Math 30820: Honors Algebra IV
Problem Set 7

1. Artin, Chapter 15, Problem 8.2.

2. Artin, Chapter 15, problem 10.1 (we asserted this in class without proof – prove it!)

3. Artin, Chapter 15, Misc Exercises, M.1.

4. Let K be a field and let $f \in K[x]$ be a monic polynomial of degree n. Let $K \subset L$ be a splitting field for f, i.e., an extension of the form $K[a_1, \ldots, a_n]$ with
$$f(x) = (x - a_1) \cdots (x - a_n).$$
Prove that the $[K : F]$ divides $n!$.

5. Let F be a field of characteristic p and let $F \subset K$ be a finite field extension such that p does not divide $[K : F]$. Prove that $F \subset K$ is a separable extension.

6. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a field automorphism.

   (a) Prove that $f(q) = q$ for all $q \in \mathbb{Q}$.

   (b) Prove that if $x > 0$, then $f(x) > 0$, and then prove that this implies that f is an increasing function. Hint: Note that all you can use are field-theoretic properties. How can you characterize positive elements of $\mathbb{R}$ just using its structure as a field?

   (c) Prove that if $|x - y| < \frac{1}{n}$ for some $n \geq 1$, then $|f(x) - f(y)| < \frac{1}{n}$, and then proe that this implies that f is continuous.

   (d) Prove that $f(x) = x$ for all $x \in \mathbb{R}$. In other words, the group of field automorphisms of $\mathbb{R}$ is the trivial group. We remark that this is something very special about $\mathbb{R}$ – the group of field automorphisms of $\mathbb{C}$ is a massive uncountable group. However, if you restrict yourself to continuous (or even just measurable!) automorphisms of $\mathbb{C}$, then there are only two: the identity, and complex conjugation.