Yet another book on algebraic topology

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Part 1

Homology

CHAPTER 1

A geometric introduction to H_0 and H_1

The homology groups of a space X are abelian groups $\mathrm{H}_k(X)$ that encode subtle information about the d-dimensional "holes" in X. Before we introduce the formidable technical apparatus needed to discuss homology in general, we give an introduction to the geometric meaning of $\mathrm{H}_0(X)$ and $\mathrm{H}_1(X)$. The definitions we give for these group do not generalize in a straightward way to the higher homology groups, but the patterns they suggest give an idea about what to expect.

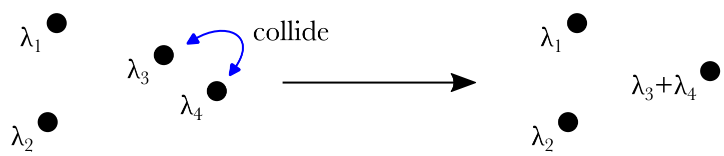
1.1. Zeroth homology

We start with $\mathrm{H}_0(X)$.

1.1.1. Intuitive description. Elements of $\mathrm{H}_0(X)$ can be thought of as formal $\mathbb{Z}$-linear combinations of points in X:

$$z = \lambda_1 p_1 + \cdots + \lambda_r p_r \quad \text{with } \lambda_1, \ldots, \lambda_r \in \mathbb{Z} \text{ and } p_1, \ldots, p_r \in X.$$

Moving the points p_i around does not change the element of $\mathrm{H}_0(X)$ that z represents. If two points collide, their coefficients add:



If a coefficient λ_i vanishes, then the point disappears.

1.1.2. Formal definition. To make this precise, we introduce the following notation:

NOTATION 1.1.1. For a set S, we write $\mathbb{Z}\langle S\rangle$ for the free abelian group consisting of formal $\mathbb{Z}$-linear combinations of elements of S. □

REMARK 1.1.2. We emphasize that even if S has some other structure (e.g., a topology) the abelian group $\mathbb{Z}\langle S\rangle$ only uses the structure of S as a set. □

The following makes the above intuitive description precise.

DEFINITION 1.1.3. Let X be a space. Then $\mathrm{H}_0(X) = \mathbb{Z}\langle X\rangle/R$, where R is the subgroup of $\mathbb{Z}\langle X\rangle$ generated by the following relations:

- for a continuous map $f\colon [0,1] \to X$, we have a relation $f(1) - f(0) \in R$.

For $z \in \mathbb{Z}\langle X\rangle$, we will write $[z]$ for the associated element of $\mathrm{H}_0(X)$. The relation $f(1) - f(0) \in R$ above therefore implies that $[f(1)] = [f(0)]$. □

REMARK 1.1.4. The topology on X plays no role in $\mathbb{Z}\langle X\rangle$, but is used to define the relations of $\mathrm{H}_0(X)$, which involve continuous maps $f\colon [0,1] \to X$. □

1.1.3. Topological meaning. It is not hard to determine the topological meaning of $\mathrm{H}_0(X)$:

LEMMA 1.1.5. *For a space X, the abelian group $\mathrm{H}_0(X)$ is isomorphic to the free abelian group with basis the path-components of X.*

PROOF. If the path components of X are $\{X_\alpha\}_{\alpha\in I}$, then it is immediate from the definition that

$$\mathrm{H}_0(X) = \bigoplus_{\alpha\in I} \mathrm{H}_0(X_\alpha).$$

This reduces us to showing that if X is path-connected, then $\mathrm{H}_0(X) \cong \mathbb{Z}$. Define a homomorphism $\epsilon\colon \mathbb{Z}\langle X\rangle \to \mathbb{Z}$ via the formula

$$\epsilon(\lambda_1 p_1 + \cdots + \lambda_r p_r) = \lambda_1 + \cdots + \lambda_r \quad \text{for } \lambda_1, \ldots, \lambda_r \in \mathbb{Z} \text{ and } p_1, \ldots, p_r \in X.$$

The homomorphism ϵ is surjective, and its kernel is generated by elements of the form $p - p'$ with $p, p' \in X$. For such a $p-p'$, since X is path-connected we can find a map $f\colon [0,1] \to X$ with $f(1) = p$ and $f(0) = p'$. The corresponding relation in $\mathrm{H}_0(X)$ shows that $[p - p'] = 0$. We conclude that ϵ factors through an isomorphism $\mathrm{H}_1(X) \cong \mathbb{Z}$. □

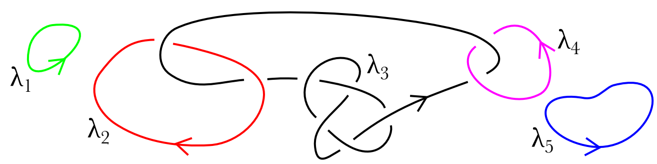
1.2. First homology

We now turn to $\mathrm{H}_1(X)$.

1.2.1. Intuitive description. A finite collection of points is a compact 0-manifold. For $\mathrm{H}_1(X)$, we go up a dimension. A compact 1-manifold is a disjoint union of circles. To avoid pathologies we will not insist that our circles in X be embedded, so elements of $\mathrm{H}_1(X)$ will be represented by formal $\mathbb{Z}$-linear combinations of maps of circles into X:

$$z = \lambda_1\gamma_1 + \cdots + \lambda_r\gamma_r \quad \text{with } \lambda_1, \ldots, \lambda_r \in \mathbb{Z} \text{ and } \gamma_1, \ldots, \gamma_r\colon S^1 \to X.$$

See here:



Each loop in this picture has an orientation coming from its parameterization $\gamma_i\colon S^1 \to X$.

Just like for $\mathrm{H}_0(X)$, homotoping the γ_i does not change the element of $\mathrm{H}_1(X)$ represented by z, and if two γ_i are homotoped to be equal their coefficients add. This implies that the knotting and linking in the above figure is irrelevant since we can homotope the γ_i through each other and themselves. However, homotopies do not exhaust the equivalence relation needed for $\mathrm{H}_1(X)$ since we also have to account for more complicated interactions between the loops.

1.2.2. Formal definition. Recall that for $\mathrm{H}_0(X)$ the relations come from maps of $[0,1]$ into X. The space $[0,1]$ is a compact oriented 1-manifold with boundary. For $\mathrm{H}_1(X)$, we replace $[0,1]$ with a compact oriented 2-manifold with boundary. Here is a formal definition:

DEFINITION 1.2.1. Let X be a space. Let $\mathcal{L}(X)$ be the set of continuous maps $\gamma\colon S^1 \to X$. Then $\mathrm{H}_1(X) = \mathbb{Z}\langle\mathcal{L}(X)\rangle/R$, where R is the subgroup of $\mathbb{Z}\langle\mathcal{L}(X)\rangle$ generated by the following relations:

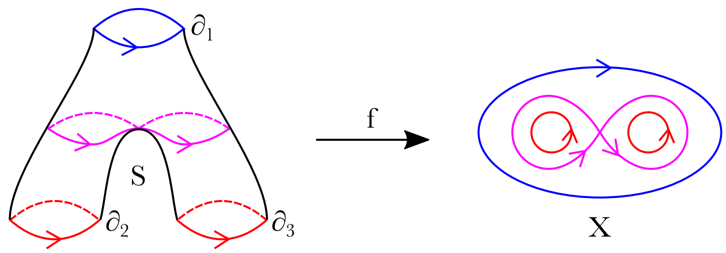
- for a compact oriented surface S with oriented boundary components $\partial_1, \ldots, \partial_r \cong S^1$ and a continuous map $f\colon S \to X$, we have a relation
$$f|_{\partial_1} + \cdots + f|_{\partial_r} \in R.$$

For $z \in \mathbb{Z}\langle\mathcal{L}(X)\rangle$, we will write $[z]$ for the associated element of $\mathrm{H}_1(X)$. □

1.2.3. Moves on homology classes. To clarify this, we describe some special cases of the relations on $\mathrm{H}_1(X)$. These should be thought of as "moves" on collections of loops in X.

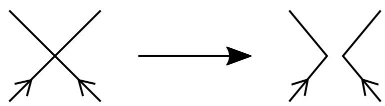
1. If $\gamma\colon S^1 \to X$ is a loop and $\overline{\gamma}\colon S^1 \to X$ is γ with the reversed orientation, then $[\gamma] = -[\overline{\gamma}]$. Indeed, let $S = S^1 \times [0,1]$ be a cylinder and let $f\colon S \to X$ be defined by $f(x,t) = \gamma(x)$. Then since in the above definition we take *oriented* boundary components the map f witnesses the relation $[\gamma + \overline{\gamma}] = 0$.
2. If $\gamma_1, \gamma_2\colon S^1 \to X$ are homotopic loops, then $[\gamma_1] = [\gamma_2]$. Indeed, let $S = S^1 \times [0,1]$ and let $f\colon S \to X$ be a homotopy from γ_1 to γ_2. Then f witnesses the fact that $[\gamma_1 + \overline{\gamma}_2] = 0$, so $[\gamma_1 - \gamma_2] = 0$.

3. If $\gamma\colon S^1 \to X$ is null-homotopic, then γ extends to a map of $\mathbb{D}^2$. Since $\mathbb{D}^2$ has only one boundary component, this implies that $[\gamma] = 0$.
4. Using a relation coming from a 3-holed sphere S, two loops in X can merge into a single loop as in the following figure showing two red loops merging to form a blue one:



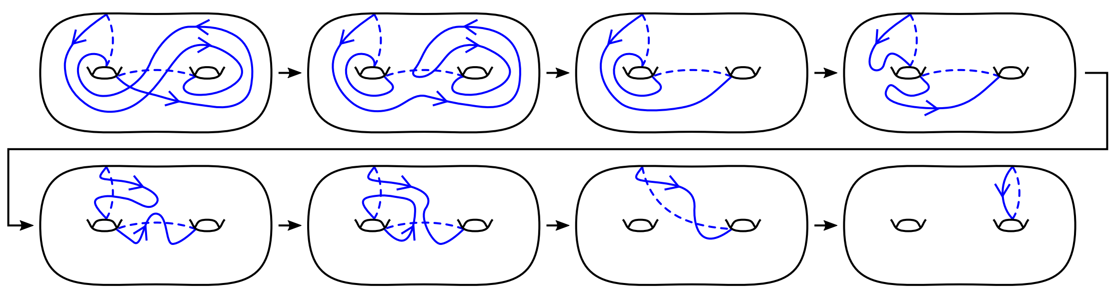
Note that as we have oriented them the oriented boundary of S is $\partial_1 \sqcup \overline{\partial}_2 \sqcup \overline{\partial}_3$, so what the above figure shows is that $[f|_{\partial_1} + f|_{\overline{\partial}_2} + f|_{\overline{\partial}_3}] = 0$, so $[f|_{\partial_1}] = [f|_{\partial_2}] + [f|_{\partial_3}]$. This ensures that as the loops $f|_{\partial_2}$ and $f|_{\partial_3}$ merge to form $f|_{\partial_1}$, their orientations match up.

Recall that our loops $\gamma\colon S^1 \to X$ need not be embedded. A special case of the fourth relation above is that if $\gamma\colon S^1 \to X$ is a loop with finitely many self-intersections and $\gamma_1, \ldots, \gamma_k\colon S^1 \to X$ are the loops obtained by resolving these self-intersections as in



then $[\gamma] = [\gamma_1] + \cdots + [\gamma_k]$. Here for ease of understanding we move the intersections off of each other, but in an arbitrary X this might not be possible.

Example 1.2.2. For a genus 2 surface Σ_2, the following is an example of applying these moves to an element of $\mathrm{H}_1(\Sigma_2)$:



In this figure, the coefficient of each loop is $+1$. □

1.2.4. Topological meaning. The following relates $\mathrm{H}_1(X)$ to the fundamental group:

Theorem 1.2.3. *Let (X, p) be a path-connected based space. Then*

$$\mathrm{H}_1(X) \cong (\pi_1(X, p_0))^{ab},$$

where ab means we are taking the abelianization of the fundamental group.

Proof. Elements of $\pi_1(X, p_0)$ are represented by based loops $\gamma\colon (S^1, 1) \to (X, p_0)$. Define a set map $\phi\colon \pi_1(X, p_0) \to \mathrm{H}_1(X)$ by letting ϕ take γ to[1] $[\gamma]$. Using the fourth move discussed above, we see that ϕ is a homomorphism. Since addition in $\mathrm{H}_1(X)$ is abelian, ϕ factors through a map

$$\Phi\colon (\pi_1(X, p_0))^{\mathrm{ab}} \to \mathrm{H}_1(X).$$

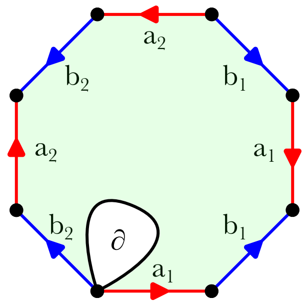
Since X is path-connected, every $\gamma\colon S^1 \to X$ can be homotoped to a based loop $\gamma\colon (S^1, 1) \to (X, p_0)$, so Φ is surjective.

[1]Here we are avoiding our usual convention of letting $[\gamma]$ denote an element of $\pi_1(X, p_0)$ to avoid a clash of notation.

We must prove that Φ is injective. Consider a based loop $\gamma\colon (S^1,1) \to (X,p_0)$ representing an element of $\ker(\phi)$. We must prove that γ vanishes in the abelianization of $\pi_1(X,p_0)$. Since $[\gamma]=0$, by definition there exist compact oriented surfaces $S_1,\ldots,S_k$ and maps $f_i\colon S_i \to X$ such that the sum of the relations corresponding to the f_i witnesses the fact that $[\gamma]=0$.

Each S_i might have multiple boundary component, but aside from one boundary component corresponding to γ all of these must cancel out when we add up the corresponding relations. We can therefore glue the cancelled-out boundary components together in pairs to construct a compact oriented genus surface S with one boundary component $\partial \cong S^1$ and $f\colon S \to X$ such that $f|_\partial = \gamma$. Discarding all components of S that do not contain ∂, we can assume that S is connected.

Let g be its genus. As the following shows, we can view S as a $4g$-gon with sides identified and with an open disc removed from its center:



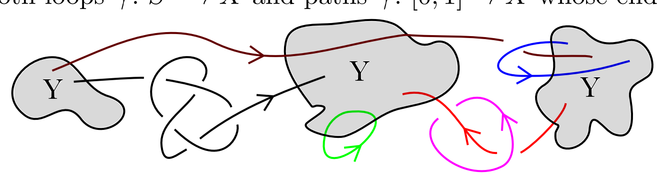
As this figure shows, we can write the loop in $\pi_1(S)$ around ∂ as a product of commutators $[a_1,b_1]\cdots[a_g,b_g]$. Mapping this over to $\pi_1(X,p_0)$ shows that γ can be written as a product of commutators, and thus vanishes in the abelianization of $\pi_1(X,p_0)$. □

1.3. Relative homology

The abelian groups $\mathrm{H}_0(X)$ and $\mathrm{H}_1(X)$ are related via *relative* homology groups. Fix a subspace Y of X.

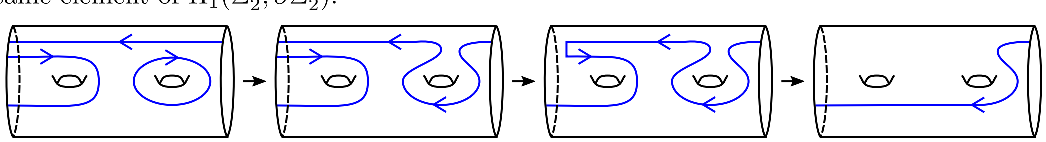
1.3.1. Relative zeroth homology. We have $\mathrm{H}_0(Y) \subset \mathrm{H}_0(X)$, and we define $\mathrm{H}_0(X,Y) = \mathrm{H}_0(X)/\mathrm{H}_0(Y)$. Informally, elements of $\mathrm{H}_0(X,Y)$ are linear combinations of points on X that disappear when they move into Y.

1.3.2. Relative first homology. The definition of $\mathrm{H}_1(X,Y)$ is a little more complicated. We will give a proper definition later, so here we only give an intuitive idea of what it is. Recall that elements of $\mathrm{H}_1(X)$ are represented by formal $\mathbb{Z}$-linear combinations of loops $\gamma\colon S^1 \to X$. For $\mathrm{H}_1(X,Y)$, we need both loops $\gamma\colon S^1 \to X$ and paths $\gamma\colon [0,1] \to X$ whose endpoints lie in Y:



These can be homotoped through paths starting and ending in Y, and a loop or arc that moves entirely inside Y disappears. We also allow moves that are similar to the ones we discussed in §1.2. For instance:

Example 1.3.1. In the genus 2 surface Σ_2^2 with two boundary components, the following represent the same element of $\mathrm{H}_1(\Sigma_2^2, \partial\Sigma_2^2)$:



In this figure, the coefficient of each loop and arc is $+1$. □

1.3.3. Long exact sequence. There are evident maps

$$H_1(Y) \to H_1(X) \to H_1(X,Y) \quad \text{and} \quad H_0(Y) \to H_0(X) \to H_0(X,Y).$$

There is also a boundary map $\partial\colon H_1(X,Y) \to H_0(Y)$ that deletes loops and take arcs $\gamma\colon [0,1] \to X$ connecting points of Y to $\gamma(1) - \gamma(0) \in H_0(Y)$. All of this is set up so that the sequence

$$H_1(Y) \to H_1(X) \to H_1(X,Y) \xrightarrow{\partial} H_0(Y) \to H_0(X) \to H_0(X,Y) \to 0. \tag{1.3.1}$$

is exact. However, the map $H_1(Y) \to H_1(X)$ need not be injective. For instance, there might be a loop $\gamma\colon S^1 \to Y$ and a compact oriented surface S with one boundary component $\partial \cong S^1$ such that there exists a map $f\colon S \to X$ with $f|_\partial = \gamma$. Once we have defined higher homology groups, $f\colon S \to X$ should represent an element of $H_2(X,Y)$. Continuing this, we should be able to extend (1.3.1) to a long exact sequence of the form

$$\cdots \to H_d(Y) \to H_d(X) \to H_d(X,Y) \to H_{d-1}(Y) \to \cdots.$$

In the next chapter we will make all of this precise.

1.4. Bordism

We close this chapter by describing a generalization of the definition we gave for $H_0(X)$ and $H_1(X)$. Though it does not give the right answer for $H_d(X)$ for all d, it provides a useful intuition.

1.4.1. Definition of bordism. Recall that the generators of $H_1(X)$ come from maps of closed connected oriented 1-manifolds (i.e, circles) into X, and the relations come from maps of compact oriented 2-manifolds with boundary. The following generalizes this:

Definition 1.4.1. Let X be a space and let $d \geq 0$. Denote by $\mathcal{M}_d(X)$ the set[2] of continuous maps $g\colon M^d \to X$ from a closed connected oriented d-manifold M^d to X. The d^{th} *bordism* group of X, denoted $\text{Bord}_d(X)$, is $\mathbb{Z}\langle \mathcal{M}_d(X)\rangle / R$ where R is the subgroup of $\mathbb{Z}\langle \mathcal{M}_d(X)\rangle$ generated by the following relations:

- for a compact oriented $(d+1)$-manifold N^{d+1} with oriented boundary components $\partial_1, \ldots, \partial_r$ and a continuous map $f\colon N^{d+1} \to X$, we have a relation

$$f|_{\partial_1} + \cdots + f|_{\partial_r} \in R. \qquad \square$$

For $g\colon M^d \to X$, we will write $[g]$ for the associated element of $\text{Bord}_d(X)$. Each Bord_d is a functor from the category of spaces to the category of abelian groups. In other words, given a map of spaces $\phi\colon X \to Y$ there is an induced map $\phi_*\colon \text{Bord}_d(X) \to \text{Bord}_d(Y)$, namely

$$\phi_*([g]) = [\phi \circ g] \quad \text{for } g\colon M^d \to X.$$

These induced maps satisfy the usual properties of a functor:

(a) for the identity map $\mathbb{1}\colon X \to X$, we have $\mathbb{1}_* = \text{id}$; and
(b) for maps of spaces $\phi\colon X \to Y$ and $\psi\colon Y \to Z$, we have $(\psi \circ \phi)_* = \psi_* \circ \phi_*$.

1.4.2. Trouble. Unfortunately, it is not true that $\text{Bord}_d(X) \cong H_d(X)$. Indeed, for a one-point space $*$, we have $H_d(*) = 0$ for $d \geq 1$, but $\text{Bord}_d(*)$ is often nonzero. In other words, there exist closed oriented d-manifolds M^d such that the constant map $g\colon M^d \to *$ is not the trivial element of $\text{Bord}_d(X)$. A bit of thought show that this means that M^d is not the boundary of a compact oriented $(d+1)$-manifold N^{d+1}. While we do not have the technology to prove this yet, one easy example is the 4-manifold $\mathbb{CP}^2$.

Remark 1.4.2. It is actually the case that $\text{Bord}_d(X) \cong H_n(X)$ for $d \leq 3$. The first difference happens for $\text{Bord}_4(X)$, where $\text{Bord}_4(X) \cong H_4(X) \oplus \mathbb{Z}$. $\square$

[2] The pedantic reader will note that as stated this is not a set, but this is easily fixed by choosing the M^d to lie in a set containing a representative of each homeomorphism class of closed connected oriented d-manifolds. We will not worry about this kind of issue in this book.

1.4.3. Intuition for higher homology. Though it is technically difficult to define it this way, one way of viewing $H_d(X)$ is that it is a "bordism group" involving maps $f\colon M^d \to X$ with M^d a certain kind of singular manifold. For instance, we would like the constant map $g\colon \mathbb{CP}^2 \to *$ to represent the trivial element of $H_4(*)$ by viewing $\mathbb{CP}^2$ as the boundary of the space

$$\mathrm{Cone}(\mathbb{CP}^2) = \mathbb{CP}^2 \times [0,1]/\sim \quad \text{where } \sim \text{ collapses } \mathbb{CP}^2 \times 1 \text{ to a single point (the "cone point").}$$

This is not a manifold at the cone point. Instead of trying to make this precise, our actual definition is in terms of simplices mapping to X and can be thought of as "simplicial bordism".

1.4.4. From bordism to homology. Bordism also gives a good geometric way to think about homology classes. Much later when we prove Poincaré Duality one thing we will show is that for a compact oriented d-manifold M^d, we have $H_d(M^d) \cong \mathbb{Z}$, generated by an element $[M^d]$ called the *fundamental class*. We will prove many special cases of this along the way; for instance, one of our earliest results will show this holds for $M^d = S^d$.

Given a map $f\colon M^d \to X$, since homology is functorial we have an induced map $f_*\colon H_d(M^d) \to H_d(X)$. We therefore get an element $f_*([M^d]) \in H_d(X)$. We will also prove that this respects the bordism relation. In other words, using bordism we can often describe homology classes using maps of manifolds into our spaces. Though this does not give a complete picture of homology, it is useful way to identify, manipulate, and visualize homology classes.

1.5. Exercises

CHAPTER 2

Axioms of homology

Constructing the homology groups of a space is fairly technical, and it turns out that the details of the construction are almost irrelevant to actually *using* homology to prove things. In this chaper, we will describe a variant on the Eilenberg–Steenrod axioms for homology, which describe enough basic properties that homology should have to calculate it for most reasonable spaces. We then give some simple calculations and examples. The construction is postponed until later in the book.

REMARK 2.0.1. We include some additional axioms beyond the usual Eilenberg–Steenrod axioms. Our goal is to include enough axioms to make it easy to develop the basic theory without getting bogged down in technicalities. □

2.1. Discussion of axioms

Fix an abelian group $\mathbf{k}$. In this section, we discuss the axioms for a homology theory over $\mathbf{k}$. In the previous chapter we were taking $\mathbf{k} = \mathbb{Z}$, and the reader should reflect on why these axioms hold for H_0 and H_1 as defined in the previous chapter.

REMARK 2.1.1. A first-time reader is advised to just assume that $\mathbf{k} = \mathbb{Z}$. It is traditional in textbooks to first develop homology over $\mathbb{Z}$, and then assert that everything works without change if you change $\mathbb{Z}$ to an arbitrary abelian group. We avoid doing this to reassure the reader that indeed the basic theory works without change for arbitrary coefficients. □

2.1.1. Pairs. To talk about relative homology, we need a language to talk about spaces equipped with a subspace.[1] A *pair* of spaces is a tuple (X, Y) with X a space and $Y \subset X$. A map of pairs $f\colon (X,Y) \to (Z,W)$ is a continuous map $f\colon X \to Z$ such that $f(Y) \subset W$. Two maps of pairs $f_0\colon (X,Y) \to (Z,W)$ and $f_1\colon (X,Y) \to (Z,W)$ are said to be homotopic if there is a homotopy $f_t\colon X \to Z$ from f_0 to f_1 such that each f_t is a map of pairs, i.e., $f_t(Y) \subset W$ for all $0 \leq t \leq 1$.

A map of pairs $f\colon (X,Y) \to (Z,W)$ is a *homotopy equivalence* of pairs if there exist a map of pairs $g\colon (X,Y) \to (Z,W)$ such that $g \circ f\colon (X,Y) \to (X,Y)$ and $f \circ g\colon (Z,W) \to (Z,W)$ are both homotopic to the identity. We will call g a *homotopy inverse* to f and say that (X,Y) is *homotopy equivalent* to (Z,W). We write this $(X,Y) \simeq (Z,W)$. If $Y = W = \emptyset$, this reduces to the usual definition of a homotopy equivalence $X \simeq Z$.

2.1.2. Basic setup. The theory of homology gives for each pair (X,Y) of spaces a sequence of abelian groups $\mathrm{H}_d(X,Y;\mathbf{k})$, one for each $d \geq 0$. These $\mathrm{H}_d(X,Y;\mathbf{k})$ are functors from the category of pairs of spaces to the category of abelian groups. In other words, for all maps of pairs $f\colon (X,Y) \to (Z,W)$ we have induced maps

$$f_*\colon \mathrm{H}_d(X,Y;\mathbf{k}) \to \mathrm{H}_d(Z,W;\mathbf{k}),$$

and these induced maps satisfy the usual properties of a functor:

(a) for the identity map $\mathbb{1}\colon (X,Y) \to (X,Y)$, we have $\mathbb{1}_* = \mathrm{id}$; and

(b) for maps of pairs $f\colon (X,Y) \to (Z,W)$ and $g\colon (Z,W) \to (U,V)$, we have $(g \circ f)_* = g_* \circ f_*$.

The homology groups are also functors of the coefficients $\mathbf{k}$. In other words, for every map of abelian group $\phi\colon \mathbf{k} \to \mathbf{k}'$ there exists maps $\phi_*\colon \mathrm{H}_d(X,Y;\mathbf{k}) \to \mathrm{H}_d(X,Y;\mathbf{k}')$ satisfying the obvious analogues of (a) and (b) above. These maps should be compatible with the ones induced by maps of pairs $f\colon (X,Y) \to (Z,W)$ in the sense that the diagram

[1] This terminology generalizes our earlier terminology for maps $f\colon (X,p) \to (Y,q)$ between spaces equipped with basepoints.

$$\begin{array}{ccc} \mathrm{H}_d(X,Y;\mathbf{k}) & \xrightarrow{f_*} & \mathrm{H}_d(Z,W;\mathbf{k}) \\ \downarrow{\phi_*} & & \downarrow{\phi_*} \\ \mathrm{H}_d(X,Y;\mathbf{k}') & \xrightarrow{f_*} & \mathrm{H}_d(Z,W;\mathbf{k}') \end{array}$$

should commute.

NOTATION 2.1.2. For a space X we let $\mathrm{H}_d(X;\mathbf{k}) = \mathrm{H}_d(X,\emptyset;\mathbf{k})$. We will omit the coefficients $\mathbf{k}$ when they are $\mathbb{Z}$, so $\mathrm{H}_d(X,Y)$ and $\mathrm{H}_d(X)$ mean $\mathrm{H}_d(X,Y;\mathbb{Z})$ and $\mathrm{H}_d(X;\mathbb{Z})$. Finally, for ease of notation we will let $\mathrm{H}_d(X,Y;\mathbf{k}) = 0$ for $d < 0$. □

2.1.3. Zeroth and first homology. The first basic property that we want from homology is that it agrees with the definition we gave in the previous chapter for $\mathrm{H}_0(X)$ and $\mathrm{H}_1(X)$. Since the definitions will not be *literally* the same, the right way to state this is to say that for $d = 0$ and $d = 1$ there is a natural isomorphism between Bord_d and H_d. This means that for all spaces X there are isomorphisms

$$\mathrm{Bord}_0(X) \xrightarrow{\cong} \mathrm{H}_0(X) \qquad \text{and} \qquad \mathrm{Bord}_1(X) \xrightarrow{\cong} \mathrm{H}_1(X)$$

that are natural in the sense that for all maps $f\colon X \to Y$, the diagrams

$$\begin{array}{ccc} \mathrm{Bord}_0(X) & \xrightarrow{\cong} & \mathrm{H}_0(X) \\ \downarrow{f_*} & & \downarrow{f_*} \\ \mathrm{Bord}_0(Y) & \xrightarrow{\cong} & \mathrm{H}_0(Y) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathrm{Bord}_1(X) & \xrightarrow{\cong} & \mathrm{H}_1(X) \\ \downarrow{f_*} & & \downarrow{f_*} \\ \mathrm{Bord}_1(Y) & \xrightarrow{\cong} & \mathrm{H}_1(X) \end{array}$$

commute. In categorical language, this means that there are natural transformations from Bord_0 to H_0 and Bord_1 to H_1 such that for all spaces X, the maps $\mathrm{Bord}_0(X) \to \mathrm{H}_0(X)$ and $\mathrm{Bord}_1(X) \to \mathrm{H}_1(X)$ are isomorphisms.

2.1.4. Homotopy invariance. The second basic property that we want from homology is that it is *homotopy invariant*. In other words, if $f_0\colon (X,Y) \to (Z,W)$ and $f_1\colon (X,Y) \to (Z,W)$ are homotopic maps of pairs, then $(f_0)_*\colon \mathrm{H}_d(X,Y;\mathbf{k}) \to \mathrm{H}_d(Z,W;\mathbf{k})$ and $(f_1)_*\colon \mathrm{H}_d(X,Y;\mathbf{k}) \to \mathrm{H}_d(Z,W;\mathbf{k})$ are equal. See Exercise 2.6.1 for how to deduce this statement for pairs from the special case of a single space.

If $f\colon (X,Y) \to (Z,W)$ is a homotopy equivalence with homotopy inverse $g\colon (X,Y) \to (Z,W)$, then homotopy invariance implies that the maps

$$f_*\colon \mathrm{H}_d(X,Y;\mathbf{k}) \to \mathrm{H}_d(Z,W;\mathbf{k}) \quad \text{and} \quad g_*\colon \mathrm{H}_d(Z,W;\mathbf{k}) \to \mathrm{H}_d(X,Y;\mathbf{k})$$

of abelian groups are inverses to each other, and in particular are isomorphisms. In other words, homology cannot tell the difference between homotopy equivalent spaces and pairs.

2.1.5. Long exact sequences. The third basic property that we want from homology is the *long exact sequence of a pair*, which was discussed at the end of the previous chapter. This says that for all pairs (X,Y), we have a long exact sequence

$$\cdots \longrightarrow \mathrm{H}_d(Y;\mathbf{k}) \longrightarrow \mathrm{H}_d(X;\mathbf{k}) \longrightarrow \mathrm{H}_d(X,Y;\mathbf{k}) \longrightarrow \mathrm{H}_{d-1}(Y;\mathbf{k}) \longrightarrow \cdots.$$

This long exact sequence should be *natural*: for a map of pairs $f\colon (X,Y) \to (Z,W)$, the diagram

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathrm{H}_d(Y;\mathbf{k}) & \longrightarrow & \mathrm{H}_d(X;\mathbf{k}) & \longrightarrow & \mathrm{H}_d(X,Y;\mathbf{k}) & \longrightarrow & \mathrm{H}_{d-1}(Y;\mathbf{k}) & \longrightarrow \cdots \\ & \downarrow{f_*} & & \downarrow{f_*} & & \downarrow{f_*} & & \downarrow{f_*} & \\ \cdots \longrightarrow & \mathrm{H}_d(W;\mathbf{k}) & \longrightarrow & \mathrm{H}_d(Z;\mathbf{k}) & \longrightarrow & \mathrm{H}_d(Z,W;\mathbf{k}) & \longrightarrow & \mathrm{H}_{d-1}(Y;\mathbf{k}) & \longrightarrow \cdots \end{array}$$

should commute.

2.1.6. Dimension axiom. The fourth basic property that we want from homology is the *dimension axiom*, which says that if $*$ is a one-point space then

$$\mathrm{H}_d(*;\mathbf{k}) = \begin{cases} \mathbf{k} & \text{if } d = 0, \\ 0 & \text{otherwise.} \end{cases}$$

REMARK 2.1.3. Recall that in the previous chapter we defined bordism groups $\mathrm{Bord}_d(X)$. Suitably extended to bordism groups of pairs $\mathrm{Bord}_d(X, Y)$, these satisfy all the axioms of a homology theory with coefficients in $\mathbb{Z}$ except for the dimension axiom. Theories of this kind are called *extraordinary homology theories*. They are very important, but are beyond the scope of this book. □

2.1.7. Additivity axiom. The fifth basic property we want is the *additivity axiom*. This says that if a space X can be written as the disjoint union of spaces

$$X = \bigsqcup_{i \in I} X_i,$$

then we have a natural isomorphism[2]

$$\mathrm{H}_d(X; \mathbf{k}) \cong \bigoplus_{i \in I} \mathrm{H}_d(X_i; \mathbf{k}) \quad \text{for all } d.$$

See Exercise 2.6.2 for the extension of this to pairs (X, Y).

2.1.8. Continuity axiom. The sixth basic property we want is the *continuity axiom*. Roughly speaking, this says that homology only depends on compact subspaces of a space. To formulate it, let X be a space. Write

$$X = \bigcup_{i=1}^{\infty} X_i \quad \text{with } X_1 \subset X_2 \subset X_3 \subset \cdots.$$

Assume that for every compact subspace K of X, there exists some $n \geq 1$ with $K \subset X_n$. For instance, this holds if the X_i are all open. It also holds if X is a CW complex and each X_i is a subcomplex of X (see Appendix 13). The continuity axiom says that the resulting maps

$$\varinjlim \mathrm{H}_d(X_i; \mathbf{k}) \to \mathrm{H}_d(X; \mathbf{k})$$

are isomorphisms. See Exercise 2.6.3 for the extension of this to pairs.

REMARK 2.1.4. If X is a CW complex and the X_i are subcomplexes, then this can be deduced from the other axioms (see Chapter 12). Most of our applications of this axiom will only use this case, and it is usually not included as an axiom of homology. However, we will use it a few times in Chapter 6 with the X_i open sets to study the topology of open subsets of $\mathbb{R}^n$. □

2.1.9. Excision axiom. The final basic property we want is the *excision axiom*. Roughly speaking, it formalizes the fact that in the relative homology groups $\mathrm{H}_d(X, Y; \mathbf{k})$ we are "ignoring" the points of Y, we should be able to remove portions of Y without changing $\mathrm{H}_d(X, Y; \mathbf{k})$. For a pair (X, Y), the excision axiom say that if $A \subset Y$ is such that $\overline{A} \subset \mathrm{Int}(Y)$, then[3]

$$\mathrm{H}_d(X \setminus A, Y \setminus A; \mathbf{k}) \cong \mathrm{H}_d(X, Y; \mathbf{k}) \quad \text{for all } d.$$

In this case, we say that A is being *excised* from (X, Y). We remark that this axiom is by far the hardest one to verify.

2.2. Existence of homology theory

The following theorem says that homology theories exist. We will prove it in Chapter 12.

THEOREM 2.2.1 (Existence of homology). *For all abelian groups* $\mathbf{k}$*, there exists a homology theory over* $\mathbf{k}$ *that is naturally isomorphic to bordism in degrees* 0 *and* 1*, is homotopy invariant, has long exact sequences of pairs, and satisfies the dimension, additivity, continuity, and excision axioms.*

---

[2]Here *natural* means that if $f\colon X \to Y$ is a map and $f_i = f|_{X_i}$ for all $i \in I$, then $f_*\colon \mathrm{H}_d(X; \mathbf{k}) \to \mathrm{H}_d(Y; \mathbf{k})$ equals $\sum_{i \in I} (f_i)_*\colon \bigoplus_{i \in I} \mathrm{H}_d(X_i; \mathbf{k}) \to \mathrm{H}_d(Y; \mathbf{k})$.

[3]It is implicit here that the isomorphism is the map induced by the map of pairs $(X \setminus A, Y \setminus A) \to (X, Y)$. Though we will not always be explicit about it, whenever we assert that two things are isomorphic and there is an obvious map between them, we mean that the obvious map induces the isomorphism.

A reader who is uncomfortable using machines whose internal details they have not verified can go and read Chapter 12 now. However, the details of the construction are not needed elsewhere, and we suggest reading this book in its linear order. As we will see, though the axioms are a little abstract they allow the calculation of the homology groups of most spaces (including all CW complexes), as well as the many applications to classical questions in geometry. The "meaning" of homology is best understood not from its definition, but from an accumulated store of examples and calculations.

Remark 2.2.2. The two difficult parts of the proof of Theorem 2.2.1 are the proofs of homotopy invariance and excision. Those are the last two things proved in Chapter 12, and a reader who wants a taste of the construction without the most technical parts of it should read that chapter up until the start of those proofs. □

Remark 2.2.3. There are actually multiple constructions of homology theories, and they do not give the same answer for all spaces. As we will prove, however, the axioms determine the homology of spaces homotopy equivalent to CW complexes. Later we will introduce one final axiom (the *weak equivalence axiom*) that pins down the answer on all spaces. We chose to postpone this final axiom to avoid getting bogged down in technicalities at this early stage. □

Remark 2.2.4. As we said above, we will omit the coefficients from our notation when they are $\mathbb{Z}$, so $\mathrm{H}_d(X)$ means $\mathrm{H}_d(X;\mathbb{Z})$. At first our calculations will be independent of the coefficients, so we will mostly do this when we are applying homology and there is no reason to use other coefficients. Eventually, however, we will do calculations where the answer is very different depending the coefficients. □

2.3. Some basic calculations: contractible spaces, reduced homology, and spheres

We now perform some basic calculations using the axioms for homology. In this section, we fix an abelian group $\mathbf{k}$.

2.3.1. Zeroth and first homology. By our work on bordism in the previous chapter, we have the following:

- $\mathrm{H}_0(X)$ is isomorphic to the free abelian group with basis the path-components of X; and
- if X is path-connected and $p_0 \in X$, then $\mathrm{H}_1(X) \cong (\pi_1(X, p_0))^{\mathrm{ab}}$.

We remark that if we omitted the assumption that our homology theory is isomorphic to bordism in degrees 0 and 1, then these would only hold for CW complexes (and proving them is not entirely straightforward).

2.3.2. Contractible spaces. The easiest spaces to handle are the contractible ones:

Lemma 2.3.1. *Let X be a contractible space. Then*

$$\mathrm{H}_d(X;\mathbf{k}) = \begin{cases} \mathbf{k} & \text{for } d = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $*$ be a one-point space. Since X is contractible, the constant map $f\colon X \to *$ is a homotopy equivalence. By homotopy invariance, $f_*\colon \mathrm{H}_d(X;\mathbf{k}) \to \mathrm{H}_d(*;\mathbf{k})$ is an isomorphism for all d. The dimension axiom then gives that $\mathrm{H}_d(X;\mathbf{k}) \cong \mathrm{H}_d(*;\mathbf{k})$ has the indicated values. □

2.3.3. Reduced homology. It is annoying that contractible spaces have non-trivial homology groups in degree 0. To fix this, we make the following definition:

Definition 2.3.2. For a non-empty space X, the *reduced homology groups* of X, denoted $\widetilde{\mathrm{H}}_d(X;\mathbf{k})$, are the kernel of the map $c_*\colon \mathrm{H}_d(X;\mathbf{k}) \to \mathrm{H}_d(*;\mathbf{k})$ induced by the constant map $c\colon X \to *$. □

We thus have $\widetilde{\mathrm{H}}_d(X;\mathbf{k}) \to \mathrm{H}_d(X;\mathbf{k})$ for $d \geq 1$. As for $d = 0$, pick a point $x_0 \in X$. Let $\iota\colon * \to X$ take $*$ to x_0 and let $c\colon X \to *$ be the constant map. We thus have $c \circ \iota = \mathbb{1}$, so the composition

$$\mathbf{k} = \mathrm{H}_0(*;\mathbf{k}) \xrightarrow{\iota_*} \mathrm{H}_0(X;\mathbf{k}) \xrightarrow{c_*} \mathrm{H}_0(*;\mathbf{k}) = \mathbf{k}$$

is the identity. In other words, c_* is a split surjection with splitting ι_*, and thus

$$\mathrm{H}_0(X;\mathbf{k}) = \widetilde{\mathrm{H}}_0(X;\mathbf{k}) \oplus \mathbf{k}.$$

In particular:

LEMMA 2.3.3. *Let X be a contractible space. Then $\widetilde{\mathrm{H}}_d(X;\mathbf{k}) = 0$ for all d.*

Also, since $\mathrm{H}_0(X)$ is isomorphic to the free abelian group with basis the path-components of X, we have:

LEMMA 2.3.4. *Let X be a space with n path components. Then $\mathrm{H}_0(X) \cong \mathbb{Z}^n$ and $\widetilde{\mathrm{H}}_0(X) \cong \mathbb{Z}^{n-1}$.*

Reduced homology only makes sense for single spaces, and we do not define $\widetilde{\mathrm{H}}_d(X, Y;\mathbf{k})$ for pairs (X, Y). However, we do have:

LEMMA 2.3.5. *Let X be a nonempty space and let $x_0 \in X$. Then $\mathrm{H}_d(X, x_0;\mathbf{k}) \cong \widetilde{\mathrm{H}}_d(X;\mathbf{k})$ for all d.*

PROOF. The long exact sequence of the pair (X, x_0) contains segments of the form

$$\mathrm{H}_d(x_0;\mathbf{k}) \longrightarrow \mathrm{H}_d(X;\mathbf{k}) \longrightarrow \mathrm{H}_d(X, x_0;\mathbf{k}) \longrightarrow \mathrm{H}_{d-1}(x_0;\mathbf{k}).$$

For $d \geq 2$, we have $\mathrm{H}_d(x_0;\mathbf{k}) = \mathrm{H}_{d-1}(x_0;\mathbf{k}) = 0$, so this implies that $\mathrm{H}_d(X, x_0;\mathbf{k}) \cong \mathrm{H}_d(X;\mathbf{k}) = \widetilde{\mathrm{H}}_d(X;\mathbf{k})$. For $d = 1$, we only have $\mathrm{H}_d(x_0;\mathbf{k}) = 0$ and we continue our long exact sequence further to the right:

$$0 \longrightarrow \mathrm{H}_1(X;\mathbf{k}) \longrightarrow \mathrm{H}_1(X, x_0;\mathbf{k}) \longrightarrow \mathrm{H}_0(x_0;\mathbf{k}) \longrightarrow \mathrm{H}_0(X;\mathbf{k}) \longrightarrow \mathrm{H}_0(X, x_0;\mathbf{k}) \longrightarrow 0.$$

By what we said above, the map $\mathrm{H}_0(x_0;\mathbf{k}) \to \mathrm{H}_0(X;\mathbf{k})$ is an injection whose cokernel is $\widetilde{\mathrm{H}}_0(X;\mathbf{k})$. The lemma follows. □

REMARK 2.3.6. For a pair (X, Y) with $Y \neq \emptyset$, there is also a long exact sequence in reduced homology of the form

$$\cdots \longrightarrow \widetilde{\mathrm{H}}_d(Y;\mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_d(X;\mathbf{k}) \longrightarrow \mathrm{H}_d(X, Y;\mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_{d-1}(Y;\mathbf{k}) \longrightarrow \cdots.$$

See Exercise 2.6.4. □

2.3.4. Spheres. Our next calculation is as follows. It gives a good indication of the power of the excision axiom.

LEMMA 2.3.7. *For $n \geq 0$, we have*

$$\widetilde{\mathrm{H}}_d(\mathbb{S}^n;\mathbf{k}) = \begin{cases} \mathbf{k} & \text{for } d = n, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. The proof is by induction on n. For the base case $n = 0$, the 0-sphere $\mathbb{S}^0$ is a discrete space consisting of two points, so $\widetilde{\mathrm{H}}_d(\mathbb{S}^0;\mathbf{k})$ is $\mathbf{k}$ for $d = 0$ and 0 otherwise. Now assume that $n \geq 1$ and that the lemma is true for $\mathbb{S}^{n-1}$. It is enough to construct an isomorphism $\widetilde{\mathrm{H}}_d(\mathbb{S}^n;\mathbf{k}) \cong \widetilde{\mathrm{H}}_{d-1}(\mathbb{S}^{n-1};\mathbf{k})$. For this, let p_1 and p_2 be the north and south poles on $\mathbb{S}^n$. Let

$$U_1 = \mathbb{S}^n \setminus p_1 \cong \mathbb{R}^n \quad \text{and} \quad U_{12} = \mathbb{S}^n \setminus \{p_1, p_2\} \cong \mathbb{R}^n \setminus 0 \simeq \mathbb{S}^{n-1}.$$

The long exact sequences in reduced homology of the pairs $(\mathbb{S}^n, U_1)$ and (U_1, U_{12}) contain the segments

$$\widetilde{\mathrm{H}}_d(U_1;\mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_d(\mathbb{S}^n;\mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_d(\mathbb{S}^n, U_1;\mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_{d-1}(U_1;\mathbf{k})$$

and

$$\widetilde{\mathrm{H}}_d(U_1;\mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_d(U_1, U_{12};\mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_{d-1}(U_{12};\mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_{d-1}(U_1;\mathbf{k}).$$

Since U_1 is contractible, we have $\widetilde{\mathrm{H}}_d(U_1;\mathbf{k}) = \widetilde{\mathrm{H}}_{d-1}(U_1;\mathbf{k}) = 0$. We therefore have isomorphisms

$$\widetilde{\mathrm{H}}_d(\mathbb{S}^n;\mathbf{k}) \cong \widetilde{\mathrm{H}}_d(\mathbb{S}^n, U_1;\mathbf{k}) \quad \text{and} \quad \widetilde{\mathrm{H}}_d(U_1, U_{12};\mathbf{k}) \cong \widetilde{\mathrm{H}}_{d-1}(U_{12};\mathbf{k}).$$

Using excision, we can excise the point p_2 from $(\mathbb{S}^n, U_1)$ and deduce that

$$\widetilde{\mathrm{H}}_d(\mathbb{S}^n, U_1;\mathbf{k}) \cong \widetilde{\mathrm{H}}_d(U_1, U_{12};\mathbf{k}).$$

Combining all of our isomorphisms with the fact that $U_{12} \simeq \mathbb{S}^{n-1}$, we conclude that

$$\widetilde{\mathrm{H}}_d(\mathbb{S}^n;\mathbf{k}) \cong \widetilde{\mathrm{H}}_d(\mathbb{S}^n, U_1;\mathbf{k}) \cong \widetilde{\mathrm{H}}_d(U_1, U_{12};\mathbf{k}) \cong \widetilde{\mathrm{H}}_{d-1}(U_{12};\mathbf{k}) \cong \widetilde{\mathrm{H}}_{d-1}(\mathbb{S}^{n-1};\mathbf{k}). \qquad \square$$

2.3.5. Infinite-dimensional spheres. Regarding $\mathbb{S}^n$ as the equator in $\mathbb{S}^{n+1}$, we have an increasing sequence

$$\mathbb{S}^0 \subset \mathbb{S}^1 \subset \mathbb{S}^2 \subset \cdots.$$

Define

$$\mathbb{S}^\infty = \bigcup_{n=0}^{\infty} \mathbb{S}^n.$$

Endow $\mathbb{S}^\infty$ with the weak topology, so $U \subset \mathbb{S}^\infty$ is open if and only if $U \cap \mathbb{S}^n$ is open for all n. We then have:

LEMMA 2.3.8. *$\widetilde{\mathrm{H}}_d(\mathbb{S}^\infty;\mathbf{k}) = 0$ for all d.*

PROOF. It is easy to see that if $K \subset \mathbb{S}^\infty$ is compact, then there exists some $n \geq 0$ such that $K \subset \mathbb{S}^n$ (see Exercise 2.6.5). We can thus apply the continuity axiom and see that

$$\widetilde{\mathrm{H}}_d(\mathbb{S}^\infty;\mathbf{k}) = \varinjlim_n \widetilde{\mathrm{H}}_d(\mathbb{S}^n;\mathbf{k}).$$

Since $\widetilde{\mathrm{H}}_d(\mathbb{S}^n;\mathbf{k}) = 0$ for $d \geq n-2$, this direct limit vanishes. $\square$

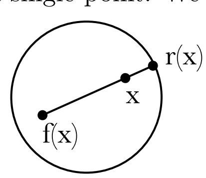
REMARK 2.3.9. In fact, $\mathbb{S}^\infty$ is contractible. See Exercise 2.6.5. $\square$

2.4. The Brouwer fixed point theorem

Our simple calculations already have the following nontrivial consequence, whose statement does not involve homology.

THEOREM 2.4.1 (Brouwer fixed point theorem). *For some $n \geq 1$, let $f \colon \mathbb{D}^n \to \mathbb{D}^n$ be a continuous map. Then f has a fixed point, i.e., there exists some $x \in \mathbb{D}^n$ with $f(x) = x$.*

PROOF. Assume that f has no fixed points. Define a function $r \colon \mathbb{D}^n \to \mathbb{S}^{n-1}$ as follows. For $x \in \mathbb{D}^n$, consider the ray starting at $f(x)$ and passing through x. This is well-defined since $f(x) \neq x$, and it intersects the boundary $\mathbb{S}^{n-1}$ in a single point. We define $r(x)$ to be that intersection point:



For $x \in \mathbb{S}^n$, we have $r(x) = x$. In other words, r is a retraction from $\mathbb{D}^n$ to its boundary $\mathbb{S}^{n-1}$. Letting $\iota \colon \mathbb{S}^{n-1} \hookrightarrow \mathbb{D}^n$ be the inclusion, the composition[4]

$$\mathbb{Z} = \widetilde{\mathrm{H}}_{n-1}(\mathbb{S}^{n-1}) \xrightarrow{\iota_*} \widetilde{\mathrm{H}}_{n-1}(\mathbb{D}^n) \xrightarrow{r_*} \widetilde{\mathrm{H}}_{n-1}(\mathbb{S}^{n-1}) = \mathbb{Z}$$

is the identity. Since $\widetilde{\mathrm{H}}_{n-1}(\mathbb{D}^n) = 0$, this is a contradiction. $\square$

[4]Here we are using our convention that we omit the coefficients from homology when they are $\mathbb{Z}$.

2.5. Local homology and the invariance of dimension

We close this chapter with:

DEFINITION 2.5.1. Let X be a space. For $p \in X$, the *local homology groups* of X at p, denoted $\mathrm{H}_d(X|_p;\mathbf{k})$, are $\mathrm{H}_d(X, X \setminus p;\mathbf{k})$. □

The reason for calling these *local* homology groups is:

LEMMA 2.5.2. *Let X be a space, let $p \in X$, and let U be a neighborhood of p. Then $\mathrm{H}_d(X|_p;\mathbf{k}) \cong \mathrm{H}_d(U|_p;\mathbf{k})$.*

PROOF. The is immediate from excision:

$$\mathrm{H}_d(X|_p;\mathbf{k}) = \mathrm{H}_d(X, X \setminus p;\mathbf{k}) \cong \mathrm{H}_d(U, U \setminus p) = \mathrm{H}_d(U|_p;\mathbf{k}).$$ □

As an example of the kind of information contained in local homology groups, we have:

LEMMA 2.5.3. *Let M be an n-manifold and let $p \in M$. Then $\mathrm{H}_d(M|_p;\mathbf{k})$ is $\mathbf{k}$ for $d = n$ and is 0 for $d \neq 0$.*

PROOF. Let U be a chart around p equipped with a homeomorphism $U \cong \mathbb{R}^n$ taking p to $0 \in \mathbb{R}^n$. Using Lemma 2.5.2, we have

$$\mathrm{H}_d(M|_p;\mathbf{k}) \cong \mathrm{H}_d(U|_p;\mathbf{k}) \cong \mathrm{H}_d(\mathbb{R}^n|_0;\mathbf{k}).$$

The long exact sequence of the pair $(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$ contains the segment

$$\widetilde{\mathrm{H}}_d(\mathbb{R}^n;\mathbf{k}) \longrightarrow \mathrm{H}_d(\mathbb{R}^n, \mathbb{R}^n \setminus 0;\mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_{d-1}(\mathbb{R}^n \setminus 0;\mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_{d-1}(\mathbb{R}^n;\mathbf{k})$$

Since $\mathbb{R}^n$ is contractible and $\mathbb{R}^n \setminus 0 \simeq \mathbb{S}^{n-1}$, we deduce that

$$\mathrm{H}_d(\mathbb{R}^n|_0;\mathbf{k}) = \mathrm{H}_d(\mathbb{R}^n, \mathbb{R}^n \setminus 0;\mathbf{k}) \cong \widetilde{\mathrm{H}}_{d-1}(\mathbb{R}^n \setminus 0;\mathbf{k}) \cong \widetilde{\mathrm{H}}_{d-1}(\mathbb{S}^{n-1};\mathbf{k}).$$

This is indeed $\mathbf{k}$ for $d = n$ and 0 otherwise. □

This has the following corollary:

COROLLARY 2.5.4 (Invariance of dimension). *Let N be an n-manifold and M be an m-manifold such that $N \cong M$. Then $n = m$.*

PROOF. Pick $p \in N$. A homeomorphism $f\colon N \to M$ induces an isomorphism $f_*\colon \mathrm{H}_n(N|_p) \xrightarrow{\cong} \mathrm{H}_n(M|_{f(p)})$. Since $\mathrm{H}_n(N|_p) \cong \mathbb{Z}$, we deduce that $\mathrm{H}_n(M|_{f(p)|}) \cong \mathbb{Z}$ and thus that $n = m$. □

REMARK 2.5.5. If N and M were smooth manifolds and $p \in N$, the derivative of a diffeomorphism $f\colon N \to M$ would be an isomorphism on tangent spaces

$$Df_p\colon T_pN \xrightarrow{\cong} T_{f(p)}M.$$

It follows that these tangent spaces have the same dimension, so N and M have the same dimension. Without smoothness, however, something like homology is needed to rule even the existence of homeomorphisms $\mathbb{R}^n \cong \mathbb{R}^m$ with $n \neq m$. □

REMARK 2.5.6. Another classic application of local homology is distinguishing the boundary of a manifold from its interior. See Exercise 2.6.6. □

2.6. Exercises

EXERCISE 2.6.1. Let $f_0\colon (X,Y) \to (Z,W)$ and $f_1\colon (X,Y) \to (Z,W)$ be homotopic maps of pairs. Prove that f_0 and f_1 induce the same map on homology. Hint: use the naturality of the long exact sequence of the pair and the homotopy invariance of the homology of spaces, along with the five-lemma. □

Exercise 2.6.2. Let (X, Y) be a pair and let

$$X = \bigsqcup_{i \in I} X_i$$

be a decomposition of X. For each i, let $Y_i = Y \cap X_i$. Prove that

$$\mathrm{H}_d(X, Y; \mathbf{k}) = \bigoplus_{i \in I} \mathrm{H}_d(X_i, Y_i; \mathbf{k}) \quad \text{for all } d.$$

□

Exercise 2.6.3. Let (X, Y) be a pair. Write

$$X = \bigcup_{i=1}^{\infty} X_i \quad \text{with } X_1 \subset X_2 \subset X_3 \subset \cdots,$$

and set $Y_i = Y \cap X_i$ for all i. Assume that for every compact subspace K of X, there exists some $n \geq 1$ with $K \subset X_n$. Prove that the resulting maps

$$\varinjlim \mathrm{H}_d(X_i, Y_i; \mathbf{k}) \to \mathrm{H}_d(X, Y; \mathbf{k}) \quad \text{for all } d$$

are isomorphisms. Hint: you will need to prove that you can apply the continuity axiom to Y as well as X. You will also need to either know or prove that direct limits are an exact functor on abelian groups, i.e., they take exact sequences to exact sequences. □

Exercise 2.6.4. Let (X, Y) be a pair. Prove that there is a long exact sequence

$$\cdots \longrightarrow \widetilde{\mathrm{H}}_d(Y; \mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_d(X; \mathbf{k}) \longrightarrow \mathrm{H}_d(X, Y; \mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_{d-1}(Y; \mathbf{k}) \longrightarrow \cdots$$

in reduced homology. □

Exercise 2.6.5. Prove the following two facts about $\mathbb{S}^\infty$:

(a) For all compact $K \subset \mathbb{S}^\infty$, there exists some $n \geq 0$ with $K \subset \mathbb{S}^n$.
(b) The space $\mathbb{S}^\infty$ is contractible (construct an explicit deformation retraction to a point, making sure to verify that it is continuous!). □

Exercise 2.6.6. Let X^n and Y^n be manifolds with boundary and let $f \colon X^n \to Y^n$ be a homeomorphism. Prove that f takes ∂X^n to ∂Y^n. Hint: use local homology. □

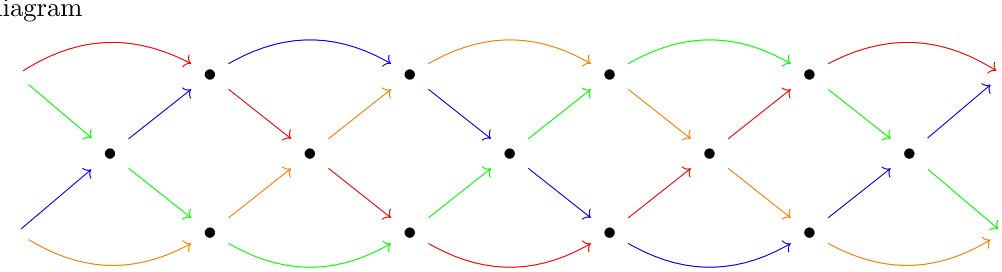
Exercise 2.6.7. Let G_1 and G_2 be graphs with no valence 2 vertices and let $f \colon G_1 \to G_2$ be a homeomorphism. Prove that f takes vertices to vertices and edges to edges. Hint: use local homology. □

Exercise 2.6.8. In this exercise, you will prove the *long exact sequence for a triple*, which says the following. Let $Z \subset Y \subset X$ be three spaces. We then have a long exact sequence of the form

$$\cdots \longrightarrow \mathrm{H}_d(Y, Z; \mathbf{k}) \longrightarrow \mathrm{H}_d(X, Z; \mathbf{k}) \longrightarrow \mathrm{H}_d(X, Y; \mathbf{k}) \longrightarrow \mathrm{H}_{d-1}(Y, Z; \mathbf{k}) \longrightarrow \cdots$$

We remark that we will give a simple direct proof of this for the homology theory we will construct in Chapter 12. The point of this exercise is to practice diagram chasing and show that this can be derived from the long exact sequence of the pair, and thus the axioms.

(a) Prove that the purported long exact sequence of a triple is a chain complex, i.e., that composing any two adjacent maps in it gives 0.
(b) Prove the *braid lemma*, which says the following. Consider a “braided” commutative diagram



where each $\bullet$ is an abelian group. The pattern continues to the left and to the right. Assume that the red, blue, orange, and green sequences are chain complexes, and that all but possibly the orange one are exact. Prove that the orange one is exact. Hint: this is a diagram chase, and the hardest part is finding good notation to express everything.

(c) Prove the long exact sequence of a triple by filling in the red, blue, and green sequences with the long exact sequence for the pairs (X, Z), (X, Y), and (Y, Z), respectively and identifying the orange sequence with the long exact sequence of the triple. $\square$

CHAPTER 3

Degree theory and its applications

Homology greatly clarifies the classical notion of the *degree* of a map between two closed oriented manifolds of the same dimension.

3.1. Classical story

We start by discussing the classical story. Since we will soon generalize this, we omit most proofs. Let M^n and N^n be closed connected oriented smooth n-manifolds and let $f\colon M^n \to N^n$ be a smooth map.

3.1.1. Regular points and local degrees. A *regular point* of f is a $p \in M^n$ such that the derivative map

$$D_p f\colon T_p M^n \to T_{f(p)} N^n$$

is surjective. Since the tangent spaces $T_p M^n$ and $T_{f(p)} N^n$ are both n-dimensional, this is equivalent to requiring $D_p f$ to be an isomorphism. These tangent spaces are oriented vector spaces. The *local degree* of f at p, denoted $\deg_p(f)$, is $+1$ if $D_p f$ is orientation-preserving and is -1 if $D_p f$ is orientation-reversing.

3.1.2. Regular values and degrees. A *regular value* of f is a point $q \in N^n$ such that each $p \in f^{-1}(q)$ is a regular point of f. Sard's theorem implies that regular values exist, and in fact are dense in N^n. Letting $q \in N^n$ be a regular value of f, the *degree* of f is

$$\deg(f) = \sum_{p \in f^{-1}(q)} \deg_p(f).$$

The fundamental theorem of degree theory says that this does not depend on the choice of regular value q. Moreover, if $f, g\colon M^n \to N^n$ are homotopic smooth maps, then $\deg(f) = \deg(g)$.

3.1.3. Surjective maps. Having nonzero degree forces a map to be surjective:

LEMMA 3.1.1. *Let M^n and N^n be closed connected oriented smooth n-manifolds and let $f\colon M^n \to N^n$ be a smooth map with $\deg(f) \neq 0$. Then f is surjective.*

PROOF. Assume that f is not surjective, and let $q \in N^n$ be a point not in its image. Trivially q is a regular value, so since $f^{-1}(q) = \emptyset$ we deduce that

$$\deg(f) = \sum_{p \in f^{-1}(q)} \deg_p(f) = 0,$$

a contradiction. □

3.1.4. Homotoping the identity map. Another application of degree is:

LEMMA 3.1.2. *Let M^n be a closed connected smooth n-manifold. Then the identity map $\mathbb{1}\colon M^n \to M^n$ is not homotopic to a constant map.*

PROOF. If M^n is orientable, then we can talk about $\deg(\mathbb{1})$ and trivially $\deg(\mathbb{1}) = 1$. Since constant maps have degree 0, this implies that $\mathbb{1}$ is not homotopic to a constant map. If M^n is not orientable, then let $\pi\colon \widetilde{M}^n \to M^n$ be its orientable double cover and let $\widetilde{\mathbb{1}}\colon \widetilde{M}^n \to \widetilde{M}^n$ be the identity map. If $\mathbb{1}\colon M^n \to M^n$ is homotopic to a constant map, then we can lift this homotopy to $\widetilde{M}^n$ and see that $\widetilde{\mathbb{1}}$ is homotopic to a constant map, which we just proved is impossible. □

3.1.5. Fundamental theorem of algebra. A final classic application of degree theory is:

THEOREM 3.1.3 (Fundamental theorem of algebra). *Let $f \in \mathbb{C}[z]$ be a nonconstant polynomial. Then there exists some $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$.*

PROOF. Regard f as a smooth map $f: \mathbb{C} \to \mathbb{C}$. Since $\mathbb{C}$ is not compact, the theory of degree does not apply directly to f. However, we can compactify $\mathbb{C}$ to $S^2 = \mathbb{C} \cup \{\infty\}$ and extend f to S^2 by letting $f(\infty) = \infty$. This is a smooth map,[1] so we can talk about $\deg(f)$.

A point $p \in \mathbb{C}$ is a regular point of f precisely when $f'(p) \neq 0$. In this case, identifying $T_p\mathbb{C}$ and $T_{f(p)}\mathbb{C}$ with $\mathbb{C}$, the derivative map

$$D_p f: T_p\mathbb{C} \to T_{f(p)}\mathbb{C}$$

is multiplication by $f'(p) \in \mathbb{C}$. In particular, it is orientation-preserving, so $\deg_p(f) = 1$.

Since $f'(z)$ has at most $(d-1)$ zeros, we see that all but finitely many points of $\mathbb{C}$ are regular values of f. Since $d \geq 1$, the polynomial $f(z)$ takes on infinitely many values, so we can choose some regular value $q \in \mathbb{C}$ such that $f^{-1}(q) \neq \emptyset$. We deduce that

$$\deg(f) = \sum_{p \in f^{-1}(q)} \deg_p(f) = \sum_{p \in f^{-1}(q)} 1 = |f^{-1}(q)| \neq 0.$$

By Lemma 3.1.1, this shows that f must be surjective, so in particular there must be some $z_0 \in \mathbb{C}$ with $f(z_0) = 0$. $\square$

3.2. Topological degree for spheres

The theory outlined in the previous section has the flaw of only applying to smooth maps.[2] Homology provides an elegant intrinsic notion of degree for continuous maps. Moreover, as we will see later in this book it provides a basic tool for making computations in homology. At this point in our development, we only have the technology to study the degree of maps between spheres. Later in this chapter we will explain what has to be done to study it in general.

REMARK 3.2.1. We will not use the definitions or results from the previous section to set up the homological notion of degree, and we will not distinguish them in our notation. Later we will prove that they are equal. $\square$

3.2.1. Definition of degree. Consider a map $f: \mathbb{S}^n \to \mathbb{S}^n$. The induced map $f_*: \widetilde{\mathrm{H}}_n(\mathbb{S}^n) \to \widetilde{\mathrm{H}}_n(\mathbb{S}^n)$ is a group homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$, and thus is multiplication by some $d \in \mathbb{Z}$. This integer d is the *degree* of f, and is denoted $\deg(f)$. Since homology is homotopy invariant, the degree of f is also unchanged under homotopies.

3.2.2. Reflections. Here is an important example of this. Recall that if H is a hyperplane in $\mathbb{R}^n$, then the *reflection* in H is the map $r: \mathbb{R}^n \to \mathbb{R}^n$ defined as follows. Let $\vec{v} \in \mathbb{R}^n$ be a unit vector orthogonal to H. Every $x \in \mathbb{R}^n$ can be written uniquely as $x = h + c\vec{v}$ for some $h \in H$ and $c \in \mathbb{R}$, and $r(x) = h - c\vec{v}$. We then have:

LEMMA 3.2.2. *Let $r: \mathbb{S}^n \to \mathbb{S}^n$ be the restriction to $\mathbb{S}^n$ of the reflection in a hyperplane H in $\mathbb{R}^{n+1}$. Then* $\deg(r) = -1$.

---

[1] To see that this is smooth at ∞, note that making the change of coordinates $z \mapsto 1/z$ to the domain and codomain of f turns it into $1/f(1/z)$. We must check that this extends to a smooth function that vanishes at $z = 0$. Writing

$$f(z) = a_n z^n + \cdots + a_1 z + a_0 \quad \text{with } a_n \neq 0,$$

this follows from the fact that

$$1/f(1/z) = \frac{1}{a_n(1/z)^n + \cdots + a_1(1/z) + a_0} = \frac{z^n}{a_n + a_{n-1}z + \cdots + a_1 z^{n-1} + a_0 z^n}.$$

[2] This can be circumvented using the fact that every continuous map between smooth manifolds can be homotoped to a smooth map, but this is an awkward kludge. Note that a version of this even shows up in the foundations of the theory: to use smooth techniques to prove that the degree is invariant under homotopy, one must prove that two homotopic smooth maps are smoothly homotopic.

PROOF. Varying H homotopes r, which does not change $\deg(r)$. This can change H to any hyperplane, so we deduce that it is enough to prove the lemma for any single reflection. We will handle the reflection

$$r(x_1, \ldots, x_{n+1}) = (-x_1, x_2, \ldots, x_{n+1}) \quad \text{for } (x_1, \ldots, x_{n+1}) \in \mathbb{S}^n.$$

The proof will be by induction on n. For the base case $n = 0$, note that $\mathbb{S}^0 \subset \mathbb{R}^1$ consists of two points 1 and -1. To keep our notation straight, let $p = 1$ and $p' = -1$. We thus have

$$r(p) = p' \quad \text{and} \quad r(p') = p.$$

Recall that $\mathrm{H}_0(\mathbb{S}^0)$ is the free abelian group on the path components of $\mathbb{S}^0$. We can thus identify $\mathrm{H}_0(\mathbb{S}^0)$ with the group $\mathbb{Z}\langle p, p'\rangle$ of formal $\mathbb{Z}$-linear combinations of p and p'. Using this identification, the group $\widetilde{\mathrm{H}}_0(\mathbb{S}^0) \cong \mathbb{Z}$ is generated by $p - p'$. We have

$$r_*(p - p') = p' - p = -(p - p').$$

In other words, r_* as as multiplication by -1 on $\widetilde{\mathrm{H}}_0(\mathbb{S}^0)$, as desired.

Now assume that $n \geq 1$ and that the lemma is true for $\mathbb{S}^{n-1}$. Let

$$q = (0, \ldots, 1) \in \mathbb{S}^n \quad \text{and} \quad q' = (0, \ldots, -1) \in \mathbb{S}^n,$$

and identify $\mathbb{S}^{n-1}$ with the subspace of $\mathbb{S}^n$ whose last coordinate is 0. We thus have $r(q) = q$ and $r(q') = q'$, and r restricts to a reflection on $\mathbb{S}^{n-1}$. Going back to our calculation of $\widetilde{\mathrm{H}}_n(\mathbb{S}^n)$ from Lemma 2.3.7, we have isomorphisms

$$\widetilde{\mathrm{H}}_n(\mathbb{S}^n) \xrightarrow{\cong} \mathrm{H}_n(\mathbb{S}^n, \mathbb{S}^n \setminus q) \xleftarrow{\cong} \mathrm{H}_n(\mathbb{S}^n \setminus q', \mathbb{S}^n \setminus \{q, q'\}) \xrightarrow{\cong} \widetilde{\mathrm{H}}_{n-1}(\mathbb{S}^n \setminus \{q, q'\}) \xleftarrow{\cong} \widetilde{\mathrm{H}}_{n-1}(\mathbb{S}^{n-1}).$$

Here the first isomorphism comes from the long exact sequence of the pair $(\mathbb{S}^n, \mathbb{S}^n \setminus q)$, the second from excision, the third from the long exact sequence of the pair $(\mathbb{S}^n \setminus q', \mathbb{S}^n \setminus \{q, q'\})$, and the fourth from the fact that $\mathbb{S}^n \setminus \{q, q'\}$ deformation retracts to $\mathbb{S}^{n-1}$. The reflection r acts on each of these homology groups, and these isomorphisms commute with the action of r. We know by induction that r acts as multiplication by -1 on $\widetilde{\mathrm{H}}_{n-1}(\mathbb{S}^{n-1})$, so the same is true for $\widetilde{\mathrm{H}}_n(\mathbb{S}^n)$, as desired. □

3.2.3. Composition. The notion of degree behaves well under composition:

LEMMA 3.2.3. *Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ and $g\colon \mathbb{S}^n \to \mathbb{S}^n$ be maps. Then*

$$\deg(f \circ g) = \deg(f) \cdot \deg(g).$$

PROOF. The map $(f \circ g)_*\colon \widetilde{\mathrm{H}}_n(\mathbb{S}^n) \to \widetilde{\mathrm{H}}_n(\mathbb{S}^n)$ equals the composition

$$\widetilde{\mathrm{H}}_n(\mathbb{S}^n) \xrightarrow{g_*} \widetilde{\mathrm{H}}_n(\mathbb{S}^n) \xrightarrow{f_*} \widetilde{\mathrm{H}}_n(\mathbb{S}^n).$$

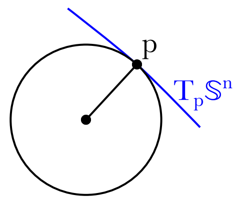
Since f_* is multiplication by $\deg(f)$ and g_* is multiplication by $\deg(g)$, we conclude that $(f \circ g)_*$ is multiplication by $\deg(f) \cdot \deg(g)$, as desired. □

3.2.4. Antipodal map. Recall that the *antipodal map* of $\mathbb{S}^n$ is the map $a\colon \mathbb{S}^n \to \mathbb{S}^n$ defined by $a(p) = -p$.

LEMMA 3.2.4. *The degree of the antipodal map $a\colon \mathbb{S}^n \to \mathbb{S}^n$ is $(-1)^{n+1}$.*

PROOF. The map a is the composition of the $(n+1)$ reflections that multiply the coordinates of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ by -1. Lemma 3.2.2 says that each of these reflections has degree -1, so Lemma 3.2.3 implies that a has degree $(-1)^{n+1}$. □

3.2.5. Hairy ball theorem. As an application of these results, we prove the following. Recall that for $p \in \mathbb{S}^n$, we can identify the tangent space $T_p\mathbb{S}^n$ with the subspace of $\mathbb{R}^{n+1}$ consisting of vectors orthogonal to p:



A vector field on $\mathbb{S}^n$ is thus the same as continuous function $\sigma\colon \mathbb{S}^n \to \mathbb{R}^{n+1}$ such that $\sigma(p)\cdot p = 0$, where $\cdot$ is the usual dot product.

Theorem 3.2.5 (Hairy ball theorem). *The sphere $\mathbb{S}^n$ has a nowhere vanishing vector field if and only if n is odd.*

Proof. Assume first that n is odd. Write $n = 2m-1$. We can then define our nowhere vanishing vector field via the map $\sigma\colon \mathbb{S}^n \to \mathbb{R}^{2m}$ given by the formula

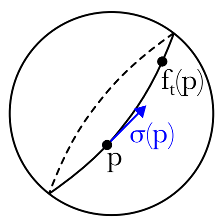
$$\sigma(x_1, \ldots, x_{2m}) = (x_2, -x_1, x_4, -x_3, \ldots, x_{2m}, -x_{2m-1}) \quad \text{for } (x_1, \ldots, x_{2m}) \in \mathbb{S}^n.$$

This is a vector field on $\mathbb{S}^n$ since

$$\sigma(x_1, \ldots, x_{2m})\cdot(x_1, \ldots, x_{2m}) = (x_1x_2 - x_2x_1) + \cdots + (x_{2m-1}x_{2m} - x_{2m}x_{2m-1}) = 0,$$

and it is nowhere vanishing since the point $(0, \ldots, 0) \in \mathbb{R}^{2m}$ does not lie on $\mathbb{S}^n$.

Assume now that $\mathbb{S}^n$ has a nowhere vanishing vector field $\sigma\colon \mathbb{S}^n \to \mathbb{R}^{2m}$. Define a family of maps $f_t\colon \mathbb{S}^n \to \mathbb{S}^n$ as follows. Consider $p \in \mathbb{S}^n$. Then $f_t(p)$ is the point at distance πt along the great circle on $\mathbb{S}^n$ starting at p in the direction $\sigma(p)$:



Since this great circle has total length 2π, the point at distance π along it is $-p$. We therefore have $f_0(p) = p$ and $f_1(p) = -p$. In other words, f is a homotopy from the identity map to the antipodal map. We conclude that the antipodal map must have degree 1. Since the antipodal map has degree $(-1)^{n+1}$, we conclude that n must be odd. □

3.3. Exercises

CHAPTER 4

Collapsing relative homology

Fix an abelian group $\mathbf{k}$. This chapter is devoted to a basic property of relative homology: for a pair (X, Y) satisfying suitable hypotheses, we have

$$\mathrm{H}_d(X, Y; \mathbf{k}) \cong \widetilde{\mathrm{H}}_d(X/Y; \mathbf{k}) \quad \text{for all } d.$$

Combined with the long exact sequence of the pair (X, Y), this gives a powerful tool for inductively understanding spaces. As an example of this, we calculate the homology groups of real and complex projective spaces.

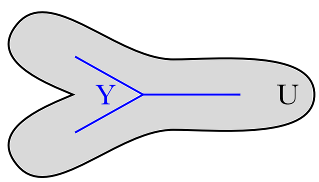
4.1. Collapsing theorem

We start by stating and proving the result alluded to above.

4.1.1. Good pairs. The hypothesis needed is:

Definition 4.1.1. A pair (X, Y) is a *good pair* if Y is a nonempty closed subset of X and there exists an open set U containing Y such that U deformation retracts to Y. □

Example 4.1.2. Letting Y be as follows, the pair $(\mathbb{R}^2, Y)$ is a good pair:



The set U is as indicated. In this case, it is easy to see that $\mathbb{R}^2/Y \cong \mathbb{R}^2$. □

Example 4.1.3. If X is a smooth manifold and Y is a smoothly embedded submanifold of X, then (X, Y) is a good pair. For U, one can take a tubular neighborhood of Y in X. □

Example 4.1.4. If X is a CW complex and Y is a subcomplex of X, then (X, Y) is a good pair. See Appendix 13. □

4.1.2. Collapsing theorem. We have:

Theorem 4.1.5. *Let (X, Y) be a good pair. Then $\mathrm{H}_d(X, Y; \mathbf{k}) \cong \widetilde{\mathrm{H}}_d(X/Y; \mathbf{k})$ for all d.*

Remark 4.1.6. The isomorphism $\mathrm{H}_d(X, Y; \mathbf{k}) \cong \widetilde{\mathrm{H}}_d(X/Y; \mathbf{k})$ is induced by the map of pairs $(X, Y) \to (X/Y, Y/Y)$. Here we are using the fact that for any nonempty space Z and any $z_0 \in Z$ we have $\mathrm{H}_d(Z, z_0; \mathbf{k}) \cong \widetilde{\mathrm{H}}_d(Z; \mathbf{k})$ for all d; see Lemma 2.3.5. □

Remark 4.1.7. Instead of requiring (X, Y) to be a good pair, we could instead have required the inclusion $Y \hookrightarrow X$ to be a *cofibration*. See Appendix 16. □

Before we prove Theorem 4.1.5, we point out a consequence of it:

Corollary 4.1.8. *Let X be a space and let $Y \subset X$ be a contractible subspace such that (X, Y) is a good pair. Then $\widetilde{\mathrm{H}}_d(X; \mathbf{k}) \cong \widetilde{\mathrm{H}}_d(X/Y; \mathbf{k})$ for all d.*

Proof. The long exact sequence for the pair (X, Y) contains the segment

$$\widetilde{\mathrm{H}}_d(Y; \mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_d(X; \mathbf{k}) \longrightarrow \mathrm{H}_d(X, Y; \mathbf{k}) \longrightarrow \mathrm{H}_{d-1}(Y; \mathbf{k}).$$

Since Y is contractible its reduced homology groups all vanish, so using Theorem 4.1.5 we have

$$\widetilde{\mathrm{H}}_d(X;\mathbf{k}) \cong \mathrm{H}_d(X,Y;\mathbf{k}) \cong \widetilde{\mathrm{H}}_d(X/Y;\mathbf{k}).$$ □

For the proof of Theorem 4.1.5, we need:

LEMMA 4.1.9. *Let (X,Y) be a pair and let Y' be a subset of X with $Y \subset Y'$ such that the induced map $\mathrm{H}_d(Y;\mathbf{k}) \to \mathrm{H}_d(Y';\mathbf{k})$ is an isomorphism for all d. Then $\mathrm{H}_d(X,Y;\mathbf{k}) \cong \mathrm{H}_d(X,Y';\mathbf{k})$ for all d.*

PROOF. Consider the map between the long exact sequences of the pairs (X,Y) and (X,Y'):

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & \mathrm{H}_d(Y;\mathbf{k}) & \longrightarrow & \mathrm{H}_d(X;\mathbf{k}) & \longrightarrow & \mathrm{H}_d(X,Y;\mathbf{k}) & \longrightarrow & \mathrm{H}_{d-1}(Y;\mathbf{k}) & \longrightarrow & \mathrm{H}_{d-1}(X;\mathbf{k}) & \longrightarrow & \cdots \\ & & \downarrow\cong & & \downarrow = & & \downarrow & & \downarrow\cong & & \downarrow = & & \\ \cdots & \longrightarrow & \mathrm{H}_d(Y';\mathbf{k}) & \longrightarrow & \mathrm{H}_d(X;\mathbf{k}) & \longrightarrow & \mathrm{H}_d(X,Y';\mathbf{k}) & \longrightarrow & \mathrm{H}_{d-1}(Y';\mathbf{k}) & \longrightarrow & \mathrm{H}_{d-1}(X;\mathbf{k}) & \longrightarrow & \cdots \end{array}$$

The indicated isomorphisms come from the hypotheses. The five lemma now implies that the maps $\mathrm{H}_d(X,Y;\mathbf{k}) \to \mathrm{H}_d(X,Y';\mathbf{k})$ are isomorphisms for all d. □

PROOF OF THEOREM 4.1.5. Let y_0 be the point Y/Y of X/Y. Lemma 2.3.5 implies that $\mathrm{H}_d(X/Y,y_0;\mathbf{k}) \cong \widetilde{\mathrm{H}}_d(X/Y;\mathbf{k})$. Letting $f\colon (X,Y) \to (X/Y,y_0)$ be the map of pairs, it is therefore enough to prove that the induced map

$$f_*\colon \mathrm{H}_d(X,Y;\mathbf{k}) \to \mathrm{H}_d(X/Y,y_0;\mathbf{k})$$

is an isomorphism.

Let U be an open neighborhood of Y in X that deformation retracts to Y. It follows that U/Y is an open neighborhood of y_0 in X/Y that deformation retracts to y_0. Let $g\colon (X,U) \to (X/Y,U/Y)$ be the map of pairs. We then have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_d(X,Y;\mathbf{k}) & \xrightarrow{f_*} & \mathrm{H}_d(X/Y,y_0;\mathbf{k}) \\ \downarrow & & \downarrow \\ \mathrm{H}_d(X,U;\mathbf{k}) & \xrightarrow{g_*} & \mathrm{H}_d(X/Y,U/Y;\mathbf{k}). \end{array}$$

Lemma 4.1.9 implies that the vertical maps are both isomorphisms, so to prove that f_* is an isomorphism it is enough to prove that g_* is an isomorphism.

Next, let $h\colon (X \setminus Y, U \setminus Y) \to (X/Y \setminus \{y_0\}, U/Y \setminus \{y_0\})$ be the map of pairs. We then have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_d(X \setminus Y, U \setminus Y;\mathbf{k}) & \xrightarrow{h_*} & \mathrm{H}_d(X/Y \setminus \{y_0\}, U/Y \setminus \{y_0\};\mathbf{k}) \\ \downarrow & & \downarrow \\ \mathrm{H}_d(X,U;\mathbf{k}) & \xrightarrow{g_*} & \mathrm{H}_d(X/Y,U/Y;\mathbf{k}). \end{array}$$

Excision implies that the vertical maps are both isomorphisms, so to prove that g_* is an isomorphism it is enough to prove that h_* is an isomorphism. But this is easy; indeed, the map $h\colon (X \setminus Y, U \setminus Y) \to (X/Y \setminus \{y_0\}, U/Y \setminus \{y_0\})$ is actually an *isomorphism* of pairs. □

4.2. Wedges product of spaces

Let $\{(X_i,p_i)\}_{i \in I}$ be a collection of based spaces. Recall that their *wedge product* is the space obtained by gluing together the basepoints of the X_i:

$$\bigvee_{i \in I} X_i = \bigsqcup_{i \in I} X_I / \sim \quad \text{where } p_i \sim p_j \text{ for all } i,j \in I.$$

The basepoints in the X_i map to a distinguished basepoint $*$ of $\bigvee_{i \in I} X_i$. For each $i_0 \in I$, there are inclusion and collapse maps

$$\iota_{i_0}\colon (X_{i_0},p_{i_0}) \hookrightarrow \left(\bigvee_{i \in I} X_i, *\right) \quad \text{and} \quad c_{i_0}\colon \left(\bigvee_{i \in I} X_i, *\right) \to (X_{i_0},p_{i_0}).$$

Here c_{i_0} collapses every term but X_{i_0} to the basepoint. These satisfy $c_{i_0} \circ \iota_{i_0} = \mathbb{1}_{X_{i_0}}$, so on reduced homology the composition

$$\widetilde{\mathrm{H}}_d(X_{i_0}; \mathbf{k}) \xrightarrow{(\iota_{i_0})_*} \widetilde{\mathrm{H}}_d(\textstyle\bigvee_{i \in I} X_i; \mathbf{k}) \xrightarrow{(c_{i_0})_*} \widetilde{\mathrm{H}}_d(X_{i_0}; \mathbf{k})$$

is the identity. In other words, $(\iota_{i_0})_*$ is a split injection. Assembling all the $(\iota_{i_0})_*$ together, we get an injective map

$$\sum_{i \in I} (\iota_i)_* \colon \bigoplus_{i \in I} \widetilde{\mathrm{H}}_d(X_i; \mathbf{k}) \to \widetilde{\mathrm{H}}_d(\bigvee_{i \in I} X_i; \mathbf{k}).$$

As a first application of Theorem 4.1.5, we prove this is an isomorphism if each p_i is a *good basepoint* for X_i, i.e., the pairs (X_i, p_i) are good pairs.

LEMMA 4.2.1. *Let $\{(X_i, p_i)\}_{i \in I}$ be a collection of based spaces such that each p_i is a good basepoint for X_i. Then the map*

$$\bigoplus_{i \in I} \widetilde{\mathrm{H}}_d(X_i; \mathbf{k}) \to \widetilde{\mathrm{H}}_d(\bigvee_{i \in I} X_i; \mathbf{k})$$

discussed above is an isomorphism for all d.

PROOF. Using the relative version of the additivity axiom along with Theorem 4.1.5, we have:

$$\bigoplus_{i \in I} \widetilde{\mathrm{H}}_d(X_i; \mathbf{k}) \cong \bigoplus_{i \in I} \mathrm{H}_d(X_i, p_i; \mathbf{k}) \cong \mathrm{H}_d(\bigsqcup_{i \in I} X_i, \bigsqcup_{i \in I} p_i; \mathbf{k}) \cong \mathrm{H}_d(\bigvee_{i \in I} X_i, *; \mathbf{k}) \cong \widetilde{\mathrm{H}}_d(\bigvee_{i \in I} X_i; \mathbf{k}).$$

Here $*$ is the distinguished basepoint of the wedge product. It is immediate that this composition of isomorphisms is given by the indicated map. □

4.3. Complex projective space

As another application of Theorem 4.1.5, we calculate the homology of complex projective space. Recall that this is the space $\mathbb{CP}^n$ of lines through the origin in $\mathbb{C}^{n+1}$. Points in $\mathbb{CP}^n$ can be expressed via *homogeneous coordinates*: for $z_0, \ldots, z_n \in \mathbb{C}$ not all 0, the point $[z_0, \ldots, z_n] \in \mathbb{CP}^n$ is the line through $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$. For $\lambda \in \mathbb{C}^\times$, we have

$$[\lambda z_0, \ldots, \lambda z_n] = [z_0, \ldots, z_n] \quad \text{for } [z_0, \ldots, z_n] \in \mathbb{CP}^n.$$

We have:

THEOREM 4.3.1. *For $n \geq 1$, we have*

$$\mathrm{H}_d(\mathbb{CP}^n; \mathbf{k}) = \begin{cases} \mathbf{k} & \text{if } 0 \leq d \leq n \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. The proof is by induction on n. The base case $n = 1$ holds since $\mathbb{CP}^1 \cong S^2 = \mathbb{C} \cup \{\infty\}$ via the homeomorphism taking $[z, 1] \in \mathbb{CP}^1$ to $z \in \mathbb{C}$ and $[0, 1] \in \mathbb{CP}^1$ to ∞. Now assume that $n \geq 2$ and that the theorem is true for $\mathbb{CP}^{n-1}$. Embed $\mathbb{CP}^{n-1}$ into $\mathbb{CP}^n$ via the map taking $[z_0, \ldots, z_{n-1}] \in \mathbb{CP}^{n-1}$ to $[z_0, \ldots, z_{n-1}, 0] \in \mathbb{CP}^n$. The complement $\mathbb{CP}^n \setminus \mathbb{CP}^{n-1}$ is all $[z_0, \ldots, z_n] \in \mathbb{CP}^n$ with $z_n \neq 0$. This is homeomorphic to $\mathbb{C}^n$ via the map taking $[z_0, \ldots, z_n] \in \mathbb{CP}^n$ with $z_n \neq 0$ to $(z_0/z_n, \ldots, z_{n-1}/z_n) \in \mathbb{C}^n$. This implies that $\mathbb{CP}^n / \mathbb{CP}^{n-1}$ is the 1-point compactification of $\mathbb{C}^n$, i.e., that

$$\mathbb{CP}^n / \mathbb{CP}^{n-1} \cong \mathbb{S}^{2n}.$$

The pair $(\mathbb{CP}^n, \mathbb{CP}^{n-1})$ is a good pair,[1] so we can apply Theorem 4.1.5 and see that

$$\mathrm{H}_d(\mathbb{CP}^n, \mathbb{CP}^{n-1}; \mathbf{k}) \cong \widetilde{\mathrm{H}}_d(\mathbb{CP}^n / \mathbb{CP}^{n-1}; \mathbf{k}) \cong \widetilde{\mathrm{H}}_d(\mathbb{S}^{2n}; \mathbf{k}) = \begin{cases} \mathbf{k} & \text{if } d = 2n, \\ 0 & \text{otherwise.} \end{cases}$$

[1]This follows from the fact that $\mathbb{CP}^n$ is a smooth manifold and $\mathbb{CP}^{n-1}$ is a smoothly embedded submanifold. More directly, the set $U = \mathbb{CP}^n \setminus [0, \ldots, 0, 1]$ is an open neighborhood of $\mathbb{CP}^{n-1}$ that deformation retracts to $\mathbb{CP}^{n-1}$ via the deformation retraction $r_t \colon U \to U$ given by the formula

$$r_t([z_0, \ldots, z_n]) = \begin{cases} [z_0/z_n, \ldots, z_{n-1}/z_n, 1 - t] & \text{if } z_n \neq 0, \\ [z_0, \ldots, z_{n-1}, 0] & \text{if } z_n = 0 \end{cases} \quad \text{for } 0 \leq t \leq 1.$$

Now consider the long exact sequence of the pair $(\mathbb{CP}^n, \mathbb{CP}^{n-1})$. It contains the segment

$$\mathrm{H}_{d+1}(\mathbb{CP}^n, \mathbb{CP}^{n-1}; \mathbf{k}) \longrightarrow \mathrm{H}_d(\mathbb{CP}^{n-1}; \mathbf{k}) \longrightarrow \mathrm{H}_d(\mathbb{CP}^n; \mathbf{k}) \longrightarrow \mathrm{H}_d(\mathbb{CP}^n, \mathbb{CP}^{n-1}; \mathbf{k})$$

For d odd, by our inductive hypothesis we have $\mathrm{H}_d(\mathbb{CP}^{n-1}; \mathbf{k}) = 0$ and $\mathrm{H}_d(\mathbb{CP}^n, \mathbb{CP}^{n-1}; \mathbf{k}) = 0$, so $\mathrm{H}_d(\mathbb{CP}^n; \mathbf{k}) = 0$. For d even but not $2n$, we have $\mathrm{H}_{d+1}(\mathbb{CP}^n, \mathbb{CP}^{n-1}; \mathbf{k}) = 0$ and $\mathrm{H}_d(\mathbb{CP}^n, \mathbb{CP}^{n-1}; \mathbf{k}) = 0$, so $\mathrm{H}_d(\mathbb{CP}^n; \mathbf{k}) \cong \mathrm{H}_d(\mathbb{CP}^{n-1}; \mathbf{k})$, which is as indicated in the theorem by induction. The only remaining case is $d = 2n$. This uses the following segment of the long exact sequence of the pair $(\mathbb{CP}^n, \mathbb{CP}^{n-1})$:

$$\mathrm{H}_{2n}(\mathbb{CP}^{n-1}; \mathbf{k}) \longrightarrow \mathrm{H}_{2n}(\mathbb{CP}^n; \mathbf{k}) \longrightarrow \mathrm{H}_{2n}(\mathbb{CP}^n, \mathbb{CP}^{n-1}; \mathbf{k}) \longrightarrow \mathrm{H}_{2n-1}(\mathbb{CP}^{n-1}; \mathbf{k})$$

By induction we have $\mathrm{H}_{2n}(\mathbb{CP}^{n-1}; \mathbf{k}) = 0$ and $\mathrm{H}_{2n-1}(\mathbb{CP}^{n-1}; \mathbf{k}) = 0$, so

$$\mathrm{H}_{2n}(\mathbb{CP}^n; \mathbf{k}) \cong \mathrm{H}_{2n}(\mathbb{CP}^n, \mathbb{CP}^{n-1}; \mathbf{k}) \cong \widetilde{\mathrm{H}}_{2n}(\mathbb{S}^{2n}; \mathbf{k}) = \mathbf{k}. \qquad \square$$

4.4. Real projective space

4.5. Exercises

CHAPTER 5

The Mayer–Vietoris theorem and its applications

Fix an abelian group $\mathbf{k}$. The Mayer–Vietoris theorem can be viewed as an analogue for homology of the Seifert–van Kampen theorem for the fundamental group. Roughly speaking, given a decomposition of a space X into two subspaces A and B it explains how $\mathrm{H}_d(X;\mathbf{k})$ is built from $\mathrm{H}_d(A;\mathbf{k})$ and $\mathrm{H}_d(B;\mathbf{k})$ and $\mathrm{H}_d(A \cap B;\mathbf{k})$.

5.1. Mayer–Vietoris

Here is a precise statement:

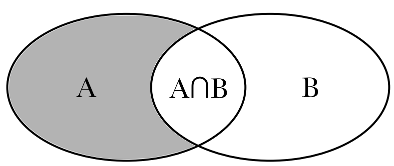
THEOREM 5.1.1 (Mayer–Vietoris). *Let X be a topological space and let $A, B \subset X$ be subspaces such that $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$. Then we have a long exact sequence*

$$\cdots \longrightarrow \mathrm{H}_d(A \cap B;\mathbf{k}) \longrightarrow \mathrm{H}_d(A;\mathbf{k}) \oplus \mathrm{H}_d(B;\mathbf{k}) \longrightarrow \mathrm{H}_d(X;\mathbf{k}) \longrightarrow \mathrm{H}_{d-1}(A \cap B;\mathbf{k}) \longrightarrow \cdots .$$

REMARK 5.1.2. There is also a version where the homology groups are replaced by reduced homology groups (see Exercise 5.6.2), as well as various relative versions (see Exercise 5.6.3). □

REMARK 5.1.3. The Mayer–Vietoris exact sequence does not generalize in a straightforward way to a cover of a space by more than two sets. Instead, what such a cover leads to is a spectral sequence called the Mayer–Vietoris Spectral Sequence. □

PROOF OF THEOREM 5.1.1. The pair $(B, A \cap B)$ can be obtained from (X, A) by removing $X \setminus B$:



Since $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$, the closure of $X \setminus B$ lies in $\mathrm{Int}(A)$. We can therefore apply excision to see that the map $(B, A \cap B) \to (X, A)$ induces isomorphisms

$$h_d \colon \mathrm{H}_d(B, A \cap B;\mathbf{k}) \xrightarrow{\cong} \mathrm{H}_d(X, A) \quad \text{for all } d.$$

These isomorphisms appear in the map between the long exact sequences for the pairs $(B, A \cap B)$ and (X, A):

$$\begin{array}{ccccccccccccc}
\cdots \xrightarrow{j_{d+1}} & \mathrm{H}_{d+1}(B, A \cap B;\mathbf{k}) & \xrightarrow{\partial_{d+1}} & \mathrm{H}_d(A \cap B;\mathbf{k}) & \xrightarrow{i_d} & \mathrm{H}_d(B;\mathbf{k}) & \xrightarrow{j_d} & \mathrm{H}_d(B, A \cap B;\mathbf{k}) & \xrightarrow{\partial_d} & \mathrm{H}_{d-1}(A \cap B;\mathbf{k}) & \xrightarrow{i_{d-1}} \cdots \\
& \cong \downarrow h_{d+1} & & \downarrow f_d & & \downarrow g_d & & \cong \downarrow h_d & & \downarrow f_{d-1} & \\
\cdots \xrightarrow{j'_{d+1}} & \mathrm{H}_{d+1}(X, A;\mathbf{k}) & \xrightarrow{\partial'_{d+1}} & \mathrm{H}_d(A;\mathbf{k}) & \xrightarrow{i'_d} & \mathrm{H}_d(X;\mathbf{k}) & \xrightarrow{j'_d} & \mathrm{H}_d(X, A;\mathbf{k}) & \xrightarrow{\partial'_d} & \mathrm{H}_{d-1}(A;\mathbf{k}) & \xrightarrow{i'_{d-1}} \cdots
\end{array}$$

From this, we can apply Lemma 5.1.4 below (the Barratt–Whitehead lemma) to obtain our Mayer–Vietoris sequence. The connecting homomorphisms $\mathrm{H}_d(X;\mathbf{k}) \to \mathrm{H}_{d-1}(A \cap B;\mathbf{k})$ are $\partial_d \circ (h_d)^{-1} \circ j'_d$. □

The above proof used:

LEMMA 5.1.4 (Barratt–Whitehead lemma). *Let*

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{j_{d+1}} & R_{d+1} & \xrightarrow{\partial_{d+1}} & I_d & \xrightarrow{i_d} & B_d & \xrightarrow{j_d} & R_d & \xrightarrow{\partial_d} & I_{d-1} & \xrightarrow{i_{d-1}} & \cdots \\
 & & \cong\downarrow h_{d+1} & & \downarrow f_d & & \downarrow g_d & & \cong\downarrow h_d & & \downarrow f_{d-1} & & \\
\cdots & \xrightarrow{j'_{d+1}} & R'_{d+1} & \xrightarrow{\partial'_{d+1}} & A_d & \xrightarrow{i'_d} & X_d & \xrightarrow{j'_d} & R'_d & \xrightarrow{\partial'_d} & A_{d-1} & \xrightarrow{i'_{d-1}} & \cdots
\end{array}$$

be a commutative diagram of abelian groups with exact rows such that each $h_d \colon R_d \to R'_d$ is an isomorphism. We then have a long exact sequence

$$\cdots \longrightarrow I_d \xrightarrow{f_d \oplus i_d} A_d \oplus B_d \xrightarrow{i'_d - g_d} X_d \longrightarrow I_{d-1} \xrightarrow{f_{d-1} \oplus i_{d-1}} \cdots$$

whose connecting homomorphisms $X_d \to I_{d-1}$ are $\partial_d \circ (h_d)^{-1} \circ j'_d$.

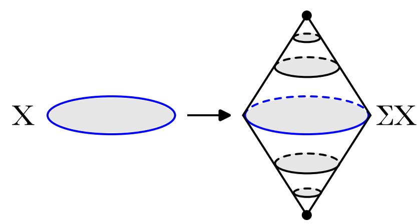
Proof. Proving that the indicated sequence is exact is a simple diagram chase that we leave as an exercise to the reader (see Exercise 5.6.1). □

5.2. Suspension

Our first application of Mayer–Vietoris is to suspensions. Let X be a topological space. Recall that the *suspension* of X is the space

$$\Sigma X = X \times [-1,1] \sim \quad \text{where } \sim \text{ collapses } X \times -1 \text{ and } X \times 1 \text{ to two points.}$$

The images of $X \times -1$ and $X \times 1$ in ΣX are called the top and bottom *suspension points*, respectively:



Example 5.2.1. For $n \geq 0$ we have $\Sigma \mathbb{S}^n \cong \mathbb{S}^{n+1}$. The suspension points correspond to the north and south poles. □

We have the following, which generalizes the isomorphisms $\widetilde{\mathrm{H}}_d(\mathbb{S}^n; \mathbf{k}) \cong \widetilde{\mathrm{H}}_{d+1}(\mathbb{S}^{n+1}; \mathbf{k})$ constructed in our calculation of the homology of spheres (c.f. Lemma 2.3.7).

Lemma 5.2.2. *For a space X, we have $\widetilde{\mathrm{H}}_d(X; \mathbf{k}) \cong \widetilde{\mathrm{H}}_{d+1}(\Sigma X; \mathbf{k})$ for all d.*

Proof. Let p and q be the top and bottom suspension points of ΣX. Define $A = \Sigma X \setminus \{p\}$ and $B = \Sigma X \setminus \{q\}$. These are open subsets of ΣX with $\Sigma X = A \cup B$, so we can apply Mayer–Vietoris and get a long exact sequence containing

$$\widetilde{\mathrm{H}}_{d+1}(A; \mathbf{k}) \oplus \widetilde{\mathrm{H}}_{d+1}(B; \mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_{d+1}(\Sigma X; \mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_d(A \cap B; \mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_d(A; \mathbf{k}) \oplus \widetilde{\mathrm{H}}_d(B; \mathbf{k}).$$

The terms involving A and B and $A \cap B$ are as follows:

- Both A and B are contractible, so their homology groups all vanish. For instance, $A = \Sigma X \setminus \{p\}$ deformation retracts to q via the deformation retraction $r_t \colon A \to A$ induced by the maps $X \times [-1,1) \to X \times [-1,1)$ taking (p,s) to $(p,(1-t)s)$.
- $A \cap B \cong X \times (-1,1)$, so $A \cap B \simeq X$.

Combining these with the above exact sequence, we see that

$$\widetilde{\mathrm{H}}_{d+1}(\Sigma X; \mathbf{k}) \cong \widetilde{\mathrm{H}}_d(A \cap B; \mathbf{k}) \cong \widetilde{\mathrm{H}}_d(X; \mathbf{k}). \qquad \square$$

5.3. Removing a point from a manifold

Removing 0 from $\mathbb{R}^n$ gives $\mathbb{R}^n \setminus 0 \simeq \mathbb{S}^{n-1}$, and thus changes the homology in degree $n-1$. Using Mayer–Vietoris, we will show that in general removing a point from an n-manifold does not change its homology up to degree $n-2$:

Lemma 5.3.1. *For some $n \geq 2$, let M be an n-manifold and let $p \in M$. Then $\mathrm{H}_d(M \setminus p; \mathbf{k}) \cong \mathrm{H}_d(M; \mathbf{k})$ for $d \leq n-2$.*

Proof. It is enough to prove a similar statement in reduced homology. Let $A = M \setminus p$ and let $B \cong \mathrm{Int}(\mathbb{D}^n)$ be an open ball around p. Then A and B form an open cover of M with

$$A \cap B \cong \mathrm{Int}(\mathbb{D}^n) \setminus 0 \simeq \mathbb{S}^{n-1}.$$

For $d \leq n-2$, the Mayer-Vietoris sequence of this cover contains

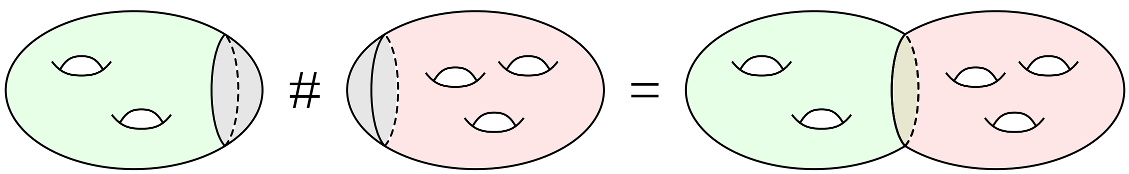
$$\widetilde{\mathrm{H}}_d(A \cap B; \mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_d(A; \mathbf{k}) \oplus \widetilde{\mathrm{H}}_d(B; \mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_d(M; \mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_{d-1}(A \cap B; \mathbf{k}).$$

Since $d \leq n-2$ and $A \cap B \simeq \mathbb{S}^{n-1}$, we have $\widetilde{\mathrm{H}}_d(A \cap B; \mathbf{k}) = 0$ and $\widetilde{\mathrm{H}}_{d-1}(A \cap B; \mathbf{k}) = 0$. It follows that

$$\widetilde{\mathrm{H}}_d(M; \mathbf{k}) \cong \widetilde{\mathrm{H}}_d(A; \mathbf{k}) \oplus \widetilde{\mathrm{H}}_d(B; \mathbf{k}) \cong \widetilde{\mathrm{H}}_d(M \setminus p; \mathbf{k}) \oplus \widetilde{\mathrm{H}}_d(\mathrm{Int}(\mathbb{D}^n); \mathbf{k}) \cong \widetilde{\mathrm{H}}_d(M \setminus p; \mathbf{k}). \qquad \square$$

5.4. Connect sums of manifolds

Let M_1 and M_2 be two smooth connected n-dimensional manifolds. Roughly speaking, the connect sum $M_1 \# M_2$ is obtained by removing open discs from M_1 and M_2 and gluing the resulting $\mathbb{S}^{n-1}$-boundary components together:



There are technical issues with making this rough description precise,[1] so we give a slightly different definition.

Definition 5.4.1. Let M_1 and M_2 be smooth connected n-dimensional manifolds, possibly with boundary. Let $p_i \in \mathrm{Int}(M_i)$ and let $f_i \colon \mathbb{R}^n \hookrightarrow \mathrm{Int}(M_i)$ be an embedding such that $f_i(0) = p_i$. If both M_1 and M_2 are oriented, then assume that f_1 preserves orientations and f_2 reverses orientations.[2] Then $M_1 \# M_2$ is the space obtained from $(M_1 \setminus p_1) \sqcup (M_2 \setminus p_2)$ by identifying[3]

$$f_1(v) \quad \text{with} \quad f_2(v/\|v\|^2) \quad \text{for } v \in \mathbb{R}^n \setminus 0. \qquad \square$$

As can be found in any book smooth manifolds, this is a well-defined smooth manifold that is orientable precisely when both M_1 and M_2 are orientable.[4] With this definition, we then have:

Theorem 5.4.2. *Let M_1 and M_2 be two smooth connected n-dimensional manifolds. Then for $1 \leq d \leq n-2$ we have*

$$\mathrm{H}_d(M_1 \# M_2; \mathbf{k}) = \mathrm{H}_d(M_1; \mathbf{k}) \oplus \mathrm{H}_2(M_2; \mathbf{k}).$$

Proof. Let $p_i \in \mathrm{Int}(M_i)$ and $f_i \colon \mathbb{R}^n \hookrightarrow \mathrm{Int}(M_i)$ be the points and maps used to define the connect sum. Let $A = M_1 \setminus p_1$ and $B = M_2 \setminus p_2$. We can identify these with open subsets of $M_1 \# M_2$ that cover $M_1 \# M_2$, and $A \cap B \cong \mathbb{R}^n \setminus 0 \simeq \mathbb{S}^{n-1}$. The Mayer–Vietoris sequence for this cover contains

$$\widetilde{\mathrm{H}}_d(A \cap B; \mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_d(A; \mathbf{k}) \oplus \widetilde{\mathrm{H}}_d(B; \mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_d(M_1 \# M_2; \mathbf{k}) \longrightarrow \widetilde{\mathrm{H}}_{d-1}(A \cap B; \mathbf{k}).$$

[1] Namely, how do you choose the gluing map? In high dimensions there is not a canonical choice. This is related to the existence of exotic smooth structures on spheres; indeed, every exotic $\mathbb{S}^n$ can be obtained by gluing two copies of $\mathbb{D}^n$ together along their boundary by a carefully chosen diffeomorphism.

[2] This ensures that $M_1 \# M_2$ is also oriented.

[3] Note that the $v/\|v\|^2$ term goes to ∞ as $v \mapsto 0$ and goes to 0 as $v \mapsto \infty$.

[4] The main point here is that the image of f_i is a tubular neighborhood of p_i, and oriented/non-oriented tubular neighborhoods are unique up to isotopy.

Since $A \cap B \simeq \mathbb{S}^{n-1}$, for $1 \le d \le n-2$ we have $\widetilde{\mathrm{H}}_d(A \cap B; \mathbf{k}) = \widetilde{\mathrm{H}}_{d-1}(A \cap B; \mathbf{k}) = 0$. We conclude that for these values of d we have

$$\mathrm{H}_d(M_1 \# M_2; \mathbf{k}) \cong \mathrm{H}_d(A; \mathbf{k}) \oplus \mathrm{H}_d(B; \mathbf{k}) = \mathrm{H}_d(M_1 \setminus p_1; \mathbf{k}) \oplus \mathrm{H}_d(M_2 \setminus p_2; \mathbf{k}) \cong \mathrm{H}_d(M_1; \mathbf{k}) \oplus \mathrm{H}_d(M_2; \mathbf{k}),$$

where the final isomorphism uses Lemma 5.3.1. □

EXAMPLE 5.4.3. Connect summing with $\mathbb{R}^n$ has the effect of deleting a point from an n-manifold. Thus

$$\mathbb{R}^n \# \mathbb{R}^n \cong \mathbb{R}^n \setminus 0 \simeq \mathbb{S}^{n-1} \quad \text{and} \quad \mathbb{S}^n \# \mathbb{R}^n \cong \mathbb{R}^n.$$

This shows that the condition $1 \le d \le n-2$ is needed in Theorem 5.4.2. Later we will prove that if M_1 and M_2 are closed orientable n-manifolds then we actually have

$$\mathrm{H}_d(M_1 \# M_2; \mathbf{k}) \cong \mathrm{H}_d(M_1; \mathbf{k}) \oplus \mathrm{H}_d(M_2; \mathbf{k}) \quad \text{for } 1 \le d \le n-1.$$

□

5.5. Smooth Jordan separation theorem

The Jordan separation theorem says that if $S \subset \mathbb{S}^n$ satisfies $S \cong \mathbb{S}^{n-1}$, then $\mathbb{S}^n \setminus S$ has two path components. We will prove this using Mayer–Vietoris in §6, but if S is smoothly embedded it has a very short proof. In fact, we can handle arbitrary $(n-1)$-dimensional submanifolds:[5]

THEOREM 5.5.1 (Smooth Jordan separation theorem). *For some $n \ge 1$, let $M^{n-1} \subset \mathbb{S}^n$ be a connected smoothly embedded $(n-1)$-dimensional submanifold. Then $\mathbb{S}^n \setminus M^{n-1}$ has two components.*

PROOF. This is trivial if $n = 1$, so we can assume that $n \ge 2$. to prove that[6] $\widetilde{\mathrm{H}}_0(\mathbb{S}^n \setminus M^{n-1}) = \mathbb{Z}$ (see Theorem 7.5.1). Let $A = \mathbb{S}^n \setminus M^{n-1}$ and let B be an open tubular neighborhood of M^{n-1}, so

$$B \cong M^{n-1} \times (-1, 1) \simeq M^{n-1}.$$

Then A and B are an open cover of $\mathbb{S}^n$, and

$$A \cap B \cong \left(M^{n-1} \times (-1,1)\right) \setminus M^{n-1} \times 0 \simeq M^{n-1} \sqcup M^{n-1}.$$

The associated Mayer–Vietoris exact sequence contains the segment

$$\widetilde{\mathrm{H}}_1(\mathbb{S}^n) \longrightarrow \widetilde{\mathrm{H}}_0(A \cap B) \longrightarrow \widetilde{\mathrm{H}}_0(A) \oplus \widetilde{\mathrm{H}}_0(B) \longrightarrow \widetilde{\mathrm{H}}_0(\mathbb{S}^n)$$

Since $\widetilde{\mathrm{H}}_1(\mathbb{S}^n)$ and $\widetilde{\mathrm{H}}_0(B)$ and $\widetilde{\mathrm{H}}_0(\mathbb{S}^k)$ vanish, we thus see that

$$\widetilde{\mathrm{H}}_0(A) \cong \widetilde{\mathrm{H}}_0(A \cap B) \cong \widetilde{\mathrm{H}}_0(M^{n-1} \sqcup M^{n-1} \cong \mathbb{Z}.$$

□

5.6. Exercises

EXERCISE 5.6.1. Prove Lemma 5.1.4 (the Barratt–Whitehead Lemma). □

EXERCISE 5.6.2. Prove Mayer–Vietoris for reduced homology: if X is a space and $A, B \subset X$ are subspaces such that $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$, then we have a long exact sequence

$$\cdots \longrightarrow \mathrm{H}_d(A \cap B; \mathbf{k}) \longrightarrow \mathrm{H}_d(A; \mathbf{k}) \oplus \mathrm{H}_d(B; \mathbf{k}) \longrightarrow \mathrm{H}_d(X; \mathbf{k}) \longrightarrow \mathrm{H}_{d-1}(A \cap B; \mathbf{k}) \longrightarrow \cdots.$$

Hint: derive it from the usual Mayer–Vietoris theorem. □

EXERCISE 5.6.3. There are several useful relative versions of the Mayer–Vietoris sequence. Prove the following by imitating our proof of the absolute version. You will need the long exact sequence of a triple (see Exercise 2.6.8).

(a) If X is a space and $A, B \subset X$ are subspaces such that $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$ and $C \subset A \cap B$, then we have a long exact sequence

$$\cdots \longrightarrow \mathrm{H}_d(A \cap B, C; \mathbf{k}) \longrightarrow \mathrm{H}_d(A, C; \mathbf{k}) \oplus \mathrm{H}_d(B, C; \mathbf{k}) \longrightarrow \mathrm{H}_d(X, C; \mathbf{k}) \longrightarrow \mathrm{H}_{d-1}(A \cap B, C; \mathbf{k}) \longrightarrow \cdots.$$

[5] Just like for spheres, it is not actually necessary that M^{n-1} is smoothly embedded. This is a consequence of the most general form of Alexander duality. We will prove a version of Alexander duality later as a consequence of Poincaré duality, but to avoid having to deal with delicate point-set issues we will not prove a version strong enough to eliminate smoothness here.

[6] Here again we are using our convention of omitting the coefficients from homology when they are $\mathbb{Z}$.

(b) If (X, Y) is a pair and $A, B \subset Y$ are subspaces such that $Y = \mathrm{Int}(A) \cup \mathrm{Int}(B)$, then we have a long exact sequence

$$\cdots \longrightarrow \mathrm{H}_d(X, A \cap B; \mathbf{k}) \longrightarrow \mathrm{H}_d(X, A; \mathbf{k}) \oplus \mathrm{H}_d(X, B; \mathbf{k}) \longrightarrow \mathrm{H}_d(X, Y; \mathbf{k}) \longrightarrow \mathrm{H}_{d-1}(X, A \cap B; \mathbf{k}) \longrightarrow \cdots .$$

We remark that there are more general relative version of Mayer–Vietoris as well, but they are more awkward to prove directly from the axioms of homology. □

CHAPTER 6

Classical geometric applications of homology

We have already given several classical applications of homology:

- the Brouwer Fixed Point theorem (Theorem 2.4.1); and
- the invariance of dimension (Corollary 2.5.4); and
- the smooth Jordan separation theorem (Theorem 5.5.1).

This chapter is devoted to some deeper applications that depend on more subtle uses of the Mayer-Vietoris exact sequence and the homology of spheres. In some sense it is a digression from our main text and a more theoretically minded reader might prefer to skip ahead, but these topics were among the original motivating examples for this theory. Fix an abelian group $\mathbf{k}$.

6.1. Removing a ball from a sphere

Removing a point from $\mathbb{S}^n$ yields the contractible space $\mathbb{R}^n$. This is not true if you thicken the point up: there exist embeddings $f\colon \mathbb{D}^m \to \mathbb{S}^n$ whose complement is not simply-connected.[1] However, the following shows that some vestige of contractibility remains:

Theorem 6.1.1. *For some $0 \leq m \leq n$, let $D \subset \mathbb{S}^n$ satisfy $D \cong \mathbb{D}^m$. Then $\widetilde{\mathrm{H}}_d(\mathbb{S}^n \setminus D; \mathbf{k}) = 0$ for all d.*

What makes this theorem difficult is that we are not assuming anything about how D is embedded in $\mathbb{S}^n$. Many natural assumptions about how D is embedded make the proof far easier. Here is an example:

Lemma 6.1.2. *For some $0 \leq m \leq n$, let $f\colon \mathbb{R}^n \hookrightarrow \mathbb{S}^n$ be an open embedding and let $D = f(\mathbb{D}^m)$, where*

$$\mathbb{D}^m = \mathbb{D}^m \times 0 \subset \mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n.$$

Then $\widetilde{\mathrm{H}}_d(\mathbb{S}^n \setminus D; \mathbf{k}) = 0$ for all d.

Proof. Set $X = \mathbb{S}^n \setminus \{f(0)\} \cong \mathbb{R}^n$. Let $A = \mathbb{S}^n \setminus D$ and $B = f(\mathbb{R}^n \setminus \{0\})$. The sets A and B are open sets that cover X, so we have a Mayer-Vietoris sequence

$$\widetilde{\mathrm{H}}_{d+1}(X; \mathbf{k}) \to \widetilde{\mathrm{H}}_d(A \cap B; \mathbf{k}) \to \widetilde{\mathrm{H}}_d(A; \mathbf{k}) \oplus \widetilde{\mathrm{H}}_d(B; \mathbf{k}) \to \widetilde{\mathrm{H}}_d(X; \mathbf{k}).$$

Since $X \cong \mathbb{R}^n$ is contractible, this gives an isomorphism

$$\widetilde{\mathrm{H}}_d(A \cap B; \mathbf{k}) \xrightarrow{\cong} \widetilde{\mathrm{H}}_d(A; \mathbf{k}) \oplus \widetilde{\mathrm{H}}_d(B; \mathbf{k}).$$

The space $B = f(\mathbb{R}^n \setminus \{0\})$ deformation retracts to $A \cap B = f(\mathbb{R}^n \setminus \mathbb{D}^m)$, so the map $\widetilde{\mathrm{H}}_d(A \cap B; \mathbf{k}) \to \widetilde{\mathrm{H}}_d(B; \mathbf{k})$ is an isomorphism. We conclude that $\widetilde{\mathrm{H}}_d(A; \mathbf{k}) = 0$, as desired. □

We now prove the general case of Theorem 6.1.1.

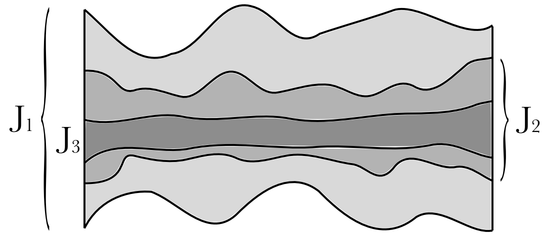
Proof of Theorem 6.1.1. The proof will be by induction on m. The base case $m = 0$ is trivial since $D \cong \mathbb{D}^0$ is a point and thus removing D from $\mathbb{S}^n$ yields the contractible space $\mathbb{R}^n$. Assume now that $m \geq 1$ and that the theorem is true for smaller m. Consider some $z \in \widetilde{\mathrm{H}}_d(\mathbb{S}^n \setminus D; \mathbf{k})$. Our goal is to prove that $z = 0$.

Assume that $z \neq 0$. Identify $\mathbb{D}^m$ with the cube I^m, and let $f\colon I^m \to D$ be a homeomorphism. For $J \subset I$, write $\mathbb{S}^n(J) = \mathbb{S}^n \setminus f(J \times I^{m-1})$. What we will do is construct a decreasing sequence

$$I = J_1 \supset J_2 \supset J_3 \supset \cdots$$

[1] For instance, the Alexander horned ball.

of closed intervals whose lengths go to 0 such that the image of z in $\widetilde{\mathrm{H}}_d(\mathbb{S}^n(J_i);\mathbf{k})$ is nonzero for all i:



This will imply the theorem. Indeed, since the J_i are compact subsets of I whose diameter goes to 0, their intersection is a single point:

$$\bigcap_{i=1}^{\infty} J_i = \{a\}.$$

We have

$$\mathbb{S}^n(J_1) \subset \mathbb{S}^n(J_2) \subset \mathbb{S}^n(J_3) \subset \cdots \quad \text{and} \quad \bigcup_{i=1}^{\infty} \mathbb{S}^n(J_i) = \mathbb{S}^n(a),$$

so by the continuity of homology we have

$$\varinjlim_i \widetilde{\mathrm{H}}_d(\mathbb{S}^n(J_i);\mathbf{k}) = \widetilde{\mathrm{H}}_d(\mathbb{S}^n(a);\mathbf{k}).$$

Since z maps to a nonzero element in each $\widetilde{\mathrm{H}}_d(\mathbb{S}^n(J_i);\mathbf{k})$, it maps to a nonzero element of $\widetilde{\mathrm{H}}_d(\mathbb{S}^n(a);\mathbf{k})$. But our inductive hypothesis says that

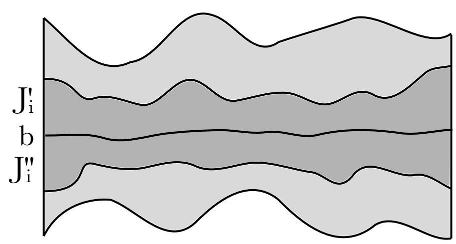
$$\widetilde{\mathrm{H}}_d(\mathbb{S}^n(a);\mathbf{k}) = \widetilde{\mathrm{H}}_d(\mathbb{S}^n \setminus f(a \times I^{m-1})) = 0,$$

contradiction.

It remains to construct the J_i. Set $J_1 = I$, and assume that we have constructed $J_i = [x, y]$ for some $i \geq 1$. Set $b = (x + y)/2$, and define

$$J_i' = [x, b] \quad \text{and} \quad J_i'' = [b, y].$$

See here:



The space $\mathbb{S}^n(b)$ is the union of the open sets $\mathbb{S}^n(J_i')$ and $\mathbb{S}^n(J_i'')$, whose intersection is $\mathbb{S}^n(J_i)$. The Mayer–Vietoris sequence of this cover takes the form

$$\widetilde{\mathrm{H}}_{d+1}(\mathbb{S}^n(b);\mathbf{k}) \to \widetilde{\mathrm{H}}_d(\mathbb{S}^n(J_i);\mathbf{k}) \to \widetilde{\mathrm{H}}_d(\mathbb{S}^n(J_i');\mathbf{k}) \oplus \widetilde{\mathrm{H}}_d(\mathbb{S}^n(J_i'');\mathbf{k}) \to \widetilde{\mathrm{H}}_d(\mathbb{S}^n(b);\mathbf{k}).$$

By our inductive hypothesis, both $\widetilde{\mathrm{H}}_{d+1}(\mathbb{S}^n(b);\mathbf{k})$ and $\widetilde{\mathrm{H}}_d(\mathbb{S}^n(b);\mathbf{k})$ vanish, so we conclude that the map

$$\widetilde{\mathrm{H}}_d(\mathbb{S}^n(J_i);\mathbf{k}) \to \widetilde{\mathrm{H}}_d(\mathbb{S}^n(J_i');\mathbf{k}) \oplus \widetilde{\mathrm{H}}_d(\mathbb{S}^n(J_i'');\mathbf{k})$$

is an isomorphism. The element z maps to a nonzero element of $\widetilde{\mathrm{H}}_d(\mathbb{S}^n(J_i);\mathbf{k})$, so it must map to a nonzero element of either $\widetilde{\mathrm{H}}_d(\mathbb{S}^n(J_i');\mathbf{k})$ or $\widetilde{\mathrm{H}}_d(\mathbb{S}^n(J_i'');\mathbf{k})$. We can therefore let J_{i+1} be either J_i' or J_i'', depending on which of these terms z maps nontrivially to. $\square$

6.2. Removing a sphere from a sphere, and the Jordan separation theorem

Our next theorem is as follows:

THEOREM 6.2.1. *For some $0 \leq m < n$, let $S \subset \mathbb{S}^n$ satisfy $S \cong \mathbb{S}^m$. Then*

$$\widetilde{\mathrm{H}}_d(\mathbb{S}^n \setminus S; \mathbf{k}) = \begin{cases} \mathbf{k} & \text{for } d = n - m - 1, \\ 0 & \text{otherwise.} \end{cases}$$

In the special case $m = n - 1$ and $\mathbf{k} = \mathbb{Z}$, we have $\widetilde{\mathrm{H}}_0(\mathbb{S}^n \setminus S) = \mathbb{Z}$. This implies that $\mathbb{S}^n \setminus S$ has two path components, and we deduce the following (c.f. Theorem 5.5.1):

COROLLARY 6.2.2 (Jordan separation theorem). *For some $n \geq 1$, let $S \subset \mathbb{S}^n$ satisfy $S \cong \mathbb{S}^{n-1}$. Then $\mathbb{S}^n \setminus S$ has two path components.*

PROOF OF THEOREM 6.2.1. The proof is by induction on m. For the base case $m = 0$, the subspace S satisfies $S \cong \mathbb{S}^0$, i.e., S has two points. It follows that $\mathbb{S}^n \setminus S \cong \mathbb{R}^n \setminus \{0\} \simeq \mathbb{S}^{n-1}$, whose reduced homology is nonzero precisely in dimension $n - 1$, as desired.

Assume now that $m \geq 1$ and that the theorem is true for smaller m. Let $f\colon \mathbb{S}^m \to S$ be a homeomorphism. Let $D_u \cong \mathbb{D}^m$ be the image in S of the closed upper hemisphere of $\mathbb{S}^m$, let $D_\ell \cong \mathbb{D}^m$ be the image of the closed lower hemisphere, and let $S' \subset S$ be the image of the equator $\mathbb{S}^{m-1} \subset \mathbb{S}^m$. We thus have $D_u \cap D_\ell = S'$. The space $\mathbb{S}^n \setminus S'$ is covered by the open sets $A = \mathbb{S}^n \setminus D_u$ and $B = \mathbb{S}^n \setminus D_\ell$, and $A \cap B = \mathbb{S}^n \setminus S$. The Mayer–Vietoris sequence thus contains the segment

$$\widetilde{\mathrm{H}}_{d+1}(A; \mathbf{k}) \oplus \widetilde{\mathrm{H}}_{d+1}(B; \mathbf{k}) \to \widetilde{\mathrm{H}}_{d+1}(\mathbb{S}^n \setminus S'; \mathbf{k}) \to \widetilde{\mathrm{H}}_d(\mathbb{S}^n \setminus S; \mathbf{k}) \to \widetilde{\mathrm{H}}_d(A; \mathbf{k}) \oplus \widetilde{\mathrm{H}}_d(B; \mathbf{k}).$$

Theorem 6.1.1 says that all the reduced homology groups of A and B vanish, so we get an isomorphism

$$\widetilde{\mathrm{H}}_{d+1}(\mathbb{S}^n \setminus S'; \mathbf{k}) \cong \widetilde{\mathrm{H}}_d(\mathbb{S}^n \setminus S; \mathbf{k})$$

Since $S' \cong \mathbb{S}^{m-1}$, our inductive hypothesis says that

$$\widetilde{\mathrm{H}}_{d+1}(\mathbb{S}^n \setminus S'; \mathbf{k}) = \begin{cases} \mathbf{k} & \text{if } d + 1 = n - (m-1) - 1, \\ 0 & \text{otherwise.} \end{cases}$$

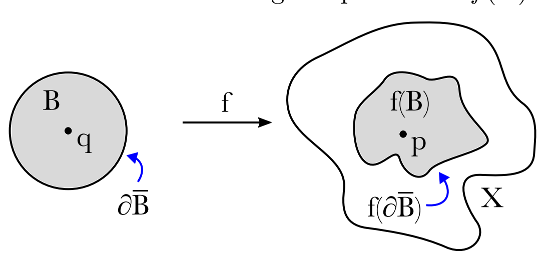
The theorem follows. □

6.3. Invariance of domain

The following theorem feels like it should be either trivial or false. Surprisingly, I am not aware of a proof that does not use something like the Jordan separation theorem.

THEOREM 6.3.1 (Invariance of domain). *For some $n \geq 1$, let $X \subset \mathbb{R}^n$ be a set such that there exists an open set $U \subset \mathbb{R}^n$ with $U \cong X$. Then X is an open subset of $\mathbb{R}^n$.*

PROOF. Embedding $\mathbb{R}^n$ into $\mathbb{S}^n$, we can assume that X is actually a subset of $\mathbb{S}^n$. Consider a point $p \in X$. Let $f\colon U \to X$ be a homeomorphism. Set $q = f^{-1}(p)$, and let $B \subset U$ be an open ball containing q whose closure $\overline{B}$ lies in U. It is enough to prove that $f(B)$ is open in $\mathbb{S}^n$:



Theorem 6.1.1 says that $\widetilde{\mathrm{H}}_0(\mathbb{S}^n \setminus f(\overline{B}); \mathbf{k}) = 0$, so $\mathbb{S}^n \setminus f(\overline{B})$ is path-connected. Since B is path-connected, its image $f(B)$ is also path-connected. Since $\partial\overline{B} \cong \mathbb{S}^{n-1}$, Corollary 6.2.2 (Jordan separation theorem) implies that $f(\partial\overline{B})$ separates $\mathbb{S}^n$ into two path components. We have

$$\mathbb{S}^n \setminus f(\partial\overline{B}) = \big(\mathbb{S}^n \setminus f(\overline{B})\big) \sqcup f(B).$$

Since both $\mathbb{S}^n \setminus f(\overline{B})$ and $f(B)$ are path-connected, these must be the two path components. Since $\mathbb{S}^n \setminus f(\partial\overline{B})$ is an open subset of the locally path connected space $\mathbb{S}^n$, its path components are also open. We conclude that $f(B)$ is open, as desired. □

This has the following curious corollary:

COROLLARY 6.3.2. *Let $f\colon M^n \to N^n$ be an embedding from a closed n-manifold M^n to a connected n-manifold N^n. Then f is a homeomorphism.*

This implies, for instance, the seemingly trivial fact that you cannot embed a closed n-manifold into $\mathbb{R}^n$.

PROOF OF COROLLARY 6.3.2. It is enough to prove that f is surjective. For this, since M^n is connected it is enough to prove that $f(M^n)$ is both open and closed. It is closed since M^n is compact, so the essential thing to prove is that it is open. For this, consider a point $p \in f(M^n)$. Set $q = f^{-1}(p)$, and choose a neighborhood W of q that is homeomorphic to an open set $U \subset \mathbb{R}^n$. Since f is a homeomorphism, $f(W)$ is also homeomorphic to $U \subset \mathbb{R}^n$, so by Theorem 6.3.1 the set $f(W)$ is an open neighborhood of q in N^n. The corollary follows. □

6.4. Commutative division algebras

Am *algebra* over a field $\mathbb{F}$ is an $\mathbb{F}$-vector space A equipped with a bilinear multiplication map $A \times A \to A$. We write this multiplicatively: for $a, b \in A$, the product of a and b is $a \cdot b$. Here are some standard properties that an algebra A might have:

- it is *associative* if $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in A$.
- it is *commutative* if $a \cdot b = b \cdot a$ for all $a, b \in A$.
- it *unital* if there is some $\mathbb{1} \in A$ called the *unit* with $\mathbb{1} \cdot a = a \cdot \mathbb{1} = a$ for all $a \in A$.

A deeper property is:

- it is a *division algebra* for all nonzero $a \in A$ and all $b \in A$, there exists a unique $c \in A$ with $a \cdot c = b$ and a unique $d \in A$ with $d \cdot a = b$.

This condition can be rephrased as saying that for all nonzero $a \in A$, the maps $A \to A$ given by left- and right-multiplication by a are bijective. If A is finite-dimensional over $\mathbb{F}$, then this is equivalent to them being injective, i.e., to A having no zero-divisors.

Finite-dimensional associative unital division algebras over $\mathbb{F}$ can be classified using algebraic tools. For instance, if they are also commutative then they are the same as finite field extensions of $\mathbb{F}$. As an example of this, $\mathbb{Q}[\sqrt{2}]$ is an associative commutative unital division algebra over $\mathbb{Q}$. It is also easy to see that the only associative commutative unital division algebras over $\mathbb{R}$ are $\mathbb{R}$ and $\mathbb{C}$. More generally, a classical theorem of Frobenius says that the only associative unital division algebra over $\mathbb{R}$ that is non-commutative is the algebra of quaternions.

These algebraic approaches all require associativity, and classifying non-associative division algebras over $\mathbb{R}$ seems to require topology. The ultimate theorem in this direction is a deep theorem of Kervaire and Milnor saying that the only finite-dimensional unital division algebras over $\mathbb{R}$ are:

- the commutative associative algebras $\mathbb{R}$ and $\mathbb{C}$; and
- the associative algebra of quaternions; and
- the non-associative algebra of octonions.

We will prove a special case of this that goes back to Hopf:

THEOREM 6.4.1. *Let A be a finite-dimensional commutative unital division algebra over $\mathbb{R}$. Then A is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. In particular, A is associative.*

PROOF. If $\dim_{\mathbb{R}}(A) = 1$, then $A = \mathbb{R}$ and there is nothing to prove. We divide the rest of the proof into two cases:

CASE 1. $\dim_{\mathbb{R}}(A) = 2$.

Let $\mathbb{1} \in A$ be the unit. Pick $a \in A$ such that $\{\mathbb{1}, a\}$ is a basis for A over $\mathbb{R}$. We can then write $a^2 = \lambda\,\mathbb{1} + \lambda' a$ for some $\lambda, \lambda' \in \mathbb{R}$. For $\kappa \in \mathbb{R}$, we have

$$(a + \kappa\,\mathbb{1})^2 = a^2 + 2\kappa a + \kappa^2\,\mathbb{1} = (\lambda\,\mathbb{1} + \lambda' a) + 2\kappa a + \kappa^2\,\mathbb{1} = (\lambda' + 2\kappa)a + (\lambda + \kappa^2)\,\mathbb{1}.$$

Replacing a with $A - \frac{\lambda'}{2}\mathbb{1}$, we can therefore ensure that $a^2 = \lambda\mathbb{1}$ for some $\lambda \in \mathbb{R}$.

If $\lambda > 0$, then we can write

$$0 = a^2 - \lambda\mathbb{1} = (a + \sqrt{\lambda}\mathbb{1})(a - \sqrt{\lambda}\mathbb{1}).$$

Since A has no zero divisors, we thus have $a = \pm\sqrt{\lambda}\mathbb{1}$, contradicting the fact that $\{\mathbb{1}, a\}$ are linearly independent. We thus have $\lambda < 0$. Replacing a with $\frac{1}{\sqrt{|\lambda|}}a$, we have $a^2 = -\mathbb{1}$. In other words, A has a basis $\{\mathbb{1}, a\}$ with $a^2 = -\mathbb{1}$, i.e., $A \cong \mathbb{C}$.

CASE 2. $\dim_{\mathbb{R}}(A) \geq 3$.

We will show that this case cannot actually happen. Assume it does. Identify A with $\mathbb{R}^n$ for some $n \geq 3$, and let $\|\cdot\|$ be the standard norm on $A = \mathbb{R}^n$. Define $f\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ via the formula

$$f(x) = \frac{x^2}{\|x^2\|} \quad \text{for all } x \in \mathbb{S}^{n-1} \subset A.$$

This makes sense since A has no zero-divisors, so $x^2 \neq 0$. Note that for $x \in \mathbb{S}^{n-1}$ we have

$$f(-x) = \frac{(-x)^2}{\|(-x)^2\|} = \frac{x^2}{\|x^2\|} = f(x).$$

It follows that f factors through a map $\overline{f}\colon \mathbb{RP}^{n-1} \to \mathbb{S}^{n-1}$.

We claim that $\overline{f}$ is injective. Indeed, consider $x, y \in \mathbb{S}^{n-1}$ with $f(x) = f(y)$. We must prove that $x = \pm y$. We have

$$\frac{x^2}{\|x^2\|} = \frac{y^2}{\|y^2\|} \quad \text{and hence} \quad x^2 = \frac{\|x^2\|}{\|y^2\|}y^2 = \lambda y^2 \quad \text{with } \lambda = \frac{\|x^2\|}{\|y^2\|} \text{ positive.}$$

This implies that

$$0 = x^2 - \lambda y^2 = (x + \sqrt{\lambda}y)(x - \sqrt{\lambda}y).$$

Since A has no zero divisors, we deduce that that $x = \pm\sqrt{\lambda}y$. Since x and y both lie on $\mathbb{S}^{n-1}$, this implies that $x = \pm y$, as desired.

Since $\overline{f}\colon \mathbb{RP}^{n-1} \to \mathbb{S}^{n-1}$ is injective, we can appeal to Corollary 6.3.2 to deduce that $\overline{f}$ is a homeomorphism. This is a contradiction;[2] for instance, since $n \geq 3$ we have $\pi_1(\mathbb{RP}^{n-1}) = \mathbb{Z}/2$. ☐

6.5. Exercises

[2] Note that it is not a contradiction for $n = 2$ since $\mathbb{RP}^1 \cong \mathbb{S}^1$.

CHAPTER 7

One-dimensional CW complexes and the zeroth homology group

We now begin several chapters that culminate in a complete description of the homology of CW complexes. This chapter starts with some basic properties of the homology groups of CW complexes. These are applied to calculate the homology groups of 1-dimensional CW complexes, and YYY

7.1. The dimension of CW complexes

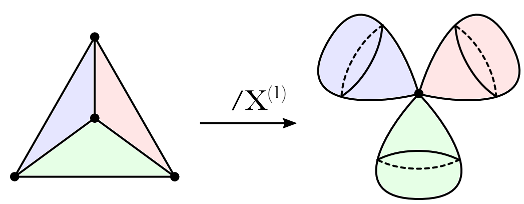
Recall that a CW complex X is at most n-dimensional if it equals its n-skeleton, i.e., $X = X^{(n)}$. The following shows that the homology of such a CW complex vanishes in dimensions greater than n:

Theorem 7.1.1. *Let X be a CW complex of dimension at most n. Then $\mathrm{H}_d(X;\mathbf{k}) = 0$ for $d \geq n+1$.*

Proof. The proof is by induction on n. The base case $n = 0$ is trivial since in this case X is a discrete space. Assume, therefore, that $n \geq 1$ and that the theorem is true for CW complexes of dimension at most $(n-1)$. Consider some $d \geq n+1$. The long exact sequence of the pair $(X, X^{(n-1)})$ contains segments of the form

$$\mathrm{H}_d(X^{(n-1)};\mathbf{k}) \longrightarrow \mathrm{H}_d(X;\mathbf{k}) \longrightarrow \mathrm{H}_d(X, X^{(n-1)};\mathbf{k})$$

Our inductive hypothesis says that $\mathrm{H}_d(X^{(n-1)};\mathbf{k}) = 0$. Also, since $X = X^{(n)}$ the quotient space $X/X^{(n-1)}$ is a wedge of n-spheres, one for each n-cell of X:



Since $d \geq n+1$, this together with Lemma 4.2.1 implies that

$$\mathrm{H}_d(X, X^{(n-1)};\mathbf{k}) \cong \widetilde{\mathrm{H}}_d(X/X^{(n-1)};\mathbf{k}) = 0.$$

The theorem follows. □

7.2. Homology carried on the skeleton

The idea of the proof of Theorem 7.1.1 above contains the germ of the idea for how we will analyze CW complexes in general. Here is another application of that idea:

Lemma 7.2.1. *Let X be a CW complex. Then the map $\mathrm{H}_d(X^{(n)};\mathbf{k}) \to \mathrm{H}_d(X;\mathbf{k})$ is an isomorphism for $d < n$ and a surjection for $d = n$.*

Proof. To simplify our notation, let $X^{(-1)} = \emptyset$. By the continuity axiom, the map

$$\varinjlim_n \mathrm{H}_d(X^{(n)};\mathbf{k}) \to \mathrm{H}_d(X;\mathbf{k})$$

is an isomorphism for all d. It follows that it is enough to prove that the map $\mathrm{H}_d(X^{(n)};\mathbf{k}) \to \mathrm{H}_d(X^{(n+1)};\mathbf{k})$ is an isomorphism for $d < n$ and a surjection for $d = n$.

The long exact sequence of the pair $(X^{(n+1)}, X^{(n)})$ contains the segment

$$\mathrm{H}_{d+1}(X^{(n+1)}, X^{(n)}; \mathbf{k}) \longrightarrow \mathrm{H}_d(X^{(n)}; \mathbf{k}) \longrightarrow \mathrm{H}_d(X^{(n+1)}; \mathbf{k}) \longrightarrow \mathrm{H}_d(X^{(n+1)}, X^{(n)}; \mathbf{k}).$$

To prove the lemma, it is therefore enough to prove that $\mathrm{H}_k(X^{(n+1)}, X^{(n)}; \mathbf{k}) = 0$ for $k \leq n$. We have

$$\mathrm{H}_k(X^{(n+1)}, X^{(n)}; \mathbf{k}) \cong \widetilde{\mathrm{H}}_k(X^{(n+1)}/X^{(n)}; \mathbf{k}).$$

Just like in the proof of Theorem 7.1.1 above, the quotient space $X^{(n+1)}/X^{(n)}$ is a wedge of $(n+1)$-spheres, one for each $(n+1)$-cell of $X^{(n+1)}$. It therefore follows from Lemma 4.2.1 that $\widetilde{\mathrm{H}}_k(X^{(n+1)}/X^{(n)}; \mathbf{k}) = 0$ for $k \leq n$, as desired. □

7.3. Representing elements of first homology

We now focus attention on homology with coefficients in $\mathbb{Z}$. This requires:

NOTATION 7.3.1. For each $d \geq 0$, fix a generator $[\mathbb{S}^d]$ for $\widetilde{\mathrm{H}}_d(\mathbb{S}^d) \cong \mathbb{Z}$. There are thus two choices for $[\mathbb{S}^d]$ corresponding to $1 \in \mathbb{Z}$ and $-1 \in \mathbb{Z}$, and for the purpose of this chapter it is not not important which choice is made. See Chapter **??** for a broader context for this. □

For a space X, one easy way to give an element of $\mathrm{H}_1(X)$ is to give a continuous map $f\colon \mathbb{S}^1 \to X$, yielding $f_*([\mathbb{S}^1]) \in \mathrm{H}_1(X)$. This only depends on the homotopy class of f. The following lemma shows that it also does not depend on the parameterization:

LEMMA 7.3.2. *Let X be a space and let $f\colon \mathbb{S}^1 \to X$ be a continuous map. For all orientation-preserving homeomorphisms $g\colon \mathbb{S}^1 \to \mathbb{S}^1$, we have $f_*([\mathbb{S}^1]) = (f \circ g)([\mathbb{S}^1])$.*

PROOF. It is enough to prove that g is homotopic to the identity. Let $\rho\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z} = \mathbb{S}^1$ be the universal cover and let $* = \rho(0) \in \mathbb{S}^1$. Homotoping g by composing it with rotations of $\mathbb{S}^1$, we can ensure that $g(*) = *$. We can then lift g to a map $\widetilde{g}\colon \mathbb{R} \to \mathbb{R}$ satisfying $\widetilde{g}(0) = 0$ and $\widetilde{g}(x+1) = \widetilde{g}(x) + 1$ for all $x \in \mathbb{R}$. For $0 \leq t \leq 1$, define $\widetilde{g}_t\colon \mathbb{R} \to \mathbb{R}$ via

$$\widetilde{g}_t(x) = (1-t)\widetilde{g}(x) + tx \quad \text{for } x \in \mathbb{R}.$$

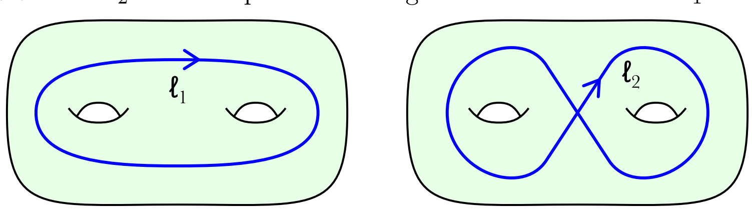
Since

$$\widetilde{g}_t(x+1) = (1-t)\widetilde{g}(x+1) + t(x+1) = (1-t)(\widetilde{g}(x)+1) + t(x+1) = \widetilde{g}_t(x) + 1,$$

this descends to a homotopy $g_t\colon \mathbb{S}^1 \to \mathbb{S}^1$ from $g_0 = g$ to $g_1 = \mathbb{1}_{\mathbb{S}^1}$. □

Using this lemma, we can specify elements of $\mathrm{H}_1(X)$ by drawing oriented but unparameterized circles ℓ in X. Letting $f\colon \mathbb{S}^1 \to \ell$ be an orientation-preserving parametization, we can then define $[\ell] = f_*([\mathbb{S}^1])$.

EXAMPLE 7.3.3. Let Σ_2 be a compact oriented genus 2 surface and let ℓ_1 and ℓ_2 be:

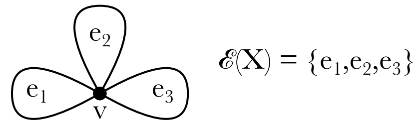


We then have $[\ell_1], [\ell_2] \in \mathrm{H}_1(\Sigma_2)$. □

7.4. One-dimensional CW complexes

Let X be a connected 1-dimensional CW complex, i.e., a connected graph. Our goal is to describe $\mathrm{H}_d(X)$.

7.4.1. One vertex. The easiest case is where X has one 0-cell v. Each 1-cell is thus a loop based at v. Let $\mathcal{E}(X)$ be the set of 1-cells of X, so X is a wedge of circles, one for each $e \in \mathcal{E}(X)$:



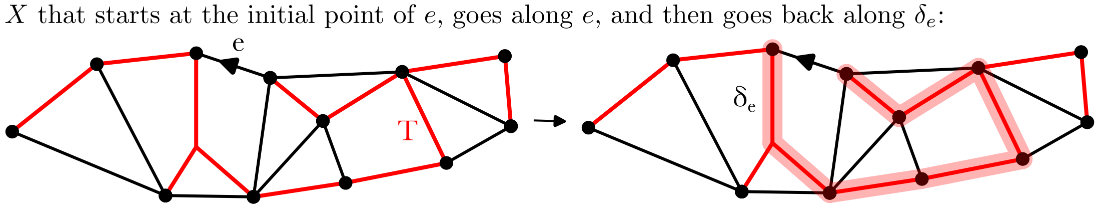
We thus have

$$\widetilde{\mathrm{H}}_d(X) \cong \widetilde{\mathrm{H}}_d(\bigvee_{e\in\mathcal{E}(X)} \mathbb{S}^1) \cong \bigoplus_{e\in\mathcal{E}(X)} \widetilde{\mathrm{H}}_d(\mathbb{S}^1) = \begin{cases} \bigoplus_{e\in\mathcal{E}(X)} \mathbb{Z} & \text{if } d=1, \\ 0 & \text{otherwise.} \end{cases}$$

Orient each e in an arbitrary way,[1] and regard e as an oriented loop in X. We thus have $[e] \in \mathrm{H}_1(X)$. Recall that for a set S, we denote by $\mathbb{Z}\langle S\rangle$ the free abelian group of formal $\mathbb{Z}$-linear combinations of elements of S. We then have $\mathrm{H}_1(X) \cong \mathbb{Z}\langle\mathcal{E}(X)\rangle$, where $e \in \mathcal{E}(X)$ corresponds to $[e] \in \mathrm{H}_1(X)$.

7.4.2. Maximal trees and loops. Now assume that X is an arbitrary connected 1-dimensional CW complex. Our identification of its homology is less canonical than the case where there is only one 0-cell, and depends on the choice of a maximal tree T in X, i.e., a connected subgraph of X with no cycles that contains every vertex of X.

Let $\mathcal{E}(X,T)$ be the set of 1-cells of X that do not lie in T. Orient each $e \in \mathcal{E}(X,T)$ in an arbitrary way, and let δ_e be a path in T from the endpoint of e to the initial point of e. Since T is contractible, this is unique up to homotopies through such paths. Define $\gamma_{T,e}$ to be the oriented loop in X that starts at the initial point of e, goes along e, and then goes back along δ_e:



Here the tree is red and the edges of $\mathcal{E}(X,T)$ are black. The element $[\gamma_{T,e}] \in \mathrm{H}_1(X)$ depends only on T and e.

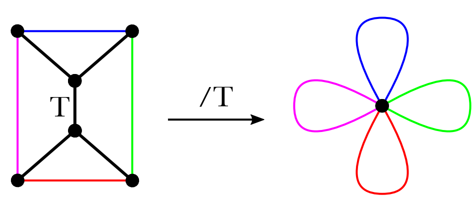
7.4.3. Homology of graph. With the above notation, we have:

THEOREM 7.4.1. *Let X be a connected 1-dimensional CW complex and let T be a maximal tree in X. Then*

$$\widetilde{\mathrm{H}}_d(X) \cong \begin{cases} \mathbb{Z}\langle\mathcal{E}(X,T)\rangle & \textit{if } d=1, \\ 0 & \textit{otherwise,} \end{cases}$$

where $e \in \mathcal{E}(X,T)$ corresponds to $[\gamma_{T,e}] \in \widetilde{\mathrm{H}}_1(X)$.

PROOF. Since T is contractible, Corollary 4.1.8 implies that $\widetilde{\mathrm{H}}_d(X) \cong \widetilde{\mathrm{H}}_d(X/T)$ for all d. The space X/T is a wedge of circles, one for each edge of $\mathcal{E}(X,T)$. See here, where the tree T is black and the edges of $\mathcal{E}(X,T)$ are in other colors:



By §7.4.1, we have

$$\widetilde{\mathrm{H}}_d(X/T) \cong \begin{cases} \mathbb{Z}\langle\mathcal{E}(X,T)\rangle & \text{if } d=1, \\ 0 & \text{otherwise,} \end{cases}$$

[1]There is actually a natural orientation since e is a 1-cell $\mathbb{D}^1_e$ attached via an attaching map $f_e\colon \partial\mathbb{D}^1_e \to X^{(0)}$, so e goes from $f_e(-1)$ to $f_e(1)$.

The isomorphism $\mathrm{H}_1(X) \to \mathrm{H}_1(X/T)$ takes $[\gamma_{T,e}]$ to the loop corresponding to e. The theorem follows. □

7.5. Zeroth homology

We are finally in a position to calculate $\mathrm{H}_0(X)$ when X has the homotopy type of a CW complex:

THEOREM 7.5.1. *Let X have the homotopy type of a CW complex and let I be the set of path components of X. Then $\mathrm{H}_0(X) \cong \mathbb{Z}^I$. In particular, if X is path-connected then $\widetilde{\mathrm{H}}_0(X) = 0$.*

PROOF. By homotopy invariance and the additivity axiom, it is enough to prove that $\widetilde{\mathrm{H}}_0(X) = 0$ if X is a path-connected CW complex. To do this, by Lemma 7.2.1 it is enough to prove that $\widetilde{\mathrm{H}}_0(X^{(1)}) = 0$. Since $X^{(1)}$ is a connected graph, this follows from Theorem 7.4.1. □

7.6. Exercises

CHAPTER 8

Two-dimensional CW complexes and the first homology group

In this chapter, we explain how to calculate $\mathrm{H}_d(X)$ for a CW complex of dimension at most 2. We will later generalize this approach to handle arbitrary CW complexes and arbitrary coefficient systems. As a byproduct of our work, we will prove that $\mathrm{H}_1(X)$ is isomorphic to the abelianization of $\pi_1(X)$ for an arbitrary connected CW complex.

8.1. Two-dimensional CW complexes

Consider a connected two-dimensional CW complex X. We showed in §8.3 how to calculate the homology groups of $X^{(1)}$, and in this section we describe how the homology changes when we attach the 2-cells to form X.

8.1.1. Main result. Let $\mathrm{C}_2^{\mathrm{cell}}(X)$ be the free abelian group consisting of formal $\mathbb{Z}$-linear combinations of 2-cells of X. For each 2-cell $\mathbb{D}_i^2$, let $f_i\colon \partial\mathbb{D}_i^2 \to X^{(1)}$ be its attaching map. Since $\partial\mathbb{D}_i^2 = \mathbb{S}^1$, we can define a map $\mathbf{b}\colon \mathrm{C}_2^{\mathrm{cell}}(X) \to \mathrm{H}_1(X^{(1)})$ via the formula

$$\mathbf{b}(\mathbb{D}_i^2) = (f_i)_*([\mathbb{S}^1]) \in \mathrm{H}_1(X^{(1)}) \quad \text{for each 2-cell } \mathbb{D}_i^2.$$

With this notation, we then have the following. There are three examples after the proof.

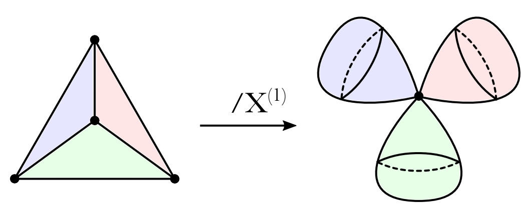
Theorem 8.1.1. *Let X be a connected 2-dimensional CW complex and let $\mathbf{b}\colon \mathrm{C}_2^{cell}(X) \to \mathrm{H}_1(X^{(1)})$ be as above. Then we have*

$$\widetilde{\mathrm{H}}_d(X) = \begin{cases} \ker(\mathbf{b}) & \text{if } d = 2, \\ \operatorname{coker}(\mathbf{b}) & \text{if } d = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since X is 2-dimensional, Theorem 7.1.1 implies that its homology groups vanish above degree 2, so we only need to calculate $\mathrm{H}_d(X)$ for $d \leq 2$. The long exact sequence of the pair $(X, X^{(1)})$ contains the segment

$$\mathrm{H}_2(X^{(1)}) \longrightarrow \mathrm{H}_2(X) \longrightarrow \mathrm{H}_2(X, X^{(1)}) \longrightarrow \mathrm{H}_1(X^{(1)}) \longrightarrow \mathrm{H}_1(X) \longrightarrow \mathrm{H}_1(X, X^{(1)}).$$

By Theorem 7.4.1, we have $\mathrm{H}_2(X^{(1)}) = 0$. Since $X = X^{(2)}$, the space $X/X^{(1)}$ is a wedge of 2-spheres, one for each 2-cell of X:



Let $[\mathbb{D}^n] \in \mathrm{H}_n(\mathbb{D}^n, \partial\mathbb{D}^n)$ be the element mapping to $[\mathbb{S}^{n-1}]$ under the isomorphism $\mathrm{H}_n(\mathbb{D}^n, \partial\mathbb{D}^n) \cong \widetilde{\mathrm{H}}_{n-1}(\partial\mathbb{D}^n)$ coming from the long exact sequence of the pair $(\mathbb{D}^n, \partial\mathbb{D}^n)$. We then have that

$$\mathrm{H}_d(X, X^{(1)}) \cong \widetilde{\mathrm{H}}_d(X/X^{(1)}) \cong \begin{cases} \mathrm{C}_2^{\mathrm{cell}}(X) & \text{if } d = 2, \\ 0 & \text{otherwise,} \end{cases}$$

where for a 2-cell $\mathbb{D}_i^2$ of X the element of $\widetilde{\mathrm{H}}_d(X, X^{(1)})$ corresponding to $\mathbb{D}_i^2 \in \mathrm{C}_2^{\mathrm{cell}}(X)$ is the image of $[\mathbb{D}^2]$ under the map of pairs $\phi_i \colon (\mathbb{D}_i^2, \partial\mathbb{D}_i^2) \to (X, X^{(1)})$ that attaches $\mathbb{D}_i^2$ to $X^{(1)}$. Plugging all of this into our exact sequence, we get

$$0 \longrightarrow \mathrm{H}_2(X) \longrightarrow \mathrm{C}_2^{\mathrm{cell}}(X) \xrightarrow{\mathbf{b}'} \mathrm{H}_1(X^{(1)}) \longrightarrow \mathrm{H}_1(X) \longrightarrow 0.$$

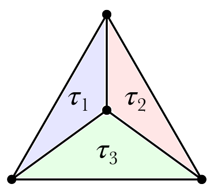
This will prove the theorem once we verify that the map $\mathbf{b}' \colon \mathrm{C}_2^{\mathrm{cell}}(X) \to \mathrm{H}_1(X^{(1)})$ in the above exact sequence is $\mathbf{b}$. For this, note that the restriction of $\phi_i \colon (\mathbb{D}_i^2, \partial\mathbb{D}_i^2) \to (X, X^{(1)})$ to $\partial\mathbb{D}_i^2$ is the attaching map $f_i \colon \partial\mathbb{D}_i^2 \to X^{(1)}$. We thus have $\mathbf{b}(\mathbb{D}_i^2) = (f_i)_*([\mathbb{S}^1])$. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{H}_2(\mathbb{D}_i^2, \partial\mathbb{D}_i^2) & \xrightarrow{\quad\cong\quad} & \mathrm{H}_1(\partial\mathbb{D}_i^2) \\ \downarrow{\scriptstyle (\phi_i)_*} & & \downarrow{\scriptstyle (f_i)_*} \\ \mathrm{H}_2(X, X^{(1)} \xrightarrow{\cong} \mathrm{C}_2^{\mathrm{cell}}(X) & \xrightarrow{\mathbf{b}'} & \mathrm{H}_1(X^{(1)}) \end{array}$$

The map $(\phi_i)_*$ takes $[\mathbb{D}_i^2]$ to the element of $\mathrm{H}_2(X, X^{(1)})$ corresponding to $\mathbb{D}_i^2 \in \mathrm{C}_2^{\mathrm{cell}}(X)$. It follows that $\mathbf{b}'(\mathbb{D}_i^2) = (f_i)_*([\mathbb{S}^1])$, as desired. □

8.1.2. Examples. To understand the previous theorem, we give two examples.

EXAMPLE 8.1.2. Let X be the following contractible 2-complex:

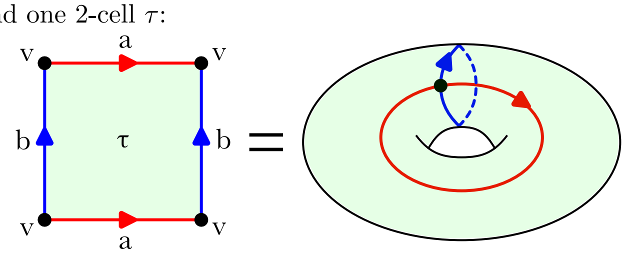


Our theorem gives an exact sequence

$$0 \longrightarrow \mathrm{H}_2(X) \longrightarrow \mathbb{Z}\langle \tau_1, \tau_2, \tau_3 \rangle \xrightarrow{\mathbf{b}} \mathrm{H}_1(X^{(1)}) \longrightarrow \mathrm{H}_1(X) \longrightarrow 0.$$

By Theorem 7.4.1, the map $\mathbf{b}$ takes the τ_i to a basis for $\mathrm{H}_1(X^{(1)})$. It follows that $\mathbf{b}$ is an isomorphism, and thus that $\mathrm{H}_2(X) = \mathrm{H}_1(X) = 0$. Of course, we already knew this since X is contractible. □

EXAMPLE 8.1.3. Consider the torus T^2. Give T^2 the CW complex structure with one 0-cell v, two 1-cells a and b, and one 2-cell τ:



Since the 1-skeleton of T^2 is a wedge of two circles, we can identify $\mathrm{H}_1((T^2)^{(1)})$ with the abelian group $\mathbb{Z}\langle a, b \rangle \cong \mathbb{Z}^2$ of formal $\mathbb{Z}$-linear combinations of a and b. We also have $\mathrm{C}_2^{\mathrm{cell}}(T^2) = \mathbb{Z}\langle \tau \rangle \cong \mathbb{Z}$. Our theorem therefore says that there is an exact sequence

$$0 \longrightarrow \mathrm{H}_2(T^2) \longrightarrow \mathbb{Z}\langle \tau \rangle \xrightarrow{\mathbf{b}} \mathbb{Z}\langle a, b \rangle \longrightarrow \mathrm{H}_1(T^2) \longrightarrow 0.$$

To determine the answer here, we must calculate $\mathbf{b}(\tau)$. We will explain how to do this in the next two sections, where we will show that $\mathbf{b}(\tau) = 0$ and thus that $\mathrm{H}_2(T^2) \cong \mathbb{Z}$ and $\mathrm{H}_1(T^2) \cong \mathbb{Z}^2$. □

8.2. The Hurewicz map

As the examples in the previous section show, for a 1-dimensional CW complex Y and a map $f \colon \mathbb{S}^1 \to Y$ we need a way to determine the image of the map $f_* \colon \mathrm{H}_1(\mathbb{S}^1) \to \mathrm{H}_1(Y)$ in terms of the generators for $\mathrm{H}_1(Y)$ given by Theorem 7.4.1. The map f gives a loop in Y, and we will actually prove something more general that will allow us to relate $\pi_1(X, p)$ and $\mathrm{H}_1(X)$ for arbitrary based CW complexes (X, p).

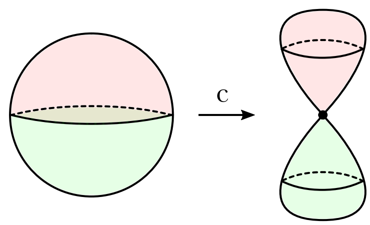
8.2.1. Homotopy groups. To put this into its proper context, we will define this relationship not just on $\pi_1(X,p)$, but on $\pi_d(X,p)$ for arbitrary $d \geq 1$. Let $* \in \mathbb{S}^d$ be a basepoint lying in the equator $\mathbb{S}^{d-1} \subset \mathbb{S}^n$. Recall that the d^{th} homotopy group of a based space (X,p) is the set $\pi_d(X,p)$ of homotopy classes of basepoint-preserving maps $(\mathbb{S}^d,*) \to (X,p)$:

$$\pi_d(X,p) = [(\mathbb{S}^d,*),(X,p)].$$

For $f\colon (\mathbb{S}^d,*) \to (X,p)$, we denote by $[f]$ the associated element of $\pi_d(X,p)$. The set $\pi_d(X,p)$ is a group. For $[f],[g] \in \pi_1(X,p)$, the element $[f]\cdot[g]$ is the homotopy class of the map

$$\mathbb{S}^d \xrightarrow{c} \mathbb{S}^d \vee \mathbb{S}^d \xrightarrow{f\vee g} X,$$

where c is the map that pinches the equator $\mathbb{S}^d$ to a point:



This is abelian if $d \geq 2$, but is typically nonabelian for $d = 1$.

8.2.2. Hurewicz map. For $d \geq 1$ and a based space (X,p), we define the *Hurewicz map* to be the set map $\mathfrak{h}\colon \pi_d(X,p) \to \mathrm{H}_d(X)$ defined via the formula

$$\mathfrak{h}([f]) = f_*([\mathbb{S}^d]) \quad \text{for } f\colon (\mathbb{S}^d,*) \to (X,p).$$

This is a homomorphism:

Lemma 8.2.1. *Let (X,p) be a based space. For all $d \geq 1$, the Hurewicz map $\mathfrak{h}\colon \pi_d(X,p) \to \mathrm{H}_d(X)$ is a group homomorphism, i.e.,*

$$\mathfrak{h}([f]\cdot[g]) = \mathfrak{h}([f]) + \mathfrak{h}([g]) \quad \textit{for all } f,g\colon (\mathbb{S}^d,*) \to (X,p).$$

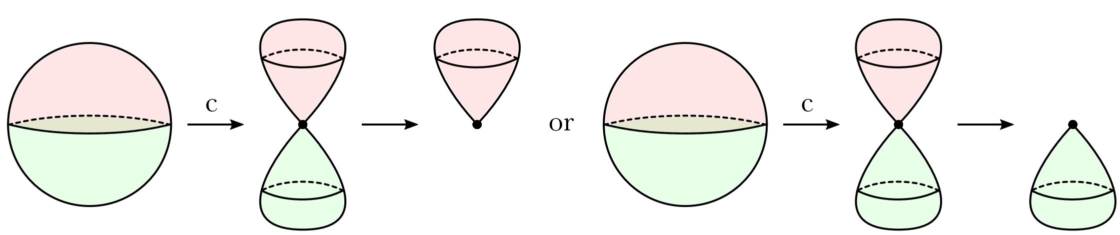
Proof. Consider $f,g\colon (\mathbb{S}^d,*) \to (X,p)$. The homology class $\mathfrak{h}([f]\cdot[g])$ is the image of $[\mathbb{S}^d]$ under the map

$$\mathrm{H}_d(\mathbb{S}^d) \xrightarrow{c_*} \mathrm{H}_d(\mathbb{S}^d \vee \mathbb{S}^d) \xrightarrow{(f\vee g)_*} \mathrm{H}_d(X),$$

where $c\colon \mathbb{S}^d \to \mathbb{S}^d \vee \mathbb{S}^d$ is the collapse map. Identifying $\mathrm{H}_d(\mathbb{S}^d \vee \mathbb{S}^d)$ with $\mathrm{H}_d(\mathbb{S}^d) \oplus \mathrm{H}_d(\mathbb{S}^d)$, this is the homomorphism

$$\mathrm{H}_d(\mathbb{S}^d) \xrightarrow{c_*} \mathrm{H}_d(\mathbb{S}^d) \oplus \mathrm{H}_d(\mathbb{S}^d) \xrightarrow{f_*+g_*} \mathrm{H}_d(X).$$

It is therefore enough to prove that $c_*([\mathbb{S}^d]) = ([\mathbb{S}^d],[\mathbb{S}^d])$. This follows from the fact that composition of c with either of the two projection maps $\mathbb{S}^d \vee \mathbb{S}^d \to \mathbb{S}^d$ induces the identity on homology. Indeed, these compositions are simply the result of collapsing either the upper or lower hemisphere of $\mathbb{S}^d$ to points:



These maps are actually homotopic to the identity via the homotopy that collapses less and less of the relevant hemisphere. □

EXAMPLE 8.2.2. Let $\mathbb{1}: \mathbb{S}^d \to \mathbb{S}^d$ be the identity map. The Hurewicz map $\mathfrak{h}: \pi_d(\mathbb{S}^d, *) \to \mathrm{H}_d(\mathbb{S}^d)$ takes $[\mathbb{1}] \in \pi_d(\mathbb{S}^d, *)$ to $[\mathbb{S}^d] \in \mathrm{H}_d(\mathbb{S}^d)$, which generates $\mathrm{H}_d(\mathbb{S}^d) = \mathbb{Z}$. In fact, as we noted long ago using degree theory one can show that $\pi_d(\mathbb{S}^d, *) = \mathbb{Z}$, and $\mathfrak{h}$ is an isomorphism. □

8.3. The first homology group of CW complexes

Since $\pi_1(X, p)$ is not abelian, the Hurewicz map $\pi_1(X, p) \to \mathrm{H}_1(X)$ cannot be an isomorphism in general. The following theorem says that this non-abelianness the only issue:

THEOREM 8.3.1 (Hurewicz for fundamental group). *Let X be a path-connected space that has the homotopy type of a CW complex. Then for each $p \in X$ the Hurewicz map $\mathfrak{h}: \pi_1(X, p) \to \mathrm{H}_1(X)$ induces an isomorphism*

$$(\pi_1(X, p))^{ab} \cong \mathrm{H}_1(X).$$

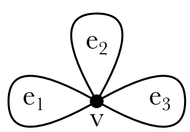
PROOF. We can assume without loss of generality that X is a CW complex and that $p \in X^{(0)}$. Lemma 7.2.1 says that the inclusion $X^{(2)} \hookrightarrow X$ induces an isomorphism $\mathrm{H}_1(X^{(2)}) \cong \mathrm{H}_1(X)$, and when we discussed how to calculate the fundamental group of a CW complex we also showed that the induced map $\pi_1(X^{(2)}, p) \to \pi_1(X, p)$ is an isomorphism. These isomorphisms are compatible with the Hurewicz map in the sense that the diagram

$$\begin{array}{ccc} \pi_1(X^{(2)}, p) & \xrightarrow{\cong} & \pi_1(X, p) \\ \downarrow{\scriptstyle \mathfrak{h}} & & \downarrow{\scriptstyle \mathfrak{h}} \\ \mathrm{H}_1(X^{(2)}) & \xrightarrow{\cong} & \mathrm{H}_1(X) \end{array}$$

commutes. We can assume therefore without loss of generality that X is 2-dimensional. We now divide the proof into two steps.

STEP 1. *The CW complex X satisfies $X = X^{(1)}$.*

Since X is connected, its 1-skeleton $X^{(1)}$ is a connected graph. Let T be a maximal tree in $X^{(1)}$. Since T is contractible, Corollary 4.1.8 implies that the map $\mathrm{H}_1(X) \to \mathrm{H}_1(X/T)$ is an isomorphism. Letting $*$ be the basepoint of X/T, the map $\pi_1(X, p) \to \pi_1(X/T, *)$ is also an isomorphism. These isomorphisms are compatible with the Hurewicz map just like above, so without loss of generality we can replace X with X/T and assume that X has only one 0-cell p. In other words, letting $\mathcal{E}(X)$ be the set of 1-cells of X the space X is a wedge of circles, one for each $e \in \mathcal{E}(X)$:



$$\mathscr{E}(X) = \{e_1, e_2, e_3\}$$

The group $\pi_1(X, p)$ is isomorphic to the free group $F(\mathcal{E}(X))$ on the set $\mathcal{E}(X)$, with $e \in \mathcal{E}(X)$ corresponding to the loop $\gamma_e: \mathbb{S}^1 \to X$ around the circle corresponding to e. As we discussed in §7.4.1, we have $\mathrm{H}_1(X) \cong \mathbb{Z}\langle \mathcal{E}(X) \rangle$. Under these identifications, the Hurewicz map $\mathfrak{h}: \pi_1(X, p) \to \mathrm{H}_1(X)$ is the evident map $F(\mathcal{E}(X)) \to \mathbb{Z}\langle \mathcal{E}(X) \rangle$, which is indeed the abelianization map.

STEP 2. *The CW complex X satisfies $X = X^{(2)}$.*

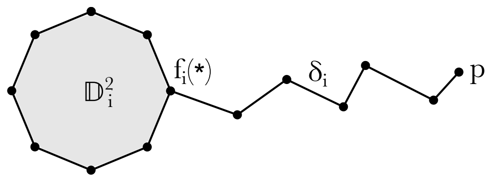
Let $\{\mathbb{D}_i^2\}_{i \in I}$ be the set of 2-cells of X. For $i \in I$, let $f_i: \partial \mathbb{D}_i^2 \to X^{(1)}$ be the attaching map of $\mathbb{D}_i^2$ and let $\overline{\mathbb{D}}_i^2$ be the image of $\mathbb{D}_i^2$ in X. By Theorem 8.1.1, the map $\mathrm{H}_1(X^{(1)}) \to \mathrm{H}_1(X)$ is surjective and its kernel is generated by elements of the form $(f_i)_*([\mathbb{S}^1])$.

By the Seifert–van Kampen theorem, attaching 2-cells to a space imposes relations on its fundamental group, so the map $\pi_1(X^{(1)}, p) \to \pi_1(X, p)$ is also surjective. We therefore have a commutative diagram

$$\begin{array}{ccccc} \pi_1(X^{(1)}, p) & \twoheadrightarrow & (\pi_1(X^{(1)}, p))^{\mathrm{ab}} & \xrightarrow{\cong} & \mathrm{H}_1(X^{(1)}) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_1(X, p) & \twoheadrightarrow & (\pi_1(X, p))^{\mathrm{ab}} & \longrightarrow & \mathrm{H}_1(X) \end{array}$$

whose horizontal rows are the Hurewicz maps. To prove that the Hurewicz map induces an isomorphism $(\pi_1(X,p))^{\mathrm{ab}} \cong \mathrm{H}_1(X)$, it is therefore enough to prove that the preimage of each $(f_i)_*([\mathbb{S}^1]) \in \mathrm{H}_1(X^{(1)})$ in $(\pi_1(X^{(1)},p))^{\mathrm{ab}}$ maps to 0 in $(\pi_1(X,p))^{\mathrm{ab}}$.

Identifying $\partial\mathbb{D}^2_i$ with $\mathbb{S}^1$, the loop $f_i\colon \mathbb{S}^1 \to X^{(1)}$ does not define an element of $\pi_1(X^{(1)},p)$ since it is not a based loop, i.e., it does not take the basepoint $* \in \mathbb{S}^1$ to p. Consider a path δ_i in $X^{(1)}$ from p to $f_i(*)$. We then have an element γ_{i,δ_i} of $\pi_1(X^{(1)},p)$ that goes along δ_i from p to $f_i(1)$, then goes around f_i, and then goes back to p along δ_i in the reverse direction:

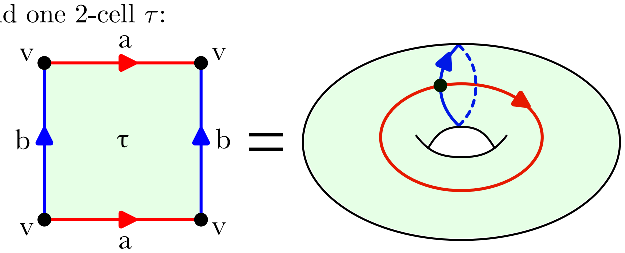


Attaching $\mathbb{D}^2_i$ to $X^{(1)}$ via f_i kills the element of $\pi_1(X^{(1)},p)$ corresponding to γ_{i,δ_i}. This maps to $(f_i)_*([\mathbb{S}^1]) \in \mathrm{H}_1(X^{(1)})$, and the theorem follows. □

8.4. Closed surfaces

We now revisit the second example from after Theorem 8.1.1 and explain how to complete the calculations in it.

Example 8.4.1. Consider the torus T^2. Give T^2 the CW complex structure with one 0-cell v, two 1-cells a and b, and one 2-cell τ:



Since the 1-skeleton of T^2 is a wedge of two circles, we can identify $\mathrm{H}_1((T^2)^{(1)})$ with the abelian group $\mathbb{Z}\langle a,b\rangle \cong \mathbb{Z}^2$. Theorem 8.1.1 gives an exact sequence

$$0 \longrightarrow \mathrm{H}_2(T^2) \longrightarrow \mathbb{Z}\langle\tau\rangle \xrightarrow{\mathbf{b}} \mathbb{Z}\langle a,b\rangle \longrightarrow \mathrm{H}_1(T^2) \longrightarrow 0.$$

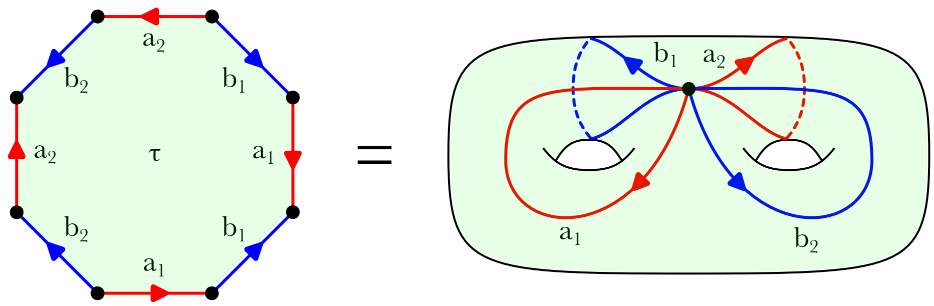
Here $\mathbf{b}(\tau)$ is the image of $[\mathbb{S}^1]$ under the attaching map $f\colon \partial\mathbb{D}^2 \to (T^2)^{(1)}$ of τ. This attaching map corresponds to the element $aba^{-1}b^{-1} \in \pi_1((T^2)^{(1)},v)$, and under the Hurewicz map this goes to

$$a + b - a - b = 0$$

in $\mathrm{H}_1((T^2)^{(1)}) = \mathbb{Z}\langle a,b\rangle$. We conclude that $\mathbf{b}(\tau) = 0$, so $\mathrm{H}_2(T^2) \cong \mathbb{Z}$ and $\mathrm{H}_1(T^2) \cong \mathbb{Z}^2$. Both a and b are oriented loops in T^2, and $[a]$ and $[b]$ form a basis for $\mathrm{H}_1(T^2)$. □

This example can be generalized to handle any 2-dimensional CW complex. We give the details for closed surfaces.

Example 8.4.2. Consider a closed oriented genus-g surface Σ_g. We can construct Σ_g by taking a $4g$-gon and identifying its sides together in pairs. For instance, Σ_2 is:



This gives a CW complex structure on Σ_g. In the general case, it has:

- one 0-cell v; and
- 1-cells $\{a_1, b_1, \ldots, a_g, b_g\}$, each going from v to v; and
- one 2-cell τ, attached to the 1-skeleton by identifying τ to a $4g$-gon and gluing its edges to the 1-skeleton according to the pattern

$$(a_1 b_1 a_1^{-1} b_1^{-1}) \cdots (a_g b_g a_g^{-1} b_g^{-1}), \tag{8.4.1}$$

just like in the above figure.

Since the 1-skeleton of Σ_g is a wedge of $2g$ circles, we can identity $\mathrm{H}_1((\Sigma_g)^{(1)})$ with the abelian group $\mathbb{Z}\langle a_1, b_1, \ldots, a_g, b_g\rangle$. Theorem 8.1.1 gives an exact sequence

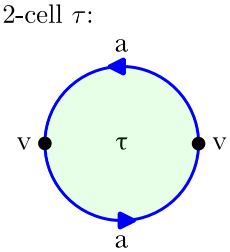
$$0 \longrightarrow \mathrm{H}_2(\Sigma_g) \longrightarrow \mathbb{Z}\langle\tau\rangle \xrightarrow{\mathbf{b}} \mathbb{Z}\langle a_1, b_1, \ldots, a_g, b_g\rangle \longrightarrow \mathrm{H}_1(\Sigma_g) \longrightarrow 0.$$

Here $\mathbf{b}(\tau)$ is the image of $[\mathbb{S}^1]$ under the attaching map $f\colon \partial\mathbb{D}^2 \to (\Sigma_g)^{(1)}$ of τ. This attaching map corresponds to the element (8.4.1) in $\pi_1((\Sigma_g)^{(1)}, v)$ and under the Hurewicz map this goes to

$$(a_1 + b_1 - a_1 - b_1) + \cdots + (a_g + b_g - a_g - b_g) = 0$$

in $\mathrm{H}_1((\Sigma_g)^{(1)})$. We conclude that $\mathbf{b}(\tau) = 0$, so $\mathrm{H}_2(\Sigma_g) \cong \mathbb{Z}$ and $\mathrm{H}_1(\Sigma_g) \cong \mathbb{Z}^{2g}$. The a_i and b_i are oriented loops on Σ_g, and the $[a_i]$ and $[b_i]$ form a basis for $\mathrm{H}_1(T^2)$. □

EXAMPLE 8.4.3. Consider the real projective plan $\mathbb{RP}^2$. Give $\mathbb{RP}^2$ the CW complex structure with one 0-cell v, one 1-cell a, and one 2-cell τ:



Since the 1-skeleton of $\mathbb{RP}^2$ is a circle, we can identify $\mathrm{H}_1(\mathbb{RP}^2)$ with the abelian group $\mathbb{Z}\langle a\rangle \cong \mathbb{Z}$. Theorem 8.1.1 gives an exact sequence

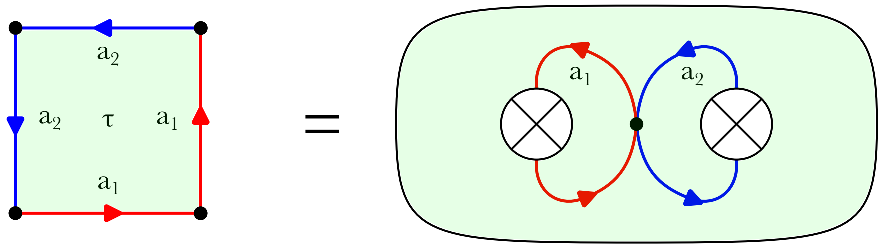
$$0 \longrightarrow \mathrm{H}_2(\mathbb{RP}^2) \longrightarrow \mathbb{Z}\langle\tau\rangle \xrightarrow{\mathbf{b}} \mathbb{Z}\langle a\rangle \longrightarrow \mathrm{H}_1(\mathbb{RP}^2) \longrightarrow 0.$$

Here $\mathbf{b}(\tau)$ is the image of $[\mathbb{S}^1]$ under the attaching map $f\colon \partial\mathbb{D}^2 \to (\mathbb{RP}^2)^{(1)}$ of τ. This attaching map corresponds to the element $aa \in \pi_1((\mathbb{RP}^2)^{(1)}, v)$, and under the Hurewicz map this goes to

$$a + a = 2a$$

in $\mathrm{H}_1((\mathbb{RP}^2)^{(1)} = \mathbb{Z}\langle a\rangle$. We conclude that $\mathbf{b}(\tau) = 2a$, so $\mathrm{H}_2(\mathbb{RP}^2) = 0$ and $\mathrm{H}_1(\mathbb{RP}^2) \cong \mathbb{Z}/2$. Regarding a as an oriented loop in $\mathbb{RP}^2$, the element $[a]$ generates $\mathrm{H}_1(\mathbb{RP}^2)$. □

EXAMPLE 8.4.4. Consider the connect sum S_r of r copies of $\mathbb{RP}^2$. We can construct S_r by taking a $2r$-gon and identifying its sides together in pairs. For instance, S_2 is:



Here the ×'s are *crosscaps*: you remove the indicated discs, and then identify antipodal points in the boundary. The loops passing through the crosscaps are orientation-reversing loops, and as you pass through the crosscap, your notions of left and right are reversed. This gives a CW complex structure on S_r. In the general case, it has:

- one 0-cell v; and
- 1-cells $\{a_1, \dots, a_r\}$, each going from v to v; and
- one 2-cell τ, attached to the 1-skeleton by identifying τ to a $2r$-gon and gluing its edges to the 1-skeleton according to the pattern

$$a_1 a_1 a_2 a_2 \cdots a_r a_r \tag{8.4.2}$$

just like in the above figure.

Since the 1-skeleton of S_r is a wedge of r circles, we can identity $\mathrm{H}_1((S_r)^{(1)})$ with the abelian group $\mathbb{Z}\langle a_1, \dots, a_r\rangle$. Theorem 8.1.1 gives an exact sequence

$$0 \longrightarrow \mathrm{H}_2(S_r) \longrightarrow \mathbb{Z}\langle\tau\rangle \xrightarrow{\mathbf{b}} \mathbb{Z}\langle a_1, \dots, a_r\rangle \longrightarrow \mathrm{H}_1(S_r) \longrightarrow 0.$$

Here $\mathbf{b}(\tau)$ is the image of $[\mathbb{S}^1]$ under the attaching map $f\colon \partial\mathbb{D}^2 \to (S_r)^{(1)}$ of τ. This attaching map corresponds to the element (8.4.2) in $\pi_1((S_r)^{(1)}, v)$ and under the Hurewicz map this goes to

$$(a_1 + a_1) + \cdots + (a_r + a_r) = 2(a_1 + \cdots + a_r)$$

in $\mathrm{H}_1((S_r)^{(1)})$. We conclude that $\mathbf{b}$ is injective, so $\mathrm{H}_2(S_r) = 0$ and

$$\mathrm{H}_1(S_r) \cong \frac{\mathbb{Z}\langle a_1, \dots, a_r\rangle}{\mathbb{Z}\langle 2(a_1 + \cdots + a_r)\rangle} \cong \mathbb{Z}/2 \oplus \mathbb{Z}^{r-1}.$$

The a_i are oriented loops on S_r, and the $[a_i]$ span $\mathrm{H}_1(S_r)$ and satisfy the single relation

$$2\left([a_1] + \cdots + [a_r]\right) = 0.$$ □

8.5. The higher-dimensional Hurewicz theorem

We now briefly discuss the higher-dimensional Hurewicz maps. Recall that a space X is n-connected if for all $d \le n$, every map $\mathbb{S}^d \to X$ extends to a map $\mathbb{D}^{d+1} \to X$. For $n = -1$ this means that X is nonempty, for $n = 0$ it means that X is nonempty and path-connected, and for $n \ge 1$ it means that X is nonempty, path-connected, and has $\pi_d(X, p) = 0$ for all $p \in X$. For the higher homotopy groups, the Hurewicz theorem is as follows:

Theorem 8.5.1 (Hurewicz theorem). *For some $n \ge 2$, let X be an $(n-1)$-connected space that has the homotopy type of a CW complex. Then for each $p \in X$ the Hurewicz map $\mathfrak{h}\colon \pi_n(X, p) \to \mathrm{H}_n(X)$ is an isomorphism.*

Proving this requires developing a bit more homotopy theory than we have at this point, so we postpone it until a later chapter. The following corollary gives one way the Hurewicz theorem can be used:

Corollary 8.5.2. *Let (X, p) be a path-connected based space that has the homotopy type of a CW complex and let $(\widetilde{X}, \widetilde{p}) \to (X, p)$ be the universal cover of X. Then $\pi_2(X, p) \cong \mathrm{H}_2(\widetilde{X})$.*

Proof. Passing to covers does not change π_2, so since $\widetilde{X}$ is 1-connected we can apply Theorem 8.5.1 and see that $\pi_2(X, p) \cong \pi_2(\widetilde{X}, \widetilde{p}) \cong \mathrm{H}_2(\widetilde{X})$. □

Remark 8.5.3. The above idea generalizes to give a powerful tool for computing homotopy groups of spaces that goes back to Serre's PhD thesis. Consider a space X with a universal cover $\widetilde{X}$. The homotopy groups of X are computed inductively, starting at π_1. What you do is construct a sequence of spaces

$$X = X[0], \widetilde{X} = X[1], X[2], \ldots$$

where $X[d]$ is a generalization of the universal cover called the *d-connected cover*.[1] It has the feature that $X[d]$ is d-connected, but has the same homotopy groups as X starting in degree $d+1$. For a basepoint $p \in X$, we can then use Theorem 8.5.1 to see that $\pi_{d+1}(X,p) \cong \mathrm{H}_d(X[d];\mathbb{Z})$. The homology groups of $X[d]$ can be calculated from $X[d-1]$ inductively using a tool called the Serre spectral sequence. The computations are not trivial, but they are remarkably effective. Most modern tools for computing homotopy groups are generalizations of this method. □

8.6. Exercises

[1]For $d \geq 2$, this is not an actual cover but something more complicated.

CHAPTER 9

CW complexes and cellular homology

We now how to generalize what we did in the previous chapter to calculate the homology of an arbitrary CW complex.

9.1. The cellular chain complex

Let X be a CW complex and let $\mathbf{k}$ be an abelian group. Let $\mathrm{C}_d^{\mathrm{cell}}(X;\mathbf{k})$ be the abelian group consisting of formal $\mathbf{k}$-linear combinations of d-cells of X. As is our usual practice, we will omit the $\mathbf{k}$ if it is $\mathbb{Z}$. We prove:

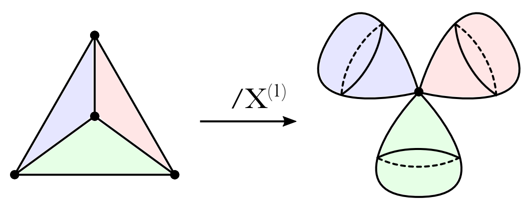
Theorem 9.1.1. *Let X be a CW complex and let $\mathbf{k}$ be an abelian group. There are then boundary maps*

$$\partial\colon\ \mathrm{C}_d^{\mathrm{cell}}(X;\mathbf{k}) \to \mathrm{C}_{d-1}^{\mathrm{cell}}(X;\mathbf{k})$$

making $(\mathrm{C}_\bullet^{\mathrm{cell}}(X;\mathbf{k}),\partial)$ into a chain complex with

$$\mathrm{H}_d(\mathrm{C}_\bullet^{\mathrm{cell}}(X;\mathbf{k})) \cong \mathrm{H}_d(X;\mathbf{k}) \quad \textit{for all } d.$$

Proof. To simplify our notation in this proof, we will omit the coefficients $\mathbf{k}$ from our notation (even when they are not $\mathbb{Z}$). As we have noted several times already, the quotient $X^{(d)}/X^{(d-1)}$ is a wedge of d-spheres, one for each d-cell of X:



Using this, we can identify

$$\mathrm{H}_d(X^{(d)},X^{(d-1)}) \cong \widetilde{\mathrm{H}}_d(X^{(d)}/X^{(d-1)}) \cong \mathrm{C}_d^{\mathrm{cell}}(X).$$

Making this identification, define ∂ to be the composition

$$\mathrm{H}_d(X^{(d)},X^{(d-1)}) \longrightarrow \mathrm{H}_{d-1}(X^{(d-1)}) \longrightarrow \mathrm{H}_{d-1}(X^{(d-1)},X^{(d-2)}).$$

This makes $\mathrm{C}_\bullet^{\mathrm{cell}}(X)$ into a chain complex since the chain of maps making up ∂^2 contains

$$\mathrm{H}_{d-1}(X^{(d-1)}) \longrightarrow \mathrm{H}_{d-1}(X^{(d-1)},X^{(d-2)}) \longrightarrow \mathrm{H}_{d-2}(X^{(d-2)}),$$

which vanishes since it forms part of the long exact sequence of the pair $(X^{(d-1)},X^{(d-2)})$.

We must now prove that $\mathrm{C}_\bullet^{\mathrm{cell}}(X)$ computes $\mathrm{H}_d(X)$. Define

$$\mathrm{Z}_d^{\mathrm{cell}}(X) = \ker\left(\mathrm{H}_d(X^{(d)},X^{(d-1)}) \xrightarrow{\partial} \mathrm{H}_{d-1}(X^{(d-1)},X^{(d-2)})\right),$$
$$\mathrm{B}_d^{\mathrm{cell}}(X) = \mathrm{Im}\left(\mathrm{H}_{d+1}(X^{(d+1)},X^{(d)}) \xrightarrow{\partial} \mathrm{H}_d(X^{(d)},X^{(d-1)})\right).$$

Our goal is to construct an isomorphism

$$\mathrm{Z}_d^{\mathrm{cell}}(X)/\,\mathrm{B}_d^{\mathrm{cell}}(X) \cong \mathrm{H}_d(X).$$

The first key observation is:

Claim 1. $\mathrm{Z}_d^{\mathrm{cell}}(X) \cong \mathrm{H}_d(X^{(d)})$.

Proof of claim. Let us give names to the maps making up ∂:

$$\mathrm{H}_d(X^{(d)}, X^{(d-1)}) \xrightarrow{\partial'} \mathrm{H}_{d-1}(X^{(d-1)}) \xrightarrow{\partial''} \mathrm{H}_{d-1}(X^{(d-1)}, X^{(d-2)}).$$

Since $X^{(d-2)}$ is $(d-2)$-dimensional, its homology vanishes above degree $(d-2)$ (see Theorem 7.1.1). In particular, it vanishes in degree $d-1$, so from the long exact sequence of the pair $(X^{(d-1)}, X^{(d-2)})$ we see that the map ∂'' above is injective. It follows that

$$\mathrm{Z}_d^{\mathrm{cell}}(X) = \ker\left(\mathrm{H}_d(X^{(d)}, X^{(d-1)}) \xrightarrow{\partial'} \mathrm{H}_{d-1}(X^{(d-1)})\right).$$

Now consider the long exact sequence of the pair $(X^{(d)}, X^{(d-1)})$:

$$\mathrm{H}_d(X^{(d-1)}) \longrightarrow \mathrm{H}_d(X^{(d)}) \longrightarrow \mathrm{H}_d(X^{(d)}, X^{(d-1)}) \xrightarrow{\partial'} \mathrm{H}_{d-1}(X^{(d-1)}).$$

Again, since $X^{(d-1)}$ is $(d-1)$-dimensional its homology vanishes in degree d. We conclude that the above can be rewritten as

$$0 \longrightarrow \mathrm{H}_d(X^{(d)}) \longrightarrow \mathrm{H}_d(X^{(d)}, X^{(d-1)}) \xrightarrow{\partial'} \mathrm{H}_{d-1}(X^{(d-1)}).$$

This implies that indeed we have $\mathrm{Z}_d^{\mathrm{cell}}(X) = \ker(\partial') \cong \mathrm{H}_d(X^{(d)})$. □

Note that in the proof of the above claim what we showed was that $\mathrm{Z}_d^{\mathrm{cell}}(X)$ equals the image of the map $\mathrm{H}_d(X^{(d)}) \to \mathrm{H}_d(X^{(d)}, X^{(d-1)})$, which is injective. During our subsequent calculations, we will silently identify $\mathrm{Z}_d^{\mathrm{cell}}(X)$ with $\mathrm{H}_d(X^{(d)})$ via this identification.

Recall that we are trying to construct an isomorphism from $\mathrm{Z}_d^{\mathrm{cell}}(X)/\,\mathrm{B}_d^{\mathrm{cell}}(X)$ to $\mathrm{H}_d(X)$. The above claim gives us a natural map

$$\mathrm{Z}_d^{\mathrm{cell}}(X) = \mathrm{H}_d(X^{(d)}) \longrightarrow \mathrm{H}_d(X)$$

that we will prove induces this isomorphism. In Lemma 7.2.1, we proved that the d^{th} homology group of X is "carried" on the d-skeleton in the sense that the map $\mathrm{H}_d(X^{(d)}) \to \mathrm{H}_d(X)$ is surjective. To prove the theorem, therefore, it is enough to prove:

Claim 2. *The kernel of the map* $\mathrm{H}_d(X^{(d)}) \to \mathrm{H}_d(X)$ *is* $\mathrm{B}_d^{\mathrm{cell}}(X)$.

Proof of claim. Lemma 7.2.1 not only says that d^{th} homology group of X is "carried" on the d-skeleton, but also that all the relations appear in the $(d+1)$-skeleton in the sense that the map $\mathrm{H}_d(X^{(d+1)}) \to \mathrm{H}_d(X)$ is an isomorphism. It is enough, therefore, to prove that the kernel of the map $\mathrm{H}_d(X^{(d)}) \to \mathrm{H}_d(X^{(d+1)})$ is $\mathrm{B}_d^{\mathrm{cell}}(X)$. Using the long exact sequence of the pair $(X^{(d+1)}, X^{(d)})$, this is equivalent to showing that $\mathrm{B}_d^{\mathrm{cell}}(X)$ is the image of the map

$$\mathrm{H}_{d+1}(X^{(d+1)}, X^{(d)}) \to \mathrm{H}_d(X^{(d)}).$$

For this, recall that $\mathrm{B}_d^{\mathrm{cell}}(X)$ is the image of the differential

$$\mathrm{H}_{d+1}(X^{(d+1)}, X^{(d)}) \longrightarrow \mathrm{H}_d(X^{(d)}) \longrightarrow \mathrm{H}_d(X^{(d)}, X^{(d-1)}).$$

Here as we said before the claim the map $\mathrm{H}_d(X^{(d)}) \to \mathrm{H}_d(X^{(d)}, X^{(d-1)})$ is injective and we are identifying $\mathrm{H}_d(X^{(d)})$ with its image, so this implies the claim. □

This completes the proof of the theorem. □

9.2. Boundary maps for integer coefficients

9.3. Boundary maps for arbitrary coefficients

9.4. Functoriality

9.5. Mayer–Vietoris for CW complexes

One nice feature of cellular homology is that it allows an easy proof of a version of Mayer–Vietoris that applies to covers of CW complexes by subcomplexes (as opposed to open sets):

THEOREM 9.5.1 (Mayer–Vietoris). *Let X be a CW complex and let $A, B \subset X$ be subcomplexes such that $X = A \cup B$. Then for all commutative rings $\mathbf{k}$, we have a long exact sequence*

$$\cdots \to \mathrm{H}_d(A \cap B; \mathbf{k}) \to \mathrm{H}_d(A; \mathbf{k}) \oplus \mathrm{H}_d(B; \mathbf{k}) \to \mathrm{H}_d(X; \mathbf{k}) \to \mathrm{H}_{d-1}(A \cap B; \mathbf{k}) \to \cdots$$

PROOF. Recall that $\mathrm{C}_d^{\mathrm{cell}}(X; \mathbf{k})$ is the free $\mathbf{k}$-module on the d-cells of X. Since every cell of X lies in A or B, the evident map

$$\mathrm{C}_d^{\mathrm{cell}}(A; \mathbf{k}) \oplus \mathrm{C}_d^{\mathrm{cell}}(B; \mathbf{k}) \to \mathrm{C}_d^{\mathrm{cell}}(X; \mathbf{k})$$

is surjective. The kernel of this map is generated by elements of the form $(\sigma, -\sigma)$ for σ a d-cell of $A \cap B$, and thus is isomorphic to $\mathrm{C}_d^{\mathrm{cell}}(A \cap B; \mathbf{k})$. The resulting short exact sequences of $\mathbf{k}$-modules commute with the differentials in the cellular chain complex, and thus assemble into a short exact sequence

$$0 \longrightarrow \mathrm{C}_\bullet^{\mathrm{cell}}(A \cap B; \mathbf{k}) \longrightarrow \mathrm{C}_\bullet^{\mathrm{cell}}(A; \mathbf{k}) \oplus \mathrm{C}_\bullet^{\mathrm{cell}}(B; \mathbf{k}) \longrightarrow \mathrm{C}_d(X; \mathbf{k}) \longrightarrow 0$$

of chain complexes. Applying the snake lemma, we obtain the desired long exact sequence. □

REMARK 9.5.2. If in the setting of Theorem 9.5.1 we have $A \cap B \neq \emptyset$, then we also get a Mayer–Vietoris sequence in reduced homology:

$$\cdots \to \widetilde{\mathrm{H}}_d(A \cap B; \mathbf{k}) \to \widetilde{\mathrm{H}}_d(A; \mathbf{k}) \oplus \widetilde{\mathrm{H}}_d(B; \mathbf{k}) \to \widetilde{\mathrm{H}}_d(X; \mathbf{k}) \to \widetilde{\mathrm{H}}_{d-1}(A \cap B; \mathbf{k}) \to \cdots .$$ □

9.6. Exercises

CHAPTER 10

Universal coefficients for homology

The universal coefficients theorem explains how $\mathrm{H}_d(X;\mathbf{k})$ for different $\mathbf{k}$ are related, at least for CW complexes. Fundamentally, it is a piece of homological algebra. In this chapter, we will freely use properties of the Tor-functor discussed in Appendix 15.

10.1. Universal coefficients, homological algebra version

Let $\mathrm{C}_\bullet$ be a chain complex of abelian groups and let A be an abelian group. One might expect that

$$\mathrm{H}_d(\mathrm{C}_\bullet \otimes A) \cong \mathrm{H}_d(\mathrm{C}_\bullet) \otimes A \quad \text{for all } d.$$

However, this is not necessarily true. Here is an example:

EXAMPLE 10.1.1. For an integer $\ell \geq 2$, let $\mathrm{C}_\bullet$ be the chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times \ell} \mathbb{Z} \longrightarrow 0$$

whose $\mathbb{Z}$-terms appear in degrees 0 and 1. Then $\mathrm{C}_\bullet \otimes \mathbb{Z}/\ell$ is the chain complex

$$0 \longrightarrow \mathbb{Z}/\ell \xrightarrow{\times 0} \mathbb{Z}/\ell \longrightarrow 0,$$

so

$$\mathrm{H}_d(\mathrm{C}_\bullet) \otimes \mathbb{Z}/\ell = \begin{cases} \mathbb{Z}/\ell & \text{if } d = 0, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathrm{H}_d(\mathrm{C}_\bullet \otimes \mathbb{Z}/\ell) = \begin{cases} \mathbb{Z}/\ell & \text{if } d = 0, 1, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

The issue is that the tensor product is right exact but not exact, i.e., if

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is a short exact sequence of abelian groups and A is another abelian group, then

$$M_1 \otimes A \longrightarrow M_2 \otimes A \longrightarrow M_3 \otimes A \longrightarrow 0$$

is exact but the map $M_1 \otimes A \to M_2 \otimes A$ need not be injective. The correction terms are given by the Tor functor discussed in Appendix 15. Namely, we have an exact sequence

$$0 \longrightarrow \mathrm{Tor}(M_1, A) \longrightarrow \mathrm{Tor}(M_2, A) \longrightarrow \mathrm{Tor}(M_3, A) \longrightarrow M_1 \otimes A \longrightarrow M_2 \otimes A \longrightarrow M_3 \otimes A \longrightarrow 0.$$

This should be viewed as the "derived tensor product", and you should expect it to appear whenever you are doing homological algebra with tensor products.

Our main result in this direction is as follows:

THEOREM 10.1.2 (Universal coefficients for chain complexes). *Let $\mathrm{C}_\bullet$ be a chain complex of free abelian groups and let A be an abelian group. Then for all d we have a natural short exact sequence*

$$0 \longrightarrow \mathrm{H}_d(\mathrm{C}_\bullet) \otimes A \longrightarrow \mathrm{H}_d(\mathrm{C}_\bullet \otimes A) \longrightarrow \mathrm{Tor}(\mathrm{H}_{d-1}(C_\bullet), A) \longrightarrow 0.$$

This exact sequence splits, but not in a natural way.

REMARK 10.1.3. The fact that this exact sequence is natural means that if $f\colon \mathrm{C}_\bullet \to \mathrm{C}'_\bullet$ is a map of chain complexes and $g\colon A \to A'$ is a map of abelian groups, then we have an induced commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}_d(\mathrm{C}_\bullet) \otimes A & \longrightarrow & \mathrm{H}_d(\mathrm{C}_\bullet \otimes A) & \longrightarrow & \mathrm{Tor}(\mathrm{H}_{d-1}(C_\bullet), A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}_d(\mathrm{C}'_\bullet) \otimes A' & \longrightarrow & \mathrm{H}_d(\mathrm{C}'_\bullet \otimes A') & \longrightarrow & \mathrm{Tor}(\mathrm{H}_{d-1}(C'_\bullet), A') & \longrightarrow & 0 \end{array}$$

whose vertical maps are induced by f and g. That it splits in a non-natural way means that we can write

$$\mathrm{H}_d(\mathrm{C}_\bullet \otimes A) \cong (\mathrm{H}_d(\mathrm{C}_\bullet) \otimes A) \oplus \mathrm{Tor}(\mathrm{H}_{d-1}(C_\bullet), A),$$

but we cannot do this simultaneously for all chain complexes and abelian groups in a way that is compatible with the above commutative diagrams. □

Proof of Theorem 10.1.2. Let $\partial_\bullet$ be the differential on $\mathrm{C}_\bullet$. For each d, define $\mathrm{Z}_d = \ker(\partial_d)$ and $\mathrm{B}_{d-1} = \mathrm{Im}(\partial_d)$. We thus have a short exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Z}_d \longrightarrow \mathrm{C}_d \xrightarrow{\partial_d} \mathrm{B}_{d-1} \longrightarrow 0. \tag{10.1.1}$$

We then have:

Claim 1. *Tensoring* (10.1.1) *with A gives a short exact sequence*

$$0 \longrightarrow \mathrm{Z}_d \otimes A \longrightarrow \mathrm{C}_d \otimes A \xrightarrow{\partial_d \otimes \mathbb{1}_A} \mathrm{B}_{d-1} \otimes A \longrightarrow 0$$

Proof of claim. Let $\iota\colon \mathrm{Z}_d \hookrightarrow \mathrm{C}_d$ be the inclusion. Tensoring with A is right exact, so we must prove that the map $\iota \otimes \mathbb{1}_A\colon \mathrm{Z}_d \otimes A \to \mathrm{C}_d \otimes A$ is injective. Since C_{d-1} is free abelian, its subgroup B_{d-1} is also free abelian. It follows that (10.1.1) splits. This implies the claim. Indeed, let $\pi\colon \mathrm{C}_d \to \mathrm{Z}_d$ be a splitting of the inclusion $\iota\colon \mathrm{Z}_d \hookrightarrow \mathrm{C}_d$, so $\pi \circ \iota = \mathbb{1}_{\mathrm{Z}_d}$. It follows that $\pi \otimes \mathbb{1}_A\colon \mathrm{C}_d \otimes A \to \mathrm{Z}_d \otimes A$ is a splitting of the map $\iota \otimes \mathbb{1}_A\colon \mathrm{Z}_d \otimes A \to \mathrm{C}_d \otimes A$, so $\iota \otimes \mathbb{1}_A$ is indeed injective. □

Observe now that the short exact sequences from this claim fit into commutative diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Z}_d \otimes A & \longrightarrow & \mathrm{C}_d \otimes A & \xrightarrow{\partial_d \otimes \mathbb{1}_A} & \mathrm{B}_{d-1} \otimes A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial_d} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathrm{Z}_{d-1} \otimes A & \longrightarrow & \mathrm{C}_{d-1} \otimes A & \xrightarrow{\partial_{d-1} \otimes \mathbb{1}_A} & \mathrm{B}_{d-2} \otimes A & \longrightarrow & 0
\end{array}$$

In other words, regarding the sequences of abelian groups $\mathrm{Z}_\bullet \otimes A$ and $\mathrm{B}_\bullet \otimes A$ as chain complexes equipped with the differential 0, we have a short exact sequence of chain complexes[1]

$$0 \longrightarrow \mathrm{Z}_\bullet \otimes A \longrightarrow \mathrm{C}_\bullet \otimes A \longrightarrow \mathrm{B}_{\bullet-1} \otimes A \longrightarrow 0.$$

Since $\mathrm{H}_d(\mathrm{Z}_\bullet \otimes A) = \mathrm{Z}_d \otimes A$ and $\mathrm{H}_d(\mathrm{B}_{\bullet-1} \otimes A) = \mathrm{B}_{d-1} \otimes A$, the associated long exact sequence provided by the snake lemma looks like

$$\cdots \xrightarrow{b_{d+1}} \mathrm{Z}_d \otimes A \longrightarrow \mathrm{H}_d(\mathrm{C}_\bullet \otimes A) \longrightarrow \mathrm{B}_{d-1} \otimes A \xrightarrow{b_d} \mathrm{Z}_{d-1} \otimes A \longrightarrow \cdots$$

Here the connecting homomorphisms are labeled as

$$b_d\colon \mathrm{B}_{d-1} \otimes A \to \mathrm{Z}_{d-1} \otimes A.$$

Letting $\iota_{d-1}\colon \mathrm{B}_{d-1} \to \mathrm{Z}_{d-1}$ be the inclusion, a bit of reflection shows that $b_d = \iota_{d-1} \otimes \mathbb{1}_A$. It follows that we can extract from the above long exact sequence a short exact sequence

$$0 \longrightarrow \mathrm{coker}(\iota_d \otimes \mathbb{1}_A) \longrightarrow \mathrm{H}_d(\mathrm{C}_\bullet \otimes A) \longrightarrow \ker(\iota_{d-1} \otimes \mathbb{1}_A) \longrightarrow 0.$$

This is the short exact sequence claimed by the theorem:

Claim 2. *For all d, we have*

$$\mathrm{coker}(\iota_d \otimes \mathbb{1}_A) = \mathrm{H}_d(\mathrm{C}_\bullet) \otimes A,$$
$$\ker(\iota_{d-1} \otimes \mathbb{1}_A) = \mathrm{Tor}(\mathrm{H}_{d-1}(\mathrm{C}_\bullet), A).$$

Proof of claim. Since C_d is a free abelian group, so are its subgroups Z_d and B_d. It follows that

$$0 \longrightarrow \mathrm{B}_d \xrightarrow{\iota_d} \mathrm{Z}_d \longrightarrow \mathrm{H}_d(\mathrm{C}_\bullet) \longrightarrow 0$$

[1]Here the notation $\mathrm{B}_{\bullet-1} \otimes A$ means the chain complex $\mathrm{B}_\bullet \otimes A$, but with a degree shift to makes its d^{th} term $\mathrm{B}_{d-1} \otimes A$.

is a free resolution of $\mathrm{H}_d(\mathrm{C}_\bullet)$. Tensoring with A, we get a chain complex that computes Tor:

$$0 \longrightarrow \mathrm{B}_d \otimes A \xrightarrow{\iota_d \otimes \mathbb{1}_A} \mathrm{Z}_d \otimes A \longrightarrow 0.$$

We conclude that

$$\operatorname{coker}(\iota_d \otimes \mathbb{1}_A) = \operatorname{Tor}_0^{\mathbb{Z}}(\mathrm{H}_d(\mathrm{C}_\bullet), A) = \mathrm{H}_d(\mathrm{C}_\bullet) \otimes A,$$
$$\ker(\iota_d \otimes \mathbb{1}_A) = \operatorname{Tor}_1^{\mathbb{Z}}(\mathrm{H}_d(\mathrm{C}_\bullet), A) = \operatorname{Tor}(\mathrm{H}_d(\mathrm{C}_\bullet), A),$$

implying the claim. □

It remains to prove that our exact sequence

$$0 \longrightarrow \mathrm{H}_d(\mathrm{C}_\bullet) \otimes A \longrightarrow \mathrm{H}_d(\mathrm{C}_\bullet \otimes A) \longrightarrow \operatorname{Tor}(\mathrm{H}_{d-1}(\mathrm{C}_\bullet), A) \longrightarrow 0$$

splits. As notation, let $\mathrm{Z}(\mathrm{C}_\bullet \otimes A)_d$ be the kernel of the differential

$$\partial_d \otimes \mathbb{1}_A \colon \mathrm{C}_d \otimes A \to \mathrm{C}_{d-1} \otimes A.$$

We thus have

$$\mathrm{H}_d(\mathrm{C}_\bullet \otimes A) = \mathrm{Z}(\mathrm{C}_\bullet \otimes A)_d / (\mathrm{B}_d \otimes A).$$

Now recall that in the proof of Claim 1 we constructed a splitting $\pi \colon \mathrm{C}_d \to \mathrm{Z}_d$ of the short exact sequence

$$0 \longrightarrow \mathrm{Z}_d \longrightarrow \mathrm{C}_d \xrightarrow{\partial_d} \mathrm{B}_{d-1} \longrightarrow 0.$$

Let $h \colon \mathrm{Z}_d \to \mathrm{H}_d(\mathrm{C}_\bullet)$ be the map taking a cycle to its homology class, and let $\mathfrak{h} \colon \mathrm{Z}(\mathrm{C}_\bullet \otimes A)_d \to \mathrm{H}_d(\mathrm{C}_\bullet) \otimes A$ be the composition

$$\mathrm{Z}(\mathrm{C}_\bullet \otimes A)_d \hookrightarrow \mathrm{C}_d \otimes A \xrightarrow{\pi \otimes \mathbb{1}_A} \mathrm{Z}_d \otimes A \xrightarrow{h \otimes \mathbb{1}_A} \mathrm{H}_d(\mathrm{C}_\bullet) \otimes A.$$

Since π restricts to the identity on Z_d, it in particular restricts to the identity on B_d. It follows that $\mathfrak{h}$ vanishes on $\mathrm{B}_d \otimes A$, and thus induces a map

$$\overline{\mathfrak{h}} \colon \mathrm{H}_d(\mathrm{C}_\bullet \otimes A) \to \mathrm{H}_d(\mathrm{C}_\bullet) \otimes A.$$

By construction, this is a splitting of (10.1). □

10.2. Exercises

CHAPTER 11

The Künneth formula for homology

11.1. Exercises

CHAPTER 12

Construction of homology

12.1. Exercises

Part 2

Appendices

CHAPTER 13

Appendix: more on the topology of CW complexes

CHAPTER 14

Appendix: Spines of noncompact manifolds

CHAPTER 15

Appendix: Tor **and** Ext

CHAPTER 16

Appendix: cofibrations