# Math 60440: Basic Topology II
# Problem Set 2

1. (a) Draw a connected directed graph with 4 vertices and 8 edges.
   (b) Write down the semisimplicial set $\mathbb{X}$ that this picture represents.
   (c) Explictly write down the simplicial chain complex

$$\cdots \xrightarrow{\partial} C_2(\mathbb{X}) \xrightarrow{\partial} C_1(\mathbb{X}) \xrightarrow{\partial} C_0(\mathbb{X}).$$

2. (a) Explictly construct a semisimplicial set $\mathbb{S}$ whose geometric realization is homeomorphic to a genus-$g$ surface.
   (b) Explictly write down the simplicial chain complex

$$\cdots \xrightarrow{\partial} C_2(\mathbb{S}) \xrightarrow{\partial} C_1(\mathbb{S}) \xrightarrow{\partial} C_0(\mathbb{S}).$$

3. Let $\mathbb{X}$ be a semisimplicial set that has finitely many simplices (i.e., such that each $\mathbb{X}_n$ is a finite set and $\mathbb{X}_n = \emptyset$ for $n$ sufficiently large). Prove that the geometric realization $|\mathbb{X}|$ is compact.

4. If $\mathbb{X}$ and $\mathbb{Y}$ are simplicial sets, then a *morphism* $f\colon \mathbb{X} \to \mathbb{Y}$ of semisimplicial sets consists of the following:

   - For each $n \geq 0$, a set map $f_n\colon \mathbb{X}_n \to \mathbb{Y}_n$ such that for each strictly increasing function $\iota\colon [m] \to [n]$, the diagram

$$\begin{array}{ccc} \mathbb{X}_n & \xrightarrow{f_n} & \mathbb{Y}_n \\ \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \iota^*} \\ \mathbb{X}_m & \xrightarrow{f_m} & \mathbb{Y}_m \end{array}$$

     commutes.

   Prove the following:

   (a) If $f\colon \mathbb{X} \to \mathbb{Y}$ is a morphism of semisimplicial sets, then there is an induced map $f\colon |\mathbb{X}| \to |\mathbb{Y}|$ on geometric realizations.
   (b) Letting $\mathbb{\Delta}^n$ denote the semisimplicial set we called the $n$-simplex, for all semisimplicial sets $\mathbb{X}$ there is a bijection between morphisms $\mathbb{\Delta}^n \to \mathbb{X}$ and elements of $\mathbb{X}_n$.

5. Let $\mathbb{X}$ be a semisimplicial set and let $f\colon Z \to |\mathbb{X}|$ be a covering space of its geometric realization. Construct a semisimplicial set $\mathbb{Z}$ with $|\mathbb{Z}| = Z$. Hint: the key thing to check here is that for all simplices $\Delta$ in the geometrical realization $|\mathbb{X}|$, the preimage $f^{-1}(\mathrm{Int}(\Delta))$ is a disjoint union of copies of copies of $\mathrm{Int}(\Delta)$. These will each correspond to a simplex of $\mathbb{Z}$.