bar{x}, \bar{a}')$ such that $[\varphi(\bar{x}, \bar{a}')] \cap S_\pi(M')$ is disjoint from $[\psi(\bar{x}, \bar{a}')] \cap S_\pi(M')$. This just means that $\varphi(\bar{x}, \bar{a}') \wedge \psi(\bar{x}, \bar{a}')$ is inconsistent with $\pi(\bar{x})$. Take $\bar{a} \in M$ such that $\bar{a} \equiv \bar{a}'$. Then $\varphi(\bar{x}, \bar{a}) \wedge \psi(\bar{x}, \bar{a})$ is still inconsistent with $\pi(\bar{x})$, so $\mu'(([\varphi(\bar{x}, \bar{a}')] \cap S_\pi(M')) \cup ([\psi(\bar{x}, \bar{a}')] \cap S_\pi(M'))) = \mu(([\varphi(\bar{x}, \bar{a})] \cap S_\pi(M)) \cup ([\psi(\bar{x}, \bar{a})] \cap S_\pi(M))) = \mu([\varphi(\bar{x}, \bar{a})] \cap S_\pi(M)) + \mu([\psi(\bar{x}, \bar{a})] \cap S_\pi(M)) = \mu'([\varphi(\bar{x}, \bar{a}')] \cap S_\pi(M')) + \mu'([\varphi(\bar{x}, \bar{a}')] \cap S_\pi(M'))$. □

Remark 4.6. Assume T to be countable, and let $\pi(\bar{x})$ be a partial type. Then $\pi(\bar{x})$ is amenable if and only if for all [sufficiently large] countable, ($\aleph_0$-)homogeneous models M, $\pi(\bar{x})$ has an extension to a Keisler measure $\mu(\bar{x})$ over M which is $\mathrm{Aut}(M)$-invariant. If T is uncountable, the same is true but with "countable, $\aleph_0$-homogeneous models" replaced by "strongly $\aleph_0$-homogeneous models of cardinality $|T|$".

Proof. For each [sufficiently large] countable homogeneous model $M \prec \mathfrak{C}$, let $\mu_M(\bar{x})$ be an $\mathrm{Aut}(M)$-invariant Keisler measure over M extending π, and let $\bar{\mu}_M$ be an arbitrary global Keisler measure extending μ_M. Working in the compact space of global Keisler measures, there is a subnet of the net $\{\bar{\mu}_M\}_M$, which converges to some $\bar{\mu}$. But then $\bar{\mu}$ is $\mathrm{Aut}(\mathfrak{C})$-invariant: For otherwise, for some formula $\phi(\bar{x}, \bar{y})$ and tuples a, b in $\mathfrak{C}$ with the same type, we have $\bar{\mu}(\phi(\bar{x}, \bar{a})) = r$ and $\bar{\mu}(\phi(\bar{x}, \bar{b})) = s$ for some $r < s$. But then we can find some countable homogeneous model M (containing $\bar{a}, \bar{b}$) such that $\bar{\mu}_M(\phi(\bar{x}, \bar{a})) < \bar{\mu}_M(\phi(\bar{x}, \bar{b}))$, contradicting the $\mathrm{Aut}(M)$-invariance of $\bar{\mu}_M$. □

Lemma 4.7. *A type $\pi(\bar{x})$ (over $\emptyset$) is amenable if and only if each formula $\varphi(\bar{x})$ (without parameters) implied by $\pi(\bar{x})$ is amenable.*

Proof. The implication ($\rightarrow$) is obvious, as $S_\pi(\mathfrak{C}) \subseteq S_\varphi(\mathfrak{C})$, and so for any formula $\psi(\bar{x}, \bar{a})$ we can define $\mu'([\psi(\bar{x}, \bar{a})] \cap S_\varphi(\mathfrak{C})) := \mu([\psi(\bar{x}, \bar{a})] \cap S_\pi(\mathfrak{C}))$, where μ is an $\mathrm{Aut}(\mathfrak{C})$-invariant, Borel probability measure on $S_\pi(\mathfrak{C})$.

($\leftarrow$). On the set of formulas (without parameters) implied by $\pi(\bar{x})$, consider an ultrafilter $\mathcal{U}$ containing for every $\varphi(\bar{x}) \dashv \pi$ the set $\{\psi(\bar{x}) : \pi(\bar{x}) \vdash \psi(\bar{x}) \vdash \varphi(\bar{x})\}$.

By assumption and Remark 4.4, for any $\varphi(\bar{x}) \dashv \pi(\bar{x})$ we have an $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive probability measure μ_φ on $\mathfrak{C}$-definable subsets of $\mathfrak{C}^{\bar{x}}$ which is concentrated on $\varphi(\bar{x})$. Put $\mathfrak{C}' := \prod_{\varphi(\bar{x})\dashv\pi(\bar{x})} \mathfrak{C}/\mathcal{U}$ and define

$$\mu' := \mathrm{st}\left(\prod_{\varphi(\bar{x})\dashv\pi(\bar{x})} \mu_\varphi/\mathcal{U}\right),$$

where st stands for the standard part map. It is clear that μ' is is a finitely additive probability measure on definable subsets of $\mathfrak{C}'^{\bar{x}}$. By the choice of $\mathcal{U}$, μ' is concentrated on $\pi(\bar{x})$. By the $\mathrm{Aut}(\mathfrak{C})$-invariance of all μ_φ, for any finite $\bar{a} \equiv \bar{b}$ from $\mathfrak{C}$, for any $\varphi(\bar{x}) \dashv \pi(\bar{x})$ and any $\psi(\bar{x}, \bar{y})$, we have $\mu_\varphi(\psi(\bar{x}, \bar{a})) = \mu_\varphi(\psi(\bar{x}, \bar{b}))$. Therefore, $\mu'(\psi(\bar{x}, \bar{a})) = \mu'(\psi(\bar{x}, \bar{b}))$ (treating $\bar{a}$ and $\bar{b}$ as tuples from $\mathfrak{C}'$). Finally, let μ be the restriction of μ' to the algebra of $\mathfrak{C}$-definable sets. We conclude that μ is an $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive probability measure on definable subsets of $\mathfrak{C}^{\bar{x}}$ which is concentrated on $\pi(\bar{x})$, which is enough by Remark 4.4. □

Lemma 4.8. *All types in $S(\emptyset)$ (possibly in unboundedly many variables) are amenable if and only if all finitary types in $S(\emptyset)$ are amenable.*

Proof. The implication ($\rightarrow$) is trivial. For the other implication, take $p(\bar{x}) \in S_{\bar{x}}(\emptyset)$. Consider the compact space $X := [0,1]^{\{\varphi(\bar{x},\bar{a}):\ \varphi(\bar{x},\bar{y}) \text{ a formula},\ \bar{a}\in\mathfrak{C}\}}$ with the pointwise convergence topology (where $\bar{x}$ is the fixed tuple of variables). Then the $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive probability measures on $\mathfrak{C}$-definable sets in variables $\bar{x}$ concentrated on $p(\bar{x})$ form a closed subset $\mathcal{M}$ of X. We can present $\mathcal{M}$ as the intersection of a directed family of closed subsets of X each of which witnessing a finite portion of information of being in $\mathcal{M}$. But each such finite portion of information involves only finitely many variables, so the corresponding closed set is nonempty by the assumption that all finitary types are amenable and Remark 4.4. By the compactness of X, we conclude that $\mathcal{M}$ is nonempty. □

Corollary 4.9. *The following conditions are equivalent.*

1. *All partial types (possibly in unboundedly many variables) over $\emptyset$ are amenable.*
2. *All complete types (possibly in unboundedly many variables) over $\emptyset$ are amenable.*
3. *All finitary complete types over $\emptyset$ are amenable.*
4. *All consistent formulas (in finitely many variables $\bar{x}$) over $\emptyset$ are amenable.*
5. $\mathrm{tp}(\bar{c}/\emptyset)$ *is amenable.*
6. $\mathrm{tp}(\bar{m}/\emptyset)$ *is amenable for some tuple $\bar{m}$ enumerating a model.*

Proof. The equivalence (1) $\leftrightarrow$ (2) is obvious (for (2) $\rightarrow$ (1) use the argument as in the proof of ($\rightarrow$) in Lemma 4.7). The equivalence (2) $\leftrightarrow$ (3) is Lemma 4.8. The equivalence (3) $\leftrightarrow$ (4) follows from Lemma 4.7. The implications (1) $\rightarrow$ (5) $\rightarrow$ (6) are trivial. Finally, (6) $\rightarrow$ (4) also follows from Lemma 4.7, because taking all possible finite subtuples $\bar{m}'$ of $\bar{m}$ and $\varphi(\bar{x}') \in \mathrm{tp}(\bar{m}'/\emptyset)$, we will get all consistent formulas over $\emptyset$. □

Definition 4.10. The theory T is *amenable* if the equivalent conditions of Corollary 4.9 hold.

By Proposition 4.5, we see that amenability of T is really a property of T, i.e. it does not depend on the choice of $\mathfrak{C}$.

Analogously, one can define the stronger notion of an extremely amenable theory.

Definition 4.11. A type $\pi(\bar{x})$ over $\emptyset$ is *extremely amenable* if there is an $\mathrm{Aut}(\mathfrak{C})$-invariant type in $S_\pi(\mathfrak{C})$. The theory T is *extremely amenable* if every type (in any number of variables) in $S(\emptyset)$ is extremely amenable.

As in the case of amenability, compactness arguments easily show that these notions are absolute (i.e. do not depend on the choice of $\mathfrak{C}$), and, in fact, they can be tested on any $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model in place of $\mathfrak{C}$; moreover, T is extremely amenable if and only if all finitary types in $S(\emptyset)$ are extremely amenable. Note that Remark 4.6 specializes to extremely amenable partial types, too. So note that for countable theories, both amenability and extreme amenability can be seen at the level of countable models. It is also easy to see that in a stable theory, a type in $S(\emptyset)$ is extremely amenable if and only if it is stationary.

Yet another equivalent approach to amenability of T is via $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive probability measures on the algebra of so-called relatively definable subsets of $\mathfrak{C}$. This will be the exact analogue of the definition of definable amenability of definable groups (via the existence of an invariant Keisler measure). We will use this approach in Subsection 4.4.

The idea of identifying $\mathrm{Aut}(\mathfrak{C})$ with the subset $\{\sigma(\bar{c}) : \sigma \in \mathrm{Aut}(\mathfrak{C})\}$ of $\mathfrak{C}^{\bar{c}}$ and considering relatively definable subsets of $\mathrm{Aut}(\mathfrak{C})$, i.e. subsets of the form $\{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \varphi(\sigma(\bar{c}), \bar{c})\}$ for a formula $\varphi(\bar{x}, \bar{c})$, already appeared in [28, Section 3]. Here, we extend this notion of relative definability to the local context and introduce an associated notion of amenability which is easily seen to be equivalent to the amenability of T [or of a certain type in the extended local version].

Let M be any model of T and let $\bar{m}$ be its enumeration.

Definition 4.12. i) By a *relatively definable subset* of $\mathrm{Aut}(M)$ we mean a subset of the form $\{\sigma \in \mathrm{Aut}(M) : M \models \varphi(\sigma(\bar{m}), \bar{m})\}$, where $\varphi(\bar{x}, \bar{y})$ is a formula without parameters.
ii) If $\bar{\alpha}$ is a tuple of some elements of M, by *relatively $\bar{\alpha}$-definable subset* of $\mathrm{Aut}(M)$ we mean a subset of the form $\{\sigma \in \mathrm{Aut}(M) : M \models \varphi(\sigma(\bar{\alpha}), \bar{m})\}$, where $\varphi(\bar{x}, \bar{y})$ is a formula without parameters.

The above definition differs from the standard terminology in which "A-definable" means "definable over A"; here, "relatively $\bar{\alpha}$-definable" has nothing to do with the parameters over which the set is relatively definable. One should keep this in mind from now on.

For a formula $\varphi(\bar{x}, \bar{y})$ and tuples $\bar{a}$, $\bar{b}$ from M corresponding to $\bar{x}$ and $\bar{y}$, respectively, we will use the following notation

$$A_{\varphi,\bar{a},\bar{b}} = \{\sigma \in \mathrm{Aut}(M) : M \models \varphi(\sigma(\bar{a}), \bar{b})\}.$$

When $\bar{x}$ and $\bar{y}$ are of the same length (by which we mean that they are also of the same sorts) and $\bar{a} = \bar{b}$, then this is set will be denoted by $A_{\varphi,\bar{a}}$.

Definition 4.13. i) The group $\mathrm{Aut}(M)$ is said to be *relatively definably amenable* if there exists a left $\mathrm{Aut}(M)$-invariant, finitely additive probability measure on the Boolean algebra of relatively definable subsets of $\mathrm{Aut}(M)$.
ii) If $\bar{\alpha}$ is a tuple of some elements of M, the group $\mathrm{Aut}(M)$ is said to be *relatively $\bar{\alpha}$-definably amenable* if there exists a left $\mathrm{Aut}(M)$-invariant, finitely additive probability measure on the Boolean algebra of relatively $\bar{\alpha}$-definable subsets of $\mathrm{Aut}(M)$.

In particular, $\mathrm{Aut}(M)$ being relatively definably amenable means exactly that it is relatively $\bar{m}$-definably amenable.

We will mostly focus on the case when $M = \mathfrak{C}$ is a monster model. But often one can work in the more general context when M is $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous, including the case of the unique countable model of an ω-categorical theory.

Remark 4.14. Let M be $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous enumerated as $\bar{m}$. Let $\bar{\alpha}$ be a tuple of some elements of M. Then $\mathrm{Aut}(M)$ is relatively $\bar{\alpha}$-definably amenable if and only if there is an $\mathrm{Aut}(M)$-invariant, (regular) Borel probability measure on $S_{\bar{\alpha}}(M)$ (equivalently, $\mathrm{tp}(\bar{\alpha}/\emptyset)$ is amenable). In particular, $\mathrm{Aut}(M)$ is relatively definable if and only if there is an $\mathrm{Aut}(M)$-invariant, (regular) Borel probability measure on $S_{\bar{m}}(M)$ (equivalently, T is amenable).

Proof. Suppose first that $\mathrm{Aut}(M)$ is relatively $\bar{\alpha}$-definably amenable, witnessed by a measure μ. For a formula $\varphi(\bar{x}, \bar{m})$ let $[\varphi(\bar{x}, \bar{m})]$ be the basic clopen set in $S_{\bar{\alpha}}(M)$ given by this formula. Define $\tilde{\mu}([\varphi(\bar{x}, \bar{m})]) := \mu(A_{\varphi,\bar{m}})$. It is clear that $\tilde{\mu}$ is an $\mathrm{Aut}(M)$-invariant, finitely additive probability measure on the algebra of clopen subsets of $S_{\bar{\alpha}}(M)$. This $\tilde{\mu}$ extends (by [36, Chapter 7.1])) to an $\mathrm{Aut}(M)$-invariant, (regular) Borel probability measure on $S_{\bar{\alpha}}(M)$.

Conversely, assume that $\tilde{\mu}$ is an $\mathrm{Aut}(M)$-invariant, Borel probability measure on $S_{\bar{\alpha}}(M)$. For any relatively $\bar{\alpha}$-definable subset $A_{\varphi,\bar{m}}$ define $\mu(A_{\varphi,\bar{m}}) := \tilde{\mu}([\varphi(\bar{x}, \bar{m})])$. By the $\aleph_0$-saturation and strong $\aleph_0$-homogeneity of M, we easily get that μ is a well-defined, $\mathrm{Aut}(M)$-invariant, finitely additive probability measure on relatively $\bar{\alpha}$-definable subsets of $\mathrm{Aut}(M)$.

The fact that the existence of an $\mathrm{Aut}(M)$-invariant, (regular) Borel probability measure on $S_{\bar{\alpha}}(M)$ is equivalent to amenability of $\mathrm{tp}(\bar{\alpha}/\emptyset)$ follows from Proposition

4.5. And then, the fact that the existence an $\mathrm{Aut}(M)$-invariant, (regular) Borel probability measure on $S_{\bar{m}}(M)$ is equivalent to amenability of T follows from Corollary 4.9. □

So the terminologies "$\mathrm{Aut}(M)$ is relatively [$\bar{\alpha}$-]definably amenable" and "T [resp. $\mathrm{tp}(\bar{\alpha}/\emptyset)$] is amenable" will be used interchangeably.

Corollary 4.15. *[For a given tuple $\bar{\alpha}$] relative [$\bar{\alpha}$-]definable amenability of an $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model M [containing $\bar{\alpha}$] does not depend on the choice of M.*

Corollary 4.16. *Let M be $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous. Then, if* $\mathrm{Aut}(M)$ *is amenable as a topological group (with the pointwise convergence topology), then it is relatively definably amenable, which in turn implies that it is relatively $\bar{\alpha}$-definably amenable for any tuple $\bar{\alpha}$ of elements M.*

Similarly, extreme amenability of $\mathrm{Aut}(M)$ *as a topological group implies extreme amenability of T.*

Proof. Amenability of $\mathrm{Aut}(M)$ implies that there is an $\mathrm{Aut}(M)$-invariant, Borel probability measure on $S_{\bar{m}}(M)$. By Remark 4.14, this implies relative definable amenability of $\mathrm{Aut}(M)$. Furthermore, since there is an obvious flow homomorphism from $S_{\bar{m}}(M)$ to $S_{\bar{\alpha}}(M)$, a measure on $S_{\bar{m}}(M)$ induces a measure on $S_{\bar{\alpha}}(M)$, and this is enough by Remark 4.14. □

As in the introduction, we will call a countable $\aleph_0$-categorical theory KPT-[extremely] amenable if the automorphism group of its unique countable model is [extremely] amenable as a topological group.

So, by Corollary 4.16, KPT-[extreme] amenability of a (countable $\aleph_0$-categorical) theory T implies [extreme] amenability of T in the new sense of this paper. In fact most, if not all, of the examples of not only KPT-extremely amenable theories (such as dense linear orderings) but also KPT-amenable (not necessarily KPT-extremely amenable) theories (such as the random graph) come from Fraïssé classes with *canonical amalgamation*, hence are extremely amenable in our sense. Only canonical amalgamation over $\emptyset$ is needed here (see the next paragraph for a justification) which says that there is a map $\otimes$ taking pairs of finite structures (A, B) from the Fraïssé class to an amalgam $A \otimes B$ (also in the Fraïssé class) which is compatible with embeddings, i.e. if $f\colon B \to C$ is an embedding of finite structure structures from the Fraïssé class, then there exists an embedding from $A \otimes B$ to $A \otimes C$ which commutes with f and with the embeddings: $A \to A \otimes B$, $B \to A \otimes B$, $A \to A \otimes C$, and $C \to A \otimes C$. A typical example is a Fraïssé class with "free amalgamation", namely adding no new relations.

Let us briefly explain why canonical amalgamation of a Fraïssé class of finite structures in a relational language [or, more generally, finitely generated structures in any language] whose Fraïssé limit M is ω-categorical implies extreme amenability. First, note that canonical amalgamation implies that for any finite tuples $\bar{d}, \bar{a}_1, \bar{b}_1, \dots, \bar{a}_n, \bar{b}_n$ from M, if the structures $\bar{a}_i$ and $\bar{b}_i$ are isomorphic (i.e. have the same quantifier-free type), then we can amalgamate structures $\bar{d}$ and $(\bar{a}_i, \bar{b}_i : i \leq n)$ into a structure $\bar{d}', \bar{a}'_1, \bar{b}'_1, \dots, \bar{a}'_n, \bar{b}'_n$ in such a way that $\bar{a}'_i$ is isomorphic with $\bar{b}'_i$ over $\bar{d}'$ for all $i \leq n$. Therefore, using ω-categoricity and quantifier elimination, one concludes by compactness that any finitary type in $S(\emptyset)$ extends to an $\mathrm{Aut}(M)$-invariant type in $S(M)$, so T is extremely amenable (since M is ω-categorical).

In [27], we proved that both KPT-amenability and KPT-extreme amenability are preserved by adding finitely many parameters. This is not the case for our notion of first order [extreme] amenability. For example, if T is the theory of two equivalence relations E_1, E_2, where E_1 has infinitely many classes, all infinite, and each E_1-class is divided into two E_2-classes, both infinite, then T is extremely amenable, but adding an (imaginary) parameter for an E_1-class destroys extreme amenability. Similar examples can be built by putting uniformly in each E_1-class some non amenable theory.

Corollary 4.17. *If* $\mathrm{Aut}(\mathfrak{C})$ *is relatively $\bar{\alpha}$-definably amenable, where $\bar{\alpha}$ is a tuple in $\mathfrak{C}$ (e.g. $\bar{\alpha} = \bar{c}$), then there exists an* $\mathrm{Aut}(\mathfrak{C})$*-invariant, finitely additive probability measure on the Boolean algebra generated by relatively $\bar{\alpha}$-type-definable sets, i.e. sets of the form* $\{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi(\sigma(\bar{a}), \bar{b}))\}$ *for some partial type $\pi(\bar{x}, \bar{y})$, where $\bar{x}$ and $\bar{y}$ are short tuples of variables, and $\bar{a}$, $\bar{b}$ are tuples from $\mathfrak{C}$ corresponding to $\bar{x}$ and $\bar{y}$, respectively, such that $\bar{a}$ is a subtuple of $\bar{\alpha}$.*

In particular, if $\mathrm{Aut}(\mathfrak{C})$ *is relatively definably amenable, then there exists an* $\mathrm{Aut}(\mathfrak{C})$*-invariant, finitely additive probability measure on the Boolean algebra generated by relatively type-definable sets (i.e., relatively $\bar{c}$-type-definable sets).*

Proof. A set X belongs to the Boolean algebra in question if and only if it is of the form $\{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathrm{tp}(\sigma(\bar{a})/A)) \in \mathcal{P}\}$, where $A \subseteq \mathfrak{C}$ is a (small) set, $\bar{a}$ is a short subtuple of $\bar{\alpha}$, and $\mathcal{P}$ is a finite Boolean combination of closed subsets of $S_{\bar{a}}(A)$. By Remark 4.14, there is and $\mathrm{Aut}(\mathfrak{C})$-invariant, (regular) Borel probability measure $\tilde{\mu}$ on $S_{\bar{\alpha}}(\mathfrak{C})$. Then define $\mu(X) := \tilde{\mu}(\pi^{-1}[\mathcal{P}])$, where $\pi \colon S_{\bar{\alpha}}(\mathfrak{C}) \to S_{\bar{a}}(A)$ is the restriction map. It is easy to check that it is a well-defined measure as required. □

Recall that in an NIP theory, for any global type p the following conditions are equivalent (see [17, Proposition 2.11]).

(1) p does not fork over $\emptyset$.
(2) The $\mathrm{Aut}(\mathfrak{C})$-orbit of p is bounded.
(3) p is Kim-Pillay invariant (i.e. invariant under $\mathrm{Autf}_{KP}(\mathfrak{C})$).
(4) p is Lascar invariant.

Proposition 4.18. *Assume NIP, and let $\bar{\alpha}$ be any tuple in $\mathfrak{C}$ (e.g. $\bar{\alpha} = \bar{c}$). Then* $\mathrm{Aut}(\mathfrak{C})$ *is relatively $\bar{\alpha}$-definably amenable if and only if there exists $p \in S_{\bar{\alpha}}(\mathfrak{C})$ with bounded* $\mathrm{Aut}(\mathfrak{C})$*-orbit.*

Proof. ($\leftarrow$). Consider any formula $\varphi(\bar{x}, \bar{b})$ with $\bar{x}$ corresponding to $\bar{\alpha}$. Recall that
$$A_{\varphi,\bar{\alpha},\bar{b}} = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \varphi(\sigma(\bar{\alpha}), \bar{b})\}.$$

Let $S_\varphi = \{\bar{b}' : \varphi(\bar{x}, \bar{b}') \in p\}$. As was recalled above, p is $\mathrm{Autf}_{KP}(\mathfrak{C})$-invariant, i.e. it is invariant over $\mathrm{bdd}^{heq}(\emptyset)$ (the hyperimaginary bounded closure of $\emptyset$). By Proposition 2.6 of [17] and NIP, there is $N < \omega$ such that
$$S_\varphi = \bigcup_{n<N} A_n \cap B^c_{n+1},$$
where each A_n and B_n is type-definable and invariant under $\mathrm{Autf}_{KP}(\mathfrak{C})$.

Let $\tilde{S}_\varphi = \{\sigma/\mathrm{Autf}_{KP}(\mathfrak{C}) : \varphi(\bar{x}, \sigma^{-1}(\bar{b})) \in p\} = \{\sigma/\mathrm{Autf}_{KP}(\mathfrak{C}) : \varphi(\bar{x}, \bar{b}) \in \sigma(p)\}$, and let $\pi \colon \mathrm{Aut}(\mathfrak{C}) \to \mathrm{Gal}_{KP}(T)$ be the quotient map.

We claim that $\tilde{S}_\varphi$ is a Borel (even constructible) subset of $\mathrm{Gal}_{KP}(T)$. Indeed, since p is $\mathrm{Autf}_{KP}(\mathfrak{C})$-invariant, we have
$$\pi^{-1}[\tilde{S}_\varphi] = \{\sigma : \sigma^{-1}(\bar{b}) \in S_\varphi\} = \bigcup_{n<N} A'_n \cap B'^c_{n+1},$$
where $A'_n = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \sigma^{-1}(\bar{b}) \in A_n\}$ and $B'_n = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \sigma^{-1}(\bar{b}) \in B_n\}$. Since A_n and B_n are $\mathrm{Autf}_{KP}(\mathfrak{C})$-invariant, we get that A'_n and B'_n are unions of cosets of $\mathrm{Autf}_{KP}(\mathfrak{C})$, so
$$\tilde{S}_\varphi = \pi[\pi^{-1}[\tilde{S}_\varphi]] = \bigcup_{n<N} \pi[A'_n] \cap \pi[B'_{n+1}]^c.$$
Moreover, $\pi^{-1}[\pi[A'_n]^{-1}] = A'^{-1}_n = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \sigma(\bar{b}) \in A_n\}$ and $\pi^{-1}[\pi[B'_n]^{-1}] = B'^{-1}_n = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \sigma(\bar{b}) \in B_n\}$, so $\pi[A'_n]$ and $\pi[B'_{n+1}]$ are closed (by type-definability of A_n and B_n). Therefore, $\tilde{S}_\varphi$ is constructible.

Let $\mathfrak{h}$ be the unique (left invariant) normalized Haar measure on the compact group $\mathrm{Gal}_{KP}(T)$. By the last paragraph, $\tilde{S}_\varphi$ is Borel, hence $\mathfrak{h}(\tilde{S}_\varphi)$ is defined, and so we can put
$$\mu(A_{\varphi,\bar{\alpha},\bar{b}}) := \mathfrak{h}(\tilde{S}_\varphi).$$
We leave as an easy exercise to check that μ is a well-defined (i.e. does not depend on the choice of φ yielding the fixed set $A = A_{\varphi,\bar{\alpha},\bar{b}}$), $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive probability measure on relatively $\bar{\alpha}$-definable subsets of $\mathrm{Aut}(\mathfrak{C})$.

($\rightarrow$). Let μ be a measure witnessing relative $\bar{\alpha}$-definable amenability. We can find $p \in S_{\bar{\alpha}}(\mathfrak{C})$ such that for every $\varphi(\bar{x}, \bar{b}) \in p$ one has $\mu(A_{\varphi,\bar{\alpha},\bar{b}}) > 0$. We claim that p does not fork over $\emptyset$, which will complete the proof by the comments preceding Proposition 4.18.

Suppose p forks over $\emptyset$. Then there is a formula $\varphi(\bar{x}, \bar{b}) \in p$ and an indiscernible sequence $\langle \bar{b}_i \rangle_{i\in\omega}$ with $\bar{b}_0 = \bar{b}$ such that $\langle \varphi(\bar{x}, \bar{b}_i) \rangle_{i\in\omega}$ is k-inconsistent for some $k \in \omega$. By left invariance of μ, we have $\mu(A_{\varphi,\bar{\alpha},\bar{b}_i}) = \epsilon > 0$ for all $i \in \omega$. So we can find a maximal n such that $\mu(A_{\varphi,\bar{\alpha},\bar{b}_0} \cap \dots \cap A_{\varphi,\bar{\alpha},\bar{b}_{n-1}}) = \delta > 0$. Let $\psi_m(\bar{x}, \bar{b}'_m) = \varphi(\bar{x}, \bar{b}_{mn}) \wedge \dots \wedge \varphi(\bar{x}, \bar{b}_{mn+n-1})$, where $\bar{b}'_m$ is the concatenation of $\bar{b}_{mn}, \dots, \bar{b}_{mn+n-1}$.

Then $\mu(A_{\psi_m,\bar{\alpha},\bar{b}'_m}) = \delta$ for all m, but $\mu(A_{\psi_{m_1},\bar{\alpha},\bar{b}'_{m_1}} \cap A_{\psi_{m_2},\bar{\alpha},\bar{b}'_{m_2}}) = 0$ whenever $m_1 \neq m_2$. This is a contradiction. $\square$

By Proposition 4.18 and the discussion preceding it, we get the following corollary, yielding a large class of amenable theories.

Corollary 4.19. *Assume T has NIP. Then, T is amenable if and only if $\emptyset$ is an extension base (i.e. any type over $\emptyset$ does not fork over $\emptyset$). In particular, stable, o-minimal, and c-minimal theories are all amenable.*

By [17, Corollary 2.10], the above characterization gives us

Corollary 4.20. *Assume T has NIP. Then amenability of T implies G-compactness.*

In Subsection 4.4, we will generalize this corollary to arbitrary theories, using different methods.

It is worth mentioning that Theorem 7.7 of [25] yields several other conditions equivalent (under NIP) to the existence of $p \in S_{\bar{c}}(\mathfrak{C})$ with bounded $\mathrm{Aut}(\mathfrak{C})$-orbit, for example: some (equivalently, every) minimal left ideal of the Ellis semigroup of the $\mathrm{Aut}(\mathfrak{C})$-flow $S_{\bar{c}}(\mathfrak{C})$ is of bounded size.

Let us finally mention in this subsection some relations between our notions of amenability and extreme amenability of a theory T and the notion of a strongly determined over $\emptyset$ theory from [19] (originating in work of Ivanov and Macpherson [20]). Decoding the definition in [19], T is strongly determined over $\emptyset$ if any complete type $p(\bar{x})$ over $\emptyset$ has an extension to a complete type $p'(\bar{x})$ over $\mathfrak{C}$ which is $\mathrm{acl}^{eq}(\emptyset)$-invariant. So clearly T extremely amenable implies T is strongly determined over $\emptyset$. Moreover, by Proposition 4.19, assuming NIP, T strongly determined over $\emptyset$ implies amenability of T. In fact, if T is NIP and KP-strong types agree with usual strong types (over $\emptyset$), then T is strongly determined over $\emptyset$ iff T is amenable.

4.3. Amenability implies G-compactness: the case of definable measures.

The main result of this section of the paper, that amenability of T implies that T is G-compact, will be proved in full generality in Subsection 4.4. However, some special cases have a relatively easy proof. One such is the NIP case above. Another case is when T is extremely amenable, where the proof of Remark 4.21 of [27] shows that in fact T is G-trivial (the Lascar group is trivial). This is made explicit in Proposition 4.31 below. Ivanov's observation in [19] that if T is strongly determined over $\emptyset$ then Lascar strong types coincide with (Shelah) strong types follows from Proposition 4.31 by working over $\mathrm{acl}^{eq}(\emptyset)$. However, deducing G-compactness of T from amenability of T in general is more complicated, and the proof in Subsection 4.4 uses Corollary 2.11 and requires adaptations of some ideas from Section 2 involving various computations concerning relatively definable subsets of $\mathrm{Aut}(\mathfrak{C})$. This subsection is devoted to a proof of the main result in the special case when amenability of T is witnessed by $\emptyset$-definable global Keisler measures, rather than just $\emptyset$-invariant Keisler measures. We will make use of CL-stability as in Section 3. But this time we will also make explicit use of results from [3].

We first discuss the relationship between our formalism and that of [3]. Start with our (classical) complete first order theory T, which we assume for convenience to be 1-sorted. This is a theory in continuous logic in the sense of [3], but where the metric is discrete and all relation symbols are $\{0,1\}$ valued, where 0 is treated as "true" and 1 as "false". The type spaces $S_n(T)$ are of course Stone spaces. What are called *definable predicates*, in finitely many variable and without parameters, in [3] are precisely CL-formulas over $\emptyset$ in the sense of Section 3, but where the range is contained in $[0,1]$. Namely, a definable predicate in n variables is given by a continuous function from $S_n(T)$ to $[0,1]$. The CL-generalization of Morleyizing T consists of adding all such definable predicates as new predicate symbols in the sense of continuous logic. So if M is a model of T, $\phi(\bar{x})$ is such a new predicate symbol, and M is a model of T, then the interpretation $\phi(M)$ of ϕ in M is the function taking an n-tuple $\bar{a}$ from M to $\phi(\mathrm{tp}(\bar{a}))$. Let us call this new theory T_{CL} (a theory of continuous logic with quantifier elimination), to which we can apply the results of [3]. As just remarked, any model M of T expands uniquely to a model of T_{CL}, but we will still call it M.

To understand *imaginaries* as in Section 5 of [3], we have to also consider definable predicates, without parameters, but in possibly infinitely (yet countably) many variables. As in Proposition 3.10 of [3], such a definable predicate in infinitely many variables can be identified with a continuous function from $S_\omega(T)$ to $[0,1]$ where $S_\omega(T)$ is the space of complete types of T in a fixed countable sequence of variables. We feel free to call such a function (and the corresponding function on ω-tuples in models of T to $[0,1]$) a CL-formula in infinitely many variables. Let us now fix a definable predicate $\phi(\bar{x},\bar{y})$, where $\bar{x}$ is a finite tuple of variables, and $\bar{y}$ is a possibly infinite (but countable) sequence of variables. A "code" for the CL-formula (with parameters $\bar{a}$ and finite tuple $\bar{x}$ of free variables) $\phi(\bar{x},\bar{a})$ will then be a CL-imaginary in the sense of [3], and all CL-imaginaries will arise in this way. The precise formalism (involving new sorts with their own distance relation) is not so important, but the point is that the code will be something fixed by precisely those automorphisms (of a saturated model) which fix the formula $\phi(\bar{x},\bar{a})$. In other words, the code will be the equivalence class of $\bar{a}$ with respect to the obvious equivalence relation $E_\phi(\bar{y},\bar{z})$, on tuples of the appropriate length. If $\bar{y}$ is a finite tuple of variables, then we will call a corresponding imaginary (i.e. code for $\phi(\bar{x},\bar{a})$) a *finitary CL-imaginary.* We will work in the saturated model $\bar{M} = \mathfrak{C}$ of T which will also be a saturated model of T_{CL}. When we speak about interdefinability of various objects, we mean a priori in the sense of automorphisms of $\bar{M}$.

The notion of hyperimaginary is well-established in (usual, classical) model theory [29]. A *hyperimaginary* is by definition $\bar{a}/E$ where $\bar{a}$ is a possibly infinite (but small compared with the saturation) tuple and E a type-definable over $\emptyset$ equivalence relation on tuples of the relevant size. It is known that up to interdefinability we may restrict to tuples of length at most ω, which we henceforth do. When the length of $\bar{a}$ is finite, we call $\bar{a}/E$ a *finitary hyperimaginary.* The following is routine, but we sketch the proof.

Lemma 4.21. *(i) Any [finitary] CL-imaginary is interdefinable with a [finitary] hyperimaginary.*
(ii) If E is a bounded type-definable over $\emptyset$ equivalence relation on finite tuples, then each class of E is interdefinable with a sequence of finitary CL-imaginaries.

Proof. (i) if $\phi(\bar{x}, \bar{y})$ is a CL-formula where $\bar{y}$ is a possibly countably infinite tuple, then the equivalence relation $E(\bar{y}, \bar{z})$ which says of $(\bar{a}, \bar{b})$ that the functions $\phi(\bar{x}, \bar{a})$ and $\phi(\bar{x}, \bar{b})$ are the same, is a type-definable over $\emptyset$ equivalence relation in T.
(ii) It is well-known that E is equivalent to a conjunction of equivalence relations each of which is defined by a countable collection of formulas over $\emptyset$ and is also bounded. So we may assume that E is defined by a countable collection of formulas. Then $\mathfrak{C}/E$ is a compact space, metrizable via an $\mathrm{Aut}(\mathfrak{C})$-invariant metric d (see [24, Section 3, p. 237]). Define $\psi(\bar{x}, \bar{y}) := d(\bar{x}/E, \bar{y}/E)$. This is clearly a CL-formula, and we see that each $\bar{a}/E$ is interdefinable with the code of $\psi(\bar{x}, \bar{a})$. □

Let $\mathrm{acl}^{eq}_{CL}(\emptyset)$ denote the collection of CL-imaginaries which have a bounded number of conjugates under $\mathrm{Aut}(\bar{M})$. Likewise $\mathrm{bdd}^{heq}(\emptyset)$ is the collection of hyperimaginaries with a bounded number of conjugates under $\mathrm{Aut}(\bar{M})$. Now, Theorem 4.15 of [29] says that any bounded hyperimaginary is interdefinable with a sequence of finitary bounded hyperimaginaries. Therefore, by Lemma 4.21, we get

Corollary 4.22. *(i) Up to interdefinability,* $\mathrm{acl}^{eq}_{CL}(\emptyset)$ *coincides with* $\mathrm{bdd}^{heq}(\emptyset)$*.*
(ii) Moreover, $\mathrm{acl}^{eq}_{CL}(\emptyset)$ *is interdefinable with the collection of finitary CL-imaginaries with a bounded number of conjugates under* $\mathrm{Aut}(\bar{M})$*.*

We now appeal to the local stability results in [3] (which go somewhat beyond what we deduced purely from Grothendieck in Section 3). Fix a finite tuple $\bar{x}$ of variables and consider $\Delta(\bar{x})$, the collection of all stable formulas (without parameters) $\phi(\bar{x}, \bar{y})$ of T_{CL}, where $\bar{y}$ varies and where stability is as defined in Section 3. For an n-tuple $\bar{b}$ and set A of parameters (including possibly CL-imaginaries), $\mathrm{tp}_\Delta(\bar{b}/A)$ is the function taking the formula $\phi(\bar{x}, \bar{a})$ to $\phi(\bar{b}, \bar{a})$ where $\phi(\bar{x}, \bar{y}) \in \Delta$ and $\phi(\bar{x}, \bar{a})$ is over A (i.e invariant under $\mathrm{Aut}(\bar{M}/A)$). By definition, a complete Δ-type over A is something of the form $\mathrm{tp}_\Delta(\bar{b}/A)$ (and $\bar{b}$ is a realization of it).

Remark 4.23. For any $\bar{b}$, $\mathrm{tp}(\bar{b}/\,\mathrm{bdd}^{heq}(\emptyset))$ (in the classical case) coincides with $\mathrm{tp}_\Delta(\bar{b}/\,\mathrm{acl}^{eq}_{CL}(\emptyset))$ in the continuous framework.

Proof. Using Corollary 4.22, the left hand side always implies the right hand side. For the other direction, since $\bar{x} \equiv_{\mathrm{bdd}^{heq}(\emptyset)} \bar{y}$ is a bounded, type-definable over $\emptyset$ equivalence relation (in fact, it is exactly E_{KP}), it is enough to show that for any bounded, type-definable over $\emptyset$ equivalence relation E, whenever $\mathrm{tp}_\Delta(\bar{b}/\mathrm{acl}^{eq}_{CL}(\emptyset)) = \mathrm{tp}_\Delta(\bar{b}'/\mathrm{acl}^{eq}_{CL}(\emptyset))$, then $E(\bar{b},\bar{b}')$. Let $\psi(\bar{x},\bar{y})$ be the CL-formula from the proof of Lemma 4.21(ii). As E is bounded, $\psi(\bar{x},\bar{y})$ is stable. The code of $\psi(\bar{x},\bar{b})$ is interdefinable with $\bar{b}/E$, hence it is in $\mathrm{acl}^{eq}_{CL}(\emptyset)$, and so $\psi(\bar{x},\bar{a})$ is over $\mathrm{acl}^{eq}_{CL}(\emptyset)$. Since clearly $\psi(\bar{b},\bar{b}) = 0$, we conclude that $\psi(\bar{b}',\bar{b}) = 0$ which means that $E(\bar{b},\bar{b}')$. □

We have already explained in Section 3 what we mean by definability of a complete Δ-type over a model M. The following is a consequence of the local theory developed in Section 7 of [3] and the discussion around glueing in Section 8 of the same paper. We restrict ourselves to the case needed, i.e. over $\emptyset$.

Fact 4.24. *Let $p(\bar{x})$ be a complete Δ-type over $\mathrm{acl}^{eq}_{CL}(\emptyset)$. Then for any model M (which note contains $\mathrm{acl}^{eq}_{CL}(\emptyset)$) there is a unique complete Δ-type $q(\bar{x})$ over M such that $q(\bar{x})$ extends $p(\bar{x})$ and q is definable over $\mathrm{acl}^{eq}_{CL}(\emptyset)$. We say $q = p|M$. In particular, if $M \prec N$, then $p|M$ is precisely the restriction of $p|N$ to M.*

Definition 4.25. We say that $\bar{b}$ is stably independent from B (or that $\bar{b}$ and B are stably independent) if $\mathrm{tp}_\Delta(\bar{b}/B)$ equals the restriction of $p|M$ to B, where M is some model containing B and $p = \mathrm{tp}_\Delta(\bar{b}/\mathrm{acl}^{eq}_{CL}(\emptyset))$.

The usual Erdös-Rado arguments, together with Fact 4.24 give:

Corollary 4.26. *Let $p(\bar{x})$ be a complete Δ-type over $\mathrm{acl}^{eq}_{CL}(\emptyset)$. Then there is an infinite sequence $(\bar{b}_i : i < \omega)$ of realizations of p which is indiscernible and such that $\bar{b}_i$ is stably independent from $\{\bar{b}_j : j < i\}$ for all i.*

The following consequence of Fact 4.24 will also be important for us:

Corollary 4.27. *Suppose we have finite tuples $\bar{b}$ and $\bar{c}$ from the (classical) model $\mathfrak{C}$. Suppose that $\bar{b}$ is stably independent from $\bar{c}$. Then for any stable CL-formula $\psi(\bar{x},\bar{y})$ (over $\emptyset$), the value of $\psi(\bar{b},\bar{c})$ depends only on $\mathrm{tp}(\bar{b}/\mathrm{bdd}^{heq}(A))$ and $\mathrm{tp}(\bar{c}/\mathrm{bdd}^{heq}(A))$ (in the sense of the classical structure $\mathfrak{C}$).*

Proof. Let $p(\bar{x}) = \mathrm{tp}(\bar{b}/\mathrm{bdd}^{heq}(A))$, which by Remark 4.23 coincides with $\mathrm{tp}_\Delta(\bar{b}/\mathrm{acl}^{eq}_{CL}(\emptyset))$. The $\psi(\bar{x},\bar{y})$-type of $p|\mathfrak{C}$ is by Fact 4.24 definable by a CL-formula $\chi(y)$ over $\mathrm{acl}^{eq}_{CL}(\emptyset) = \mathrm{bdd}^{heq}(\emptyset)$. So assuming the stable independence of $\bar{b}$ and $\bar{c}$, by definition and Fact 4.24, the value of $\psi(\bar{b},\bar{c})$ is equal to $\chi(\bar{c})$, which by Remark 4.23 depends only on $\mathrm{tp}(\bar{c}/\mathrm{bdd}^{heq}(\emptyset))$. If $\bar{b}$ is replaced by another realization $\bar{b}'$ of p which is stably independent from another realization $\bar{c}'$ of $\mathrm{tp}(\bar{c}/\mathrm{bdd}^{heq}(\emptyset))$, then the above shows that $\psi(\bar{b}',\bar{c}') = \chi(\bar{c}') = \chi(\bar{c}) = \psi(\bar{b},\bar{c})$. □

Proposition 4.28. *Let $\mu(\bar{x})$ be a global $\emptyset$-definable Keisler measure. Let $\bar{a}$ and $\bar{b}$ be tuples of the same length from $\mathfrak{C}$, with the same type over $\mathrm{bdd}^{heq}(\emptyset)$, and* *stably independent. Let $p(\bar{x}, \bar{a})$ be a complete type over $\bar{a}$ which is "μ-wide" in the sense that every formula in $p(\bar{x}, \bar{a})$ gets μ-measure > 0. Then the partial type $p(\bar{x}, \bar{a}) \cup p(\bar{x}, \bar{b})$ is also μ-wide (again in the sense that every formula implied by it has μ-measure > 0).*

Proof. By definition, we have to show that if $\phi(\bar{x}, \bar{a})$ is a formula with μ-measure > 0, then $\phi(\bar{x}, \bar{a}) \wedge \phi(\bar{x}, \bar{b})$ has μ-measure > 0. By $\emptyset$-definability of μ, the function $\psi(\bar{y}, \bar{z})$ defined to be $\mu(\phi(\bar{x}, \bar{y}) \wedge \phi(\bar{x}, \bar{z}))$ is definable over $\emptyset$, i.e. is a CL-formula without parameters. Moreover, by Proposition 2.25 of [15], $\psi(\bar{y}, \bar{z})$ is stable. Bearing in mind Remark 4.23, let, by Corollary 4.26, $(\bar{a}_i : i < \omega)$ be an indiscernible sequence of realizations of $\mathrm{tp}(\bar{a}/\operatorname{bdd}^{heq}(\emptyset))$ such that $\bar{a}_j$ and $\bar{a}_i$ are stably independent for all $i < j$ (equivalently for some $i < j$). Since μ is $\mathrm{Aut}(\mathfrak{C})$-invariant, we see that $\mu(\phi(\bar{x}, \bar{a}_i))$ is positive and constant for all i, and $\mu(\phi(\bar{x}, \bar{a}_i) \wedge \phi(\bar{x}, \bar{a}_j))$ is positive (and constant) for $i \neq j$. In particular, $\psi(\bar{a}_0, \bar{a}_1) > 0$. By Corollary 4.27, $\psi(\bar{a}, \bar{b}) > 0$, which is what we had to prove. □

Proposition 4.29. *Suppose that amenability of (the classical, first order theory) T is witnessed by $\emptyset$-definable Keisler measures. Namely, for every formula $\phi(\bar{x})$ over $\emptyset$ there is a global $\emptyset$-definable Keisler measure $\mu(\bar{x})$ concentrating on $\phi(\bar{x})$. Then T is G-compact.*

Proof. We have to show that if $\bar{b}$, $\bar{c}$ are tuples of the same (but possibly infinite) length and with the same type over $\operatorname{bdd}^{heq}(\emptyset)$, then they have the same Lascar strong type.

Assume first that $\bar{b}$ and $\bar{c}$ are stably independent in the sense of Definition 4.25. (If the length of these tuples is infinite, we mean that any two finite corresponding subtuples of $\bar{a}$ and $\bar{b}$ are stably independent.) Fix a model M_0 and enumerate it. We will find a copy M of M_0 such that $\mathrm{tp}(\bar{b}/M) = \mathrm{tp}(\bar{c}/M)$ (which immediately yields that $\bar{b}$ and $\bar{c}$ have the same Lascar strong type). By compactness, given a consistent formula $\phi(\bar{y})$ in finitely many variables, it suffices to find some realization $\bar{m}$ of $\phi(\bar{y})$ such that $\mathrm{tp}(\bar{b}/\bar{m}) = \mathrm{tp}(\bar{c}/\bar{m})$. Again by compactness, we may assume that $\bar{b}$, $\bar{c}$ are finite tuples. By assumption, let $\mu(\bar{y})$ be a global Keisler measure concentrating on $\phi(\bar{y})$ and which is $\emptyset$-definable. Let $p(\bar{y}, \bar{b})$ be a complete type over $\bar{b}$ which is μ-wide. By Proposition 4.28, $p(\bar{y}, \bar{b}) \cup p(\bar{y}, \bar{c})$ is also μ-wide, in particular consistent. So let $\bar{m}$ realize it.

In general, given (possibly infinite) tuples $\bar{b}$, $\bar{c}$ with the same type over $\operatorname{bdd}^{heq}(\emptyset)$, let $\bar{d}$ have the same type over $\operatorname{bdd}^{heq}(\emptyset)$ and be stably independent from $\{\bar{b}, \bar{c}\}$ (by Fact 4.24, uniqueness, and compactness). By what we have just shown, $\bar{b}$ and $\bar{d}$ have the same Lascar strong type, and $\bar{c}$ and $\bar{d}$ have the same Lascar strong type. So $\bar{b}$ and $\bar{c}$ do, too. □

4.4. **Amenability implies G-compactness: the general case.** Let T be an arbitrary theory, $\mathfrak{C} \models T$ a monster model, and $\bar{c}$ an enumeration of $\mathfrak{C}$. The goal of this subsection is to prove

Theorem 4.30. *If T is amenable, then T is G-compact. In fact, the diameter of each Lascar strong type (over $\emptyset$) is bounded by* 4.

Recall, once again, that in [27] it was deduced from Conjecture 0.1 for groups with a basis of open sets at 1 consisting of open subgroups that if M is a countable ω-categorical structure and $\mathrm{Aut}(M)$ is amenable (as a topological group), then $\mathrm{Th}(M)$ is G-compact. By Corollary 4.16, we see that Theorem 4.30 is a generalization of this result.

Before we start our analysis towards the proof of Theorem 4.30, let us first note an analogous statement for extreme amenability, which is much easier to prove.

Proposition 4.31. *If $p(\bar{x}) \in S(\emptyset)$ is extremely amenable, then $p(\bar{x})$ is a single Lascar strong type. Moreover, the Lascar diameter of $p(\bar{x})$ is at most* 2.

In particular, if T is extremely amenable, then the Lascar strong types coincide with complete types (over $\emptyset$), i.e. the Lascar Galois group $\mathrm{Gal}_L(T)$ is trivial.

Proof. Choose $\mathfrak{C}$ so that $\bar{x}$ is short in $\mathfrak{C}$. Let $q \in S_p(\mathfrak{C})$ be invariant under $\mathrm{Aut}(\mathfrak{C})$. Fix $\bar{\alpha} \models q$ (in a bigger model). Take a small $M \prec \mathfrak{C}$ and choose $\bar{\beta} \in \mathfrak{C}$ such that $\bar{\beta} \models q|_M$. Then $\bar{\alpha}\, E_L\, \bar{\beta}$. But also, for any $\sigma \in \mathrm{Aut}(\mathfrak{C})$, $\sigma(\bar{\beta}) \models \sigma(q)|_{\sigma[M]} = q|_{\sigma[M]}$, and so $\sigma(\bar{\beta})\, E_L\, \bar{\alpha}$. Therefore, $\sigma(\bar{\beta})\, E_L\, \bar{\beta}$ for any $\sigma \in \mathrm{Aut}(\mathfrak{C})$, which shows that $p(\bar{x})$ is a single Lascar strong type.

For the "moreover part" notice that, in the above argument, both $d_L(\bar{\alpha}, \bar{\beta})$ and $d_L(\sigma(\bar{\beta}), \bar{\alpha})$ are bounded by 1. □

Recall from Corollary 4.17 that by a *relatively type-definable subset* of $\mathrm{Aut}(\mathfrak{C})$ we mean a subset of the form $A_{\pi,\bar{a},\bar{b}} := \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi(\sigma(\bar{a}), \bar{b}))\}$ for some partial type $\pi(\bar{x}, \bar{y})$ (without parameters), where $\bar{x}$ and $\bar{y}$ are short tuples of variables and $\bar{a}$, $\bar{b}$ are from $\mathfrak{C}$. (Note that although here we allow repetitions in the tuple $\bar{a}$, whereas in Corollary 4.17 $\bar{a}$ was a subtuple of $\bar{c}$, both versions yield the same class of relatively type-definable sets.) Without loss $\bar{x}$ is of the same length as $\bar{y}$ and $\bar{a} = \bar{b}$, and then we write $A_{\bar{\pi},\bar{a}}$. In fact, the following remark is very easy.

Remark 4.32. For any partial types $\pi_1(\bar{x}_1, \bar{y}_1)$ and $\pi_2(\bar{x}_2, \bar{y}_2)$ and tuples $\bar{a}_1, \bar{a}_2, \bar{b}_1, \bar{b}_2$ in $\mathfrak{C}$ corresponding to $\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2$, one can find partial types $\pi_1'(\bar{x}, \bar{y})$ and $\pi_2'(\bar{x}, \bar{y})$ with $\bar{x}$ of the same length (by which we also mean of the same sorts) as $\bar{y}$ and a tuple $\bar{a}$ in $\mathfrak{C}$ corresponding to $\bar{x}$ such that $A_{\pi_1,\bar{a}_1,\bar{b}_1} = A_{\pi_1',\bar{a}}$ and $A_{\pi_2,\bar{a}_2,\bar{b}_2} = A_{\pi_2',\bar{a}}$.

For a short tuple $\bar{\alpha}$ and a short tuple of parameters $\bar{b}$, a subset of $\mathrm{Aut}(\mathfrak{C})$ is called *relatively $\bar{\alpha}$-type-definable over $\bar{b}$* if it is of the form $A_{\pi,\bar{\alpha},\bar{b}}$ for some partial type $\pi(\bar{x}, \bar{y})$.

The next fact was observed in [25].

Fact 4.33 (Proposition 5.2 of [25]). *If G is a closed, bounded index subgroup of* $\mathrm{Aut}(\mathfrak{C})$ *(with* $\mathrm{Aut}(\mathfrak{C})$ *equipped with the pointwise convergence topology), then* $\mathrm{Autf}_L(\mathfrak{C}) \leq G$.

Using an argument similar to the proof of Fact 4.33, we will first show

Proposition 4.34. *If G is a relatively type-definable, bounded index subgroup of* $\mathrm{Aut}(\mathfrak{C})$*, then* $\mathrm{Autf}_{KP}(\mathfrak{C}) \leq G$*.*

Proof. Let σ_i, $i < \lambda$, by a set of representatives of the left cosets of G in $\mathrm{Aut}(\mathfrak{C})$ (so λ is bounded). Then

$$G' := \bigcap_{\sigma \in \mathrm{Aut}(\mathfrak{C})} G^{\sigma} = \bigcap_{i<\lambda} G^{\sigma_i}$$

is a normal, bounded index subgroup of $\mathrm{Aut}(\mathfrak{C})$ (where $G^{\sigma} := \sigma G \sigma^{-1}$).

Let us show now that G' is relatively type-definable. We have $G = A_{\pi,\bar{a}} = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi(\sigma(\bar{a}), \bar{a})\}$ for some type $\pi(\bar{x}, \bar{y})$ (with short $\bar{x}$, $\bar{y}$) and tuple $\bar{a}$ in $\mathfrak{C}$. Then $G^{\sigma_i} = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi(\sigma(\sigma_i(\bar{a})), \sigma_i(\bar{a}))\}$, so putting $\bar{a}' = \langle \sigma_i(\bar{a})\rangle_{i<\lambda}$, $\bar{x}' = \langle \bar{x}_i \rangle_{i<\lambda}$, $\bar{y}' = \langle \bar{y}_i\rangle_{i<\lambda}$ (where $\bar{x}_i$ and $\bar{y}_i$ are copies of $\bar{x}$ and $\bar{y}$, respectively) and $\pi'(\bar{x}', \bar{y}') = \bigcup_{i<\lambda} \pi(\bar{x}_i, \bar{y}_i)$ (as a set of formulas), we see that

$$G' = A_{\pi',\bar{a}'} = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi'(\sigma(\bar{a}'), \bar{a}')\},$$

which is clearly relatively type-definable.

The orbit equivalence relation E of the action of G' on the set of realizations of $\mathrm{tp}(\bar{a}'/\emptyset)$ is a bounded equivalence relation. This relation is type-definable, because

$$\bar{\alpha}\, E\, \bar{\beta} \iff (\exists g \in G')(g(\bar{\alpha}) = \bar{\beta}) \iff (\exists \bar{b}')(\pi'(\bar{b}', \bar{a}') \wedge \bar{a}'\bar{\alpha} \equiv \bar{b}'\bar{\beta}).$$

But E is also invariant (as G' is a normal subgroup of $\mathrm{Aut}(\mathfrak{C})$), so E is type-definable over $\emptyset$. Therefore, E is refined by E_{KP}.

Now, take any $\sigma \in \mathrm{Autf}_{KP}(\mathfrak{C})$. By the last conclusion, there is $\tau \in G'$ such that $\sigma(\bar{a}') = \tau(\bar{a}')$. Then $\tau^{-1}\sigma(\bar{a}') = \bar{a}'$ and $\sigma = \tau(\tau^{-1}\sigma)$. Since the above formula for G' shows that $G' \cdot \mathrm{Fix}(\bar{a}') = G'$, we get $\sigma \in G'$. Thus, $\mathrm{Autf}_{KP}(\mathfrak{C}) \leq G' \leq G$. □

Corollary 4.35. *If $\{C_i : i \in \omega\}$ is a family of relatively definable, generic, symmetric subsets of* $\mathrm{Aut}(\mathfrak{C})$ *such that $C_{i+1}^2 \subseteq C_i$ for all $i \in \omega$, then $\bigcap_{i\in\omega} C_i$ is a subgroup of* $\mathrm{Aut}(\mathfrak{C})$ *containing* $\mathrm{Autf}_{KP}(\mathfrak{C})$*.*

Proof. It is clear that $\bigcap_{i\in\omega} C_i$ is a subgroup of $\mathrm{Aut}(\mathfrak{C})$, and it is easy to show that it has bounded index (at most $2^{\aleph_0}$). Moreover, it is clearly relatively type-definable. Thus, the fact that it contains $\mathrm{Autf}_{KP}(\mathfrak{C})$ follows from Proposition 4.34. □

Lemma 4.36. *i) Let $\pi(\bar{x}, \bar{y})$ be a partial type (over $\emptyset$) and $\bar{a}$, $\bar{b}$ short tuples from $\mathfrak{C}$ corresponding to $\bar{x}$ and $\bar{y}$, respectively. Then $A^{-1}_{\pi,\bar{a},\bar{b}} = A_{\pi',\bar{b},\bar{a}}$, where $\pi'(\bar{y}, \bar{x}) = \pi(\bar{x}, \bar{y})$.*

ii) Let $n \geq 2$ be a natural number. Let $\bar{x}$, $\bar{y}$ and $\bar{x}_1, \dots, \bar{x}_n$ be disjoint, short tuples of variables of the same length. Then there exists a partial type $\Phi_n(\bar{x}, \bar{y}, \bar{x}_1, \dots, \bar{x}_n)$ such that for every partial types $\pi_1(\bar{x}_1, \bar{y}), \dots, \pi_n(\bar{x}_n, \bar{y})$ and tuple $\bar{a}$ corresponding to $\bar{x}$ one has

$$A_{\pi_1,\bar{a}} \cdot \ldots \cdot A_{\pi_n,\bar{a}} = A_{\pi,\bar{a}},$$

where

$$\pi(\bar{x}, \bar{y}) = (\exists \bar{x}_1, \dots, \bar{x}_n)(\pi_1(\bar{x}_1, \bar{y}) \wedge \dots \wedge \pi_n(\bar{x}_n, \bar{y}) \wedge \Phi_n(\bar{x}, \bar{y}, \bar{x}_1, \dots, \bar{x}_n)).$$

Proof. (i) follows immediately from the fact that for any $\sigma \in \mathrm{Aut}(\mathfrak{C})$

$$\mathfrak{C} \models \pi(\sigma(\bar{a}), \bar{b}) \iff \mathfrak{C} \models \pi(\bar{a}, \sigma^{-1}(\bar{b})) \iff \mathfrak{C} \models \pi'(\sigma^{-1}(\bar{b}), \bar{a}).$$

(ii) We will show that for $n = 2$ the type $\Phi_2(\bar{x}, \bar{y}, \bar{x}_1, \bar{x}_2) := (\bar{x}\bar{x}_1 \equiv \bar{x}_2\bar{y})$ and for $n \geq 3$ the type $\Phi_n(\bar{x}, \bar{y}, \bar{x}_1, \dots, \bar{x}_n)$ defined as

$$(\exists \bar{z}_1, \dots, \bar{z}_{n-2})(\bar{x}\bar{z}_{n-2} \equiv \bar{x}_n\bar{y} \wedge \bar{z}_{n-2}\bar{z}_{n-3} \equiv \bar{x}_{n-1}\bar{y} \wedge \dots \wedge \bar{z}_2\bar{z}_1 \equiv \bar{x}_3\bar{y} \wedge \bar{z}_1\bar{x}_1 \equiv \bar{x}_2\bar{y})$$

is as required.

First, let us see that $A_{\pi_1,\bar{a}} \cdot \ldots \cdot A_{\pi_n,\bar{a}} \subseteq A_{\pi,\bar{a}}$. Take σ from the left hand side, i.e. $\sigma = \sigma_1 \cdot \ldots \cdot \sigma_n$, where $\models \pi_i(\sigma_i(\bar{a}), \bar{a})$. Then $\models \pi(\sigma(\bar{a}), \bar{a})$ is witnessed by $\bar{x}_i := \sigma_i(\bar{a})$ for $i = 1, \dots, n$ and $\bar{z}_i := (\sigma_1 \dots \sigma_{i+1})(\bar{a})$ for $i = 1, \dots, n-2$. So $\sigma \in A_{\pi,\bar{a}}$.

Finally, we will justify that $A_{\pi_1,\bar{a}} \cdot \ldots \cdot A_{\pi_n,\bar{a}} \supseteq A_{\pi,\bar{a}}$. Take any σ such that $\models \pi(\sigma(\bar{a}), \bar{a})$. Let $\bar{a}_1, \dots, \bar{a}_n$ be witnesses for $\bar{x}_1, \dots, \bar{x}_n$, and $\bar{b}_1, \dots, \bar{b}_{n-2}$ be witnesses for $\bar{z}_1, \dots, \bar{z}_{n-2}$, i.e.:

(1) $\models \pi_i(\bar{a}_i, \bar{a})$ for $i = 1, \dots, n$, and
(2) $\sigma(\bar{a})\bar{b}_{n-2} \equiv \bar{a}_n\bar{a} \wedge \bar{b}_{n-2}\bar{b}_{n-3} \equiv \bar{a}_{n-1}\bar{a} \wedge \dots \wedge \bar{b}_2\bar{b}_1 \equiv \bar{a}_3\bar{a} \wedge \bar{b}_1\bar{a}_1 \equiv \bar{a}_2\bar{a}$.

By (2), there are $\tau_1, \dots, \tau_{n-1} \in \mathrm{Aut}(\mathfrak{C})$ mapping the right hand sides of the equivalences in (2) to the left hand sides. Then $\tau_1(\bar{a}_n) = \sigma(\bar{a})$, so $\tau_1^{-1}\sigma(\bar{a}) = \bar{a}_n$, so $\tau_1^{-1}\sigma \in A_{\pi_n,\bar{a}}$ by (1). Next, $\tau_1(\bar{a}) = \bar{b}_{n-2} = \tau_2(\bar{a}_{n-1})$, so $\tau_2^{-1}\tau_1(\bar{a}) = \bar{a}_{n-1}$, so $\tau_2^{-1}\tau_1 \in A_{\pi_{n-1},\bar{a}}$ by (1). We continue in this way, obtaining in the last step: $\tau_{n-1}(\bar{a}) = \bar{a}_1$, so $\tau_{n-1} \in A_{\pi_1,\bar{a}}$ by (1). Therefore,

$$\sigma = \tau_{n-1}(\tau_{n-1}^{-1}\tau_{n-2}) \dots (\tau_2^{-1}\tau_1)(\tau_1^{-1}\sigma) \in A_{\pi_1,\bar{a}} \cdot \ldots \cdot A_{\pi_n,\bar{a}}.$$

□

Corollary 4.37. *Let $\pi_1(\bar{x}, \bar{y}), \dots, \pi_n(\bar{x}, \bar{y})$ be partial types, $\bar{a}$ a tuple corresponding to $\bar{x}$ and $\bar{y}$, and $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$.*
(i) Then

$$A^{\epsilon_1}_{\pi_1,\bar{a}} \cdot \ldots \cdot A^{\epsilon_n}_{\pi_n,\bar{a}} = \bigcap \left\{ A^{\epsilon_1}_{\varphi_1,\bar{a}} \cdot \ldots \cdot A^{\epsilon_n}_{\varphi_n,\bar{a}} : \pi_1 \vdash \varphi_1, \dots, \pi_n \vdash \varphi_n \right\}.$$

ii) If $A^{\epsilon_1}_{\pi_1,\bar{a}} \cdot \ldots \cdot A^{\epsilon_n}_{\pi_n,\bar{a}}$ is contained in a relatively definable subset A of $\mathrm{Aut}(\mathfrak{C})$, then there are $\varphi_i(\bar{x}, \bar{y})$ implied by $\pi_i(\bar{x}, \bar{y})$ for $i = 1, \dots, n$, such that $A^{\epsilon_1}_{\varphi_1,\bar{a}} \cdot \ldots \cdot A^{\epsilon_n}_{\varphi_n,\bar{a}} \subseteq A$.

Lemma 4.38. *Let $p(\bar{x}) \in S(\emptyset)$ with $\bar{x}$ short, $q \in S_p(\mathfrak{C})$, $M \prec \mathfrak{C}$ small, and $\bar{\alpha} \models q|_M$. Then $A_{q|_{\bar{\alpha}},\bar{\alpha}} A_{q|_{\bar{\alpha}},\bar{\alpha}} A^{-1}_{q|_{\bar{\alpha}},\bar{\alpha}} A^{-1}_{q|_{\bar{\alpha}},\bar{\alpha}} \bar{\alpha} \subseteq \{\bar{\beta} \subset \mathfrak{C} : d_L(\bar{\alpha}, \bar{\beta}) \leq 4\} \subseteq [\bar{\alpha}]_{E_L}$.*

Proof. Let us start from the following

Claim 1: For any $\bar{\beta} \models q|_{\bar{\alpha}}$, $d_L(\bar{\beta}, \bar{\alpha}) \leq 1$.

Proof. Take $\bar{\gamma} \models q|_{M\bar{\alpha}}$. Then $d_L(\bar{\gamma}, \bar{\alpha}) \leq 1$, so the conclusion follows from the fact that $\bar{\beta} \equiv_{\bar{\alpha}} \bar{\gamma}$. □(claim)

Now, consider any $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in A_{q|_{\bar{\alpha}}, \bar{\alpha}}$. Then $\sigma_i(\bar{\alpha}) \models q|_{\bar{\alpha}}$, so, by the claim, we get $d_L(\sigma_i(\bar{\alpha}), \bar{\alpha}) \leq 1$. Therefore, $d_L(\sigma_4^{-1}(\bar{\alpha}), \bar{\alpha}) \leq 1$, so $d_L(\sigma_3^{-1}\sigma_4^{-1}(\bar{\alpha}), \sigma_3^{-1}(\bar{\alpha})) \leq 1$, so $d_L(\sigma_3^{-1}\sigma_4^{-1}(\bar{\alpha}), \bar{\alpha}) \leq 2$, so $d_L(\sigma_2\sigma_3^{-1}\sigma_4^{-1}(\bar{\alpha}), \sigma_2(\bar{\alpha})) \leq 2$, so $d_L(\sigma_2\sigma_3^{-1}\sigma_4^{-1}(\bar{\alpha}), \bar{\alpha}) \leq 3$, so $d_L(\sigma_1\sigma_2\sigma_3^{-1}\sigma_4^{-1}(\bar{\alpha}), \sigma_1(\bar{\alpha})) \leq 3$, so $d_L(\sigma_1\sigma_2\sigma_3^{-1}\sigma_4^{-1}(\bar{\alpha}), \bar{\alpha}) \leq 4$. $\square$

Lemma 4.39. *Assume* $\mathrm{Aut}(\mathfrak{C})$ *is relatively definably amenable. By Corollary 4.17, take the induced* $\mathrm{Aut}(\mathfrak{C})$*-invariant, finitely additive, probability measure* μ *on the Boolean algebra* $\mathcal{A}$ *generated by relatively type-definable subsets of* $\mathrm{Aut}(\mathfrak{C})$*. Suppose* $A \subseteq \mathrm{Aut}(\mathfrak{C})$ *is relatively type-definable with* $\mu(A) > 0$ *and* $A^4 := AAA^{-1}A^{-1} \subseteq A'$ *for some relatively definable* $A' \subseteq \mathrm{Aut}(\mathfrak{C})$*. Then there exists a relatively type-definable, generic, symmetric* $Y \subseteq \mathrm{Aut}(\mathfrak{C})$ *such that* $Y^8 \subseteq A'$*.*

Proof. By Lemma 4.36, relatively type-definable sets are closed under taking products and inversions, and one can easily check that also under left translations.

Claim 1: There exists a generic and symmetric set $S \subseteq \mathrm{Aut}(\mathfrak{C})$ such that:

(1) $S^{16} \subseteq AAA^{-1}A^{-1}$,
(2) $S = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathrm{tp}(\sigma(\bar{a})/\bar{a}) \in \mathcal{P}\}$ for some $\mathcal{P} \subseteq S_{\bar{a}}(\bar{a})$ and some short tuple $\bar{a}$ (which is a tuple of finitely many conjugates by elements of $\mathrm{Aut}(\mathfrak{C})$ of the tuple over which A is relatively type-definable).

Proof. Apply Corollary 2.11 for $G := \mathrm{Aut}(\mathfrak{C})$, A from the statement of Proposition 4.39, $\mathcal{B} := \{A\}$, and $N = 16$. As a result, we obtain a set $B' = A \cap \sigma_1[A] \cap \dots \cap \sigma_n[A]$ for some σ_i's in $\mathrm{Aut}(\mathfrak{C})$ such that for some $l \in \mathbb{N}_{>0}$, $S := \mathrm{St}_{\mathcal{L}_{l-1}}(B')$ is generic, symmetric and satisfies $S^{16} \subseteq AAA^{-1}A^{-1}$. Since A is relatively type-definable over some short tuple $\bar{\alpha}$, so is B', but over $\bar{a} := \bar{\alpha}\sigma_1(\bar{\alpha}) \dots \sigma_n(\bar{\alpha})$. Hence, by the definition of S, we easily get that

$$\mathrm{Aut}(\mathfrak{C}/\bar{a}) \cdot S \cdot \mathrm{Aut}(\mathfrak{C}/\bar{a}) = S,$$

which means that $S = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathrm{tp}(\sigma(\bar{a})/\bar{a}) \in \mathcal{P}\}$ for some $\mathcal{P} \subseteq S_{\bar{a}}(\bar{a})$. $\square$(claim)

Take any $p \in \mathcal{P}$. We can write $p = p(\bar{x}, \bar{a})$ for the obvious complete type $p(\bar{x}, \bar{y})$ over $\emptyset$. Then $(A_{p,\bar{a}} \cdot A_{p,\bar{a}}^{-1})^8 \subseteq (SS^{-1})^8 = S^{16} \subseteq AAA^{-1}A^{-1} \subseteq A'$. Hence, by Corollary 4.37(ii), there is $\psi_p(\bar{x}, \bar{y}) \in p(\bar{x}, \bar{y})$ for which $(A_{\psi_p,\bar{a}} \cdot A_{\psi_p,\bar{a}}^{-1})^8 \subseteq A'$.

Now, the complement of $\bigcup_{p\in\mathcal{P}} A_{\psi_p,\bar{a}}$ equals $\bigcap_{p\in\mathcal{P}} A_{\neg\psi_p,\bar{a}}$ which is clearly relatively type-definable. Thus, $\bigcup_{p\in\mathcal{P}} A_{\psi_p,\bar{a}} \in \mathcal{A}$. On the other hand, $S \subseteq \bigcup_{p\in\mathcal{P}} A_{\psi_p,\bar{a}}$ and S being generic implies that $\bigcup_{p\in\mathcal{P}} A_{\psi_p,\bar{a}}$ is generic. Therefore, $\mu(\bigcup_{p\in\mathcal{P}} A_{\psi_p,\bar{a}}) > 0$.

Let $\tilde{\mu}$ be the $\mathrm{Aut}(\mathfrak{C})$-invariant, (regular) Borel probability measure on $S_{\bar{c}}(\mathfrak{C})$ from which μ is induced. Then $\tilde{\mu}(\bigcup_{p\in\mathcal{P}}[\psi_p]) > 0$, so, by regularity, there is a compact $K \subseteq \bigcup_{p\in\mathcal{P}}[\psi_p]$ of positive measure. But K is covered by finitely many clopen sets $[\psi_p]$ one of which must be of positive measure, i.e. $\tilde{\mu}([\psi_p]) > 0$ for some $p \in \mathcal{P}$. Then $\mu(A_{\psi_p,\bar{a}}) > 0$. This implies that $Y := A_{\psi_p,\bar{a}} \cdot A_{\psi_p,\bar{a}}^{-1}$ is generic, and it is clearly symmetric. By Lemma 4.36, it is also relatively type-definable. Moreover, by the choice of ψ_p, $Y^8 \subseteq A'$, so we are done. $\square$

Corollary 4.40. *Assume* $\mathrm{Aut}(\mathfrak{C})$ *is relatively definably amenable. By Corollary 4.17, take the induced* $\mathrm{Aut}(\mathfrak{C})$*-invariant, finitely additive, probability measure* μ *on the Boolean algebra* $\mathcal{A}$ *generated by relatively type-definable subsets of* $\mathrm{Aut}(\mathfrak{C})$*. Suppose* $A \subseteq \mathrm{Aut}(\mathfrak{C})$ *is relatively type-definable and* $\mu(A) > 0$*. Then* $\mathrm{Autf}_{KP}(\mathfrak{C}) \subseteq AAA^{-1}A^{-1}$.

Proof. Take any A' relatively definable, symmetric, and such that $AAA^{-1}A^{-1} \subseteq A'$. Put $C_0 := A'$.

By Lemma 4.39, we obtain a relatively type-definable, generic, symmetric Y such that $(Y^4)^2 \subseteq A'$. So, by Corollary 4.37, there is a relatively definable, symmetric Y' satisfying $Y^4 \subseteq Y'$ and $Y'^2 \subseteq A'$. Put $C_1 := Y'$.

Next, we apply Lemma 4.39 to Y in place of A and Y' in place of A', and we obtain a relatively type-definable, generic, symmetric Z such that $(Z^4)^2 \subseteq Y'$. So, by Corollary 4.37, there is a relatively definable, symmetric Z' satisfying $Z^4 \subseteq Z'$ and $Z'^2 \subseteq Y'$. Put $C_2 := Z'$.

Continuing in this way, we obtain a family $\{C_i : i \in \omega\}$ of relatively definable, generic, symmetric subsets of $\mathrm{Aut}(\mathfrak{C})$ such that $C_{i+1}^2 \subseteq C_i$ for every $i \in \omega$. By Corollary 4.35, $\mathrm{Autf}_{KP}(\mathfrak{C}) \subseteq \bigcap_{i\in\omega} C_i \subseteq A'$. Since A' was an arbitrary relatively definable, symmetric set containing A^4, we get $\mathrm{Autf}_{KP}(\mathfrak{C}) \subseteq A^4$. $\square$

We have now all the ingredients to prove the main result of this section.

Proof of Theorem 4.30. By Corollary 4.17, a measure $\tilde{\mu}$ on $S_{\bar{c}}(\mathfrak{C})$ witnessing relative definable amenability of $\mathrm{Aut}(\mathfrak{C})$ induces an $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive, probability measure μ on the Boolean algebra $\mathcal{A}$ generated by relatively type-definable subsets of $\mathrm{Aut}(\mathfrak{C})$.

Consider any $p(\bar{x}) = \mathrm{tp}(\bar{\alpha}/\emptyset) \in S(\emptyset)$ with a short subtuple $\bar{\alpha}$ of $\bar{c}$. Choose a μ-wide type $q \in S_p(\mathfrak{C})$, i.e. $\tilde{\mu}([\varphi(\bar{x}', \bar{b})]) > 0$ (equivalently, $\mu(A_{\varphi,\bar{\alpha},\bar{b}}) > 0$) for any $\varphi(\bar{x}, \bar{b}) \in q$ (where $\bar{x}' \supset \bar{x}$ is the tuple of variables corresponding to $\bar{c}$). Take a small model $M \prec \mathfrak{C}$. Applying an appropriate automorphism of $\mathfrak{C}$ to q and M, and using $\mathrm{Aut}(\mathfrak{C})$-invariance of μ, we can assume that $\bar{\alpha} \models q|_M$.

Consider any $\varphi(\bar{x}, \bar{\alpha}) \in q|_{\bar{\alpha}}$. Then $\mu(A_{\varphi,\bar{\alpha}}) > 0$, so, by Corollary 4.40, we conclude that $\mathrm{Autf}_{KP}(\mathfrak{C}) \subseteq A_{\varphi,\bar{\alpha}}A_{\varphi,\bar{\alpha}}A^{-1}_{\varphi,\bar{\alpha}}A^{-1}_{\varphi,\bar{\alpha}}$. Therefore, by Lemma 4.37(i), we get

$$\mathrm{Autf}_{KP}(\mathfrak{C}) \subseteq \bigcap_{\varphi(\bar{x},\bar{\alpha})\in q|_{\bar{\alpha}}} A_{\varphi,\bar{\alpha}}A_{\varphi,\bar{\alpha}}A^{-1}_{\varphi,\bar{\alpha}}A^{-1}_{\varphi,\bar{\alpha}} = A_{q|_{\bar{\alpha}},\bar{\alpha}}A_{q|_{\bar{\alpha}},\bar{\alpha}}A^{-1}_{q|_{\bar{\alpha}},\bar{\alpha}}A^{-1}_{q|_{\bar{\alpha}},\bar{\alpha}}.$$

On the other hand, Lemma 4.38 tells us that

$$A_{q|_{\bar{\alpha}},\bar{\alpha}}A_{q|_{\bar{\alpha}},\bar{\alpha}}A^{-1}_{q|_{\bar{\alpha}},\bar{\alpha}}A^{-1}_{q|_{\bar{\alpha}},\bar{\alpha}}\bar{\alpha} \subseteq \{\bar{\beta} : d_L(\bar{\alpha}, \bar{\beta}) \leq 4\} \subseteq [\bar{\alpha}]_{E_L}.$$

Therefore, $[\bar{\alpha}]_{E_{KP}} = [\bar{\alpha}]_{E_L}$ has diameter at most 4. $\square$

Theorem 4.30 is a global result. It is natural to ask whether we can extend it to a local version (as in Proposition 4.31).

Question 4.41. *Is it true that if $p(\bar{x}) \in S(\emptyset)$ is amenable, then the Lascar strong types on $p(\bar{x})$ coincide with Kim-Pillay strong types? Does amenability of $p(\bar{x})$ imply that the Lascar diameter of $p(\bar{x})$ is at most 4?*

One could think that the above arguments should yield the positive answer to these questions. The problem is that, assuming only amenability of $p(\bar{x})$, we have the induced measure μ but defined only on the Boolean algebra of relatively $\bar{\alpha}$-type-definable subsets of $\mathrm{Aut}(\mathfrak{C})$, for a fixed $\bar{\alpha} \models p$. So, for the recursive proof of Corollary 4.40 to go through, starting from a set $A \subseteq \mathrm{Aut}(\mathfrak{C})$ relatively $\bar{\alpha}$-type-definable [where for the purpose of answering Question 4.41 via an argument as in the proof of Theorem 4.30, we can additionally assume that A is defined over $\bar{\alpha}$] of positive measure, we need to produce the desired Y also relatively $\bar{\alpha}$-type-definable [over $\bar{\alpha}$] (in order be able to continue our recursion). But this requires a strengthening of Lemma 4.39 to the version where for A relatively $\bar{\alpha}$-type-definable of positive measure one wants to obtain the desired Y which is also relatively $\bar{\alpha}$-type-definable; the variant with A and Y defined over $\bar{\alpha}$ would also be sufficient. Trying to follow the lines of the proof of Lemma 4.39, even if A is defined over $\bar{\alpha}$, Claim 1 requires a longer tuple $\bar{a}$ which produces the desired set Y which is relatively $\bar{a}$-type-definable, and this is the only obstacle to answer positively Question 4.41 via the above arguments.

5. Final remarks

5.1. **Connected components and approximate subgroups.** We clarify the connections between the question of the equality of connected components $G^{000} = G^{00}$, and definable approximate subgroups.

Here, we will work in the simpler case where no definable topology is present. Also, we work in a saturated model and over a fixed small set of parameters (even a small model). Definability, connected components, etc. will be relative to this set of parameters.

We consider to begin with a definable group G and a definable, symmetric subset X of G. $\langle X \rangle$ denotes the subgroup H, say, of G generated by X (an ind-definable subgroup) and X is said to be an *approximate subgroup* of G if X is generic in H, namely a bounded number of translates of X cover H. (It may be of interest to consider the same notion for type-definable X.) In this context, and under various auxiliary amenability-type hypotheses, one proves the "stabilizer theorem"

$$(\diamond)\ H^{00} \subseteq X^4.$$

This leads to a connection with locally compact groups L, and through them Lie groups. (See [15], [31], [35], and most relevant to us [30].) Massicot and Wagner conjecture that "even without the definable amenability assumption a suitable Lie model exists".

In this paper, we have restricted to the case where the ind-definable group H is actually definable, hence may be assumed, notationally, to be G. In this case, the locally compact group L is compact. This case is not ruled out as trivial, and indeed is of considerable interest; for instance some of the first theorems in this line, by Gowers and Helfgott, asserted in effect that generic definable subsets of certain pseudofinite groups generated the group in boundedly many steps (3 or 4), and were in turn important in further developments by Bourgain-Gamburd and many others. Remaining in this definable context, it is known that G^{000} is generated by a certain partial type, generic in G, namely $\mathcal{P} = \{a_1 a_2^{-1} : (a_1, a_2, ...) \text{ is indiscernible}\}$. Writing $\mathcal{P}$ as an intersection of definable, symmetric subsets P_n, then each P_n is an approximate subgroup of G. Hence, if the basic result ($\diamond$) holds for every generic, definable, symmetric subset X of G, it follows that $G^{00} \subseteq \mathcal{P}^4$, hence $G^{000} = G^{00}$.

More generally, in the ind-definable setting above (where $H = \langle X \rangle$) we can consider the same notions, and again the truth of ($\diamond$) for all generic, definable, symmetric subsets of H implies that $H^{000} = H^{00}$.

So, we see that any example where the relevant connected components differ must include definable approximate subgroups where ($\diamond$) fails (even with 4 replaced by any definite integer).

Starting from another angle, let $f : G \to H$ be a quasi-homomorphism, namely a map such that $f(xy)f(y)^{-1}f(x)^{-1}$ has finite image. Then any approximate subgroup Γ of H pulls back to an approximate subgroup $f^{-1}[\Gamma]$ of G. Examples in [11] (with $H = \mathbb{Z}^*$) show that even if Γ is a subgroup of H, $f^{-1}[\Gamma]$ need not be a subgroup or even rapidly generate one. Thus, the source of approximateness of $f^{-1}[\Gamma]$ (or for that matter of f itself as a subset of $G \times H$) does not appear to lie in Lie groups.

Regarding the Massicot-Wagner conjecture mentioned above, this shows at least that any connection to a suitable "Lie model" would have to differ substantially from the one proved in the amenable cases.

5.2. **Connected components and complexity.** Let us consider these notions from the point of view of descriptive set theory (see for example [32] for the terms below.) Fix a countable language L with distinguished sort G (with a binary operation), and consider the space of complete theories T (with G a group). For now, G^{000} etc. will mean $G^{000}_{\emptyset}$ etc.

The condition $G = G^{000}$ is at the finite level of the Borel hierarchy ("arithmetic"), and is in fact a countable union of closed sets. This can be seen as follows. First it is known that $G = G^{000}$ is equivalent to $\mathcal{P}^n = G$ for some n, where $\mathcal{P}$ is as in the previous subsection. We can now unwind the statement $\mathcal{P}^n = G$, using compactness: for any approximation I_k to indiscernibility,

$$T \models (\forall x)(\exists y_{i1}, y_{i2})_{i \leq n} \left(x = y_{11}y_{12}^{-1} \dots y_{n1}y_{n2}^{-1} \wedge \bigwedge_i I_k(y_{i1}, y_{i2}) \right).$$

The condition $G = G^{00}$ is Π^1_1, because the negation is equivalent to the existence of a proper type-definable subgroup containing $\mathcal{P}$, i.e. a sequence of consistent formulas $\phi_n(x)$ with $\phi_n(x) \wedge \phi_n(y) \to \phi_{n-1}(xy^{-1})$ in T, and such that $\phi_n(x)$ is generic but not equal to G.

The condition $G^{00} = G^{000}$ is also Π^1_1, as one sees by combining the two analyses above for the statement $G^{00} \subseteq \mathcal{P}^m$. This suffices to show that $G^{00} = G^{000}$ is indeed a property of T itself and does not depend on the ambient model of set theory.

It would be instructive to know if the conditions $G = G^{00}$ or $G^{00} = G^{000}$ are actually Δ^1_1. This seems related to a better understanding of the automatic definability aspects of ($\diamond$) (from the previous subsection). In the amenable setting, if there exists any generic Y with $Y^8 \subseteq X^4$, then there is a definable such Y. What happens in the non-amenable setting?

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