Math 60440: Basic Topology II
Problem Set 1

1. Let (X, x_0) be a based space. Recall that when we defined the multiplication on $\pi_n(X, x_0)$, we stacked things using the first coordinate of maps $(I^n, \partial I^n) \to (X, x_0)$. Prove that using any of the other coordinates instead would result in the same group structure.

2. Let (X, x_0) be a path-connected based space. Prove that $\pi_k(X, x_0)$ is the trivial group if and only if every map $S^k \to X$ extends to a map of the disc D^{k+1}. Notice that these maps make no reference to the basepoint (hint: last semester you should have proved this for π_1, and you should meditate on that proof).

3. A space X is an *H-space* if there exists a point $e \in X$ (the *identity*) and a continuous map $m \colon X \times X \to X$ such that $m(x, e) = m(e, x) = x$ for all $x \in X$. For example, X might be a Lie group and m might be the multiplication. Let X be an H-space with identity e and let $f, g \colon (I^n, \partial I^n) \to (X, e)$ be continuous maps. Define $h \colon (I^n, \partial I^n) \to (X, e)$ via the formula $h(x) = m(f(x), g(x))$. Prove that h is homotopic to the map used to define $f \cdot g$ in the definition of $\pi_n(X, e)$.

4. Let $f \colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a based covering map. Prove that the induced map $f_* \colon \pi_n(\widetilde{X}, \widetilde{x}_0) \to \pi_n(X, x_0)$ is an isomorphism for $n \geq 2$.

5. Let (X, x_0) be a space equipped with a basepoint. Define a set map

$$\phi \colon \pi_1(X, x_0) \to \mathrm{Bord}_1(X)$$

as follows. Consider $x \in \pi_1(X, x_0)$, and represent x by a map $\gamma \colon S^1 \to X$ taking $1 \in S^1$ to x_0. We then define $\phi(x)$ be the bordism class of γ.

   (a) Prove that ϕ is a group homomorphism.

   (b) Prove that ϕ is surjective (basepoints are important here!).

   (c) Since $\mathrm{Bord}_1(X)$ is abelian, the map ϕ factors through a surjective map $\overline{\phi} \colon (\pi_1(X, x_0))^{\text{abelianize}} \to \mathrm{Bord}_1(X)$. Prove that $\overline{\phi}$ is an isomorphism (hint: you have to prove injectivity. Try to prove that if $\gamma \in \pi_1(X, x_0)$ is taken to 0 by ϕ, then γ can be written as a product of commutators. This should have something to do with the boundary loop on a surface!)