Math 60440: Basic Topology II
Midterm Exam

Do the following problems. You have unlimited time and may consult Hatcher's book and your notes. You may not consult any other sources or speak to anyone but me about the exam. It is due by 5pm on the Thursday after break.

1. Regard S^1 as the unit circle in $\mathbb{C}$. Let $X = S^1 \times S^1$ and $Y = S^1 \times S^1$. Set $Z = X \sqcup Y / \sim$, where $\sim$ identifies $(p, 1) \in X$ with $(p, 1) \in Y$ for all $p \in S^1$. Calculate $\mathrm{H}_n(Z)$ for all n.

2. (a) Let X be a topological space and let $\{U_1, \ldots, U_n\}$ be an open cover of X such that for all $1 \leq i_1 < \cdots < i_k \leq n$, the intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ is either empty or has trivial reduced homology groups. Prove that $\widetilde{\mathrm{H}}_i(X) = 0$ for $i \geq n-1$.

   (b) Give examples for each n showing that the result in the previous part cannot be improved to $i \geq n-2$.

3. Let M^n be a compact n-manifold. Assume that the suspension ΣM is a manifold. Prove that M is a homology n-sphere, i.e. we have

$$\mathrm{H}_k(M) = \begin{cases} \mathbb{Z} & \text{if } k = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

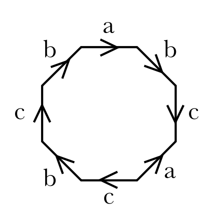
   We remark that in fact one can show that $M \cong S^n$, though this requires more technology.

4. Let

$$X = \{(x, y, z) \in \mathbb{R}^3 \mid xyz = 0\}.$$

   Prove that X is not a manifold.

5. Let X be the space obtained by identifying the sides of an octagon like this:



   (a) Describe a CW complex structure on X (be careful with the vertices!).

   (b) Write down the cellular chain complex for X.

   (c) Calculate $\mathrm{H}_n(X; \mathbb{Z}/3)$ and $\mathrm{H}_n(X; \mathbb{Q})$ for all $n \geq 0$.