Math 60440: Basic Topology II
Problem Set 3

1. Let $\mathbb{X}$ be a semisimplicial set, let Y be a topological space, and let $f\colon |\mathbb{X}| \to Y$ be a continuous map. Using f, construct chain complex map $C_\bullet(\mathbb{X}) \to C_\bullet(Y)$, where $C_\bullet(Y)$ is the singular chain complex of Y (note that I am not specifying any properties for this, but there is one that is "natural" and you will not get credit for doing something silly like writing down the chain complex map that takes everything to 0).

2. Let $\mathbb{X}$ be a 1-dimensional semisimplical set, i.e., a directed graph. Assume that the graph corresponding to $\mathbb{X}$ is connected and has finitely many edges and vertices. Prove the following:

   (a) If $\mathbb{X}$ is a tree (i.e., a connected graph with no loops), then prove that $\mathrm{H}_n(\mathbb{X})$ is $\mathbb{Z}$ for $n = 0$ and 0 for $n \geq 1$. Hint: prove it by induction on the number of vertices.

   (b) For a general $\mathbb{X}$, a *maximal tree* in $\mathbb{X}$ is a sub-semisimplicial set $\mathbb{T}$ of $\mathbb{X}$ that is a tree and contains every vertex of $\mathbb{X}$. Prove that $\mathbb{X}$ has a maximal tree.

   (c) If $\mathbb{T}$ is a maximal tree of $\mathbb{X}$, then prove that $\mathrm{H}_1(\mathbb{X}) \cong \mathbb{Z}^m$ where m is the number of edges of $\mathbb{X}$ that do not lie in $\mathbb{T}$.

3. Compute the homology of the following chain complex:

$$0 \to \mathbb{Z} < U, L > \xrightarrow{d_2} \mathbb{Z} < a, b, c > \xrightarrow{d_1} \mathbb{Z} < v, w > \to 0,$$

   where

$$\begin{aligned} d_2(U) &= -a + b + c \\ d_2(L) &= a - b + c \end{aligned}$$

   and

$$\begin{aligned} d_1(a) &= w - v \\ d_1(b) &= w - v \\ d_1(c) &= 0. \end{aligned}$$

4. For each $k \geq 1$, define a chain complex $D_\bullet^k$ by letting

$$D_n^k = \begin{cases} \mathbb{Z} & \text{if } n = k, k-1, \\ 0 & \text{otherwise} \end{cases}$$

   and letting the differential $D_k^k \to D_{k-1}^k$ be the identity (and all other differentials be 0).

   (a) Calculate $\mathrm{H}_n(D_\bullet^k)$.

(b) Prove that for all chain complexes $C_\bullet$, the set of chain complex maps $D^k_\bullet \to C_\bullet$ is in bijection with C_k.

5. Let $f\colon (C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$ be a homomorphism between chain complexes. Define $(\mathrm{Con}(f)_\bullet, e_\bullet)$ (the *mapping cone* of f) via the formulas

$$\mathrm{Con}(f)_n = C_{n-1} \oplus C'_n$$

and

$$e_n\colon \mathrm{Con}(f)_n \to \mathrm{Con}(f)_{n-1} \quad \text{is} \quad e_n(x,y) = (-d_{n-1}(x), f(x) + d'_n(y)).$$

Prove the following:

(a) $(\mathrm{Con}(f)_\bullet, e_\bullet)$ is a chain complex.
(b) The natural inclusion $(C'_\bullet, d'_\bullet) \to (\mathrm{Con}(f)_\bullet, e_\bullet)$ is a homomorphism of chain complexes.

6. Let $A_0, A_1, A_2, \ldots$ be a sequence of finitely generated abelian groups. Construct a chain complex $C_\bullet$ with the following properties:

- Each C_n is a finitely generated free abelian group, i.e. $C_n \cong \mathbb{Z}^{k_n}$ for some $k_n \geq 0$, and $C_n = 0$ for $n < 0$.
- $\mathrm{H}_n(C_\bullet) \cong A_n$ for all $n \geq 0$.