Math 30820: Honors Algebra IV
Problem Set 3

1. Artin, §14.1, problems 1.1, 1.2, 1.4.

2. Artin, §14.2, problems 2.1, 2.4.

3. Let R be a ring and let $f\colon M \to N$ be a homomorphism of R-modules. Assume that $\ker(f)$ and $\operatorname{im}(f)$ are finitely generated R-modules. Prove that M is a finitely generated R-module.

4. Let R be an integral domain and let M be an R-module. An element $m \in M$ is called a *torsion element* if there exists some nonzero $r \in R$ with $rm = 0$. Let $\operatorname{Tor}(M)$ be the set of all torsion elements.

   (a) Prove that $\operatorname{Tor}(M)$ is an R-submodule of M.

   (b) Prove that $M/\operatorname{Tor}(M)$ is torsion-free, i.e., that $\operatorname{Tor}(M/\operatorname{Tor}(M)) = 0$.

   (c) Regard $\mathbb{C}$ as a module over $\mathbb{Z}[i]$ via the usual complex multiplication, so for $r \in \mathbb{Z}[i]$ and $m \in \mathbb{C}$ we have $rm \in \mathbb{C}$ defined as usual. Thus $\mathbb{Z}[i] \subset \mathbb{C}$ is a $\mathbb{Z}[i]$-submodule, so we can define the quotient module $M = \mathbb{C}/\mathbb{Z}[i]$. **Question**: determine $\operatorname{Tor}(M)$.