Math 10860: Honors Calculus II, Spring 2021
Homework 6

Instructions: Some of this homework is a list of integrals to compute. You could in principle do each of these by entering the integrand into `Mathematica` (or some similar program), noting the result, and then verifying it by differentiation. This is **not** what I'm intending. I want to see you tackle these integrals using integration by parts, or integration by an appropriate substitution. For each integral, you should say clearly what method/substitution you are using in each step; other than that, no great explanation is need.

Warning: I promise that unless explicitly stated otherwise, all the integrals below *have elementary primitives*. I don't promise that the homework is typo-free, and unfortunately even a tiny typo can turn a do-able integration into an impossible one; so alert me if you think that there is a problem with any of these!

1. Here are a few "standard" integration formulae, randomly culled from the back page of a calculus textbook. Verify that each of them is correct. Part of this entails checking that both the function being integrated and the proposed antiderivative have the same domain (part of the full answer will be a statement of that domain); the other part entails checking that at each point in the domain of the proposed antiderivative, the the derivative of the proposed antiderivative is the function being integrated. These should be very easy, but a little care might be required for the antiderivatives that involve absolute values.

   (a) $\int \cot x\ dx = \log|\sin x|$.

   Solution:

   Now let us analyze the domains. Let $Dom(f)$ be the domain of function f. We have

$$Dom(\cot(x)) = \{x \in \mathbb{R} : x \neq k\pi,\ k \in \mathbb{Z}\}$$
$$Dom(\ln(x)) = \{x \in \mathbb{R} : x > 0\}$$

   Because $\sin(x)$ is defined for the entire real line, we do not need to worry about where the sine function is undefined. Thus, whenever $\sin(x) \leq 0$, $\ln(\sin(x))$ is undefined. This occurs on the intervals $[k\pi, (k+1)\pi]$ where k is odd. Thus, we have

$$Dom(\ln(\sin(x)) = \{x \in \mathbb{R} : x \notin [k\pi, (k+1)\pi],\ k \text{ is an odd integer}\}$$

   The way to fix the fact that the two domains are not equal to each other is to instead write $\sin(x)$ as $|\sin(x)|$ instead. Thus, our solution is $\ln(|\sin(x)|)$.

   From this domain analysis, we can state

$$\frac{d}{dx}\ln(|\sin(x)|) = \frac{\sin(x)}{\cos}(x) = \cot(x)$$

(b) $\int \sec x\ dx = \log|\sec x + \tan x|$.

Solution:

Now let us analyze the domains. We have

$$Dom(\sec(x)) = \{x \in \mathbb{R} : x \neq \frac{k\pi}{2},\ k \text{ is an odd number}\}$$

To find the domain of $\ln(\tan(x)+\sec(x))$ we need to first see where $\tan(x)+\sec(x)$ are undefined, and then see where the expression is less than or equal to 0. Note that

$$\tan(x) + \sec(x) = \frac{\sin(x)+1}{\cos(x)}$$

and so to find where the expression is undefined, we need to find the values where $\cos(x) = 0$. These values are of the form $x = k\frac{\pi}{2}$ where k is an odd integer. Now we need to find where the expression is less than or equal to 0. This occurs when $\frac{\sin(x)}{\cos(x)} \leq -\frac{1}{\cos(x)}$, which occurs on intervals $\left(k\frac{\pi}{2}, (k+2)\frac{\pi}{2}\right)$ where $k = 1, 5, 9, ...$ Thus, we have that

$$Dom(\ln(\sec(x) + \tan(x)) = \{x \in \mathbb{R} : x \notin \left[k\frac{\pi}{2}, (k+2)\frac{\pi}{2}\right],\ k = 1, 5, 9, ...\}$$

In order to make the domains equal, we need to find a way to ensure that $\tan(x)+\sec(x)$ remains positive. This can be done if we have $|\tan(x)+\sec(x)|$ and so our final answer is $\ln(|\sec(x) + \tan(x)|)$.

From this analysis we can state

$$\ln|\sec(x) + \tan(x)| = \frac{1}{\sec(x) + \tan(x)} \cdot \sec(x)(\tan(x) + \sec(x)) = \sec(x)$$

(c) For an arbitrary real a (positive or negative), $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|$.

Solution:

Now we perform a domain analysis. Note that

$$Dom\left(\frac{1}{x^2 - a^2}\right) = \{x \in \mathbb{R} : x \neq \pm a\}$$

Now, to find the domain of $\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)$ we need to first see where $\frac{x-a}{x+a}$ is undefined and where it is less than or equal to 0. It is undefined when $x = -a$, and it is less than or equal to 0 when $x \in [-a, a]$. Thus we have

$$Dom\left(\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)\right) = \{x \in \mathbb{R} : x \in [-a, a]\}$$

In order to match up the domains, we need to find a way to keep $\frac{x-a}{x+a}$ positive. To do this, we simply apply the absolute value onto it. Thus, our answer is $\frac{1}{2a}\ln\left(\left|\frac{x-a}{x+a}\right|\right)$.

From here we can state

$$\frac{d}{dx}\ln\left(\left|\frac{x-a}{x+a}\right|\right) = \frac{x+a}{x-a}\cdot\frac{2a}{(x+a)^2}$$
$$= \frac{2a}{(x-a)(x+a)}$$
$$= \frac{2a}{x^2-a^2}$$

and diviind by $2a$ gives our desired result.

2. This problem concerns a very important non-elementary function, called the *gamma function*.

   (a) Show that for $x > 0$,

$$\int_0^\infty e^{-t}t^{x-1}dt$$

   is finite. The value of this integral, for each such x, is denoted $\Gamma(x)$. Note: the gamma function can also be defined for $x \leq 0$, as long as x is not an integer — see the graph at `https://en.wikipedia.org/wiki/Gamma_function`.

   Solution: First, we need to prove two facts. First, note

$$\int_0^\infty e^{-xt}\,dt = \lim_{\alpha\to\infty}\int_0^\alpha e^{-xt}\,dt$$
$$= \lim_{\alpha\to\infty}\left(-\frac{e^{xt}}{x}\right)\Big|_0^\alpha$$
$$= \frac{1}{x}\left(\lim_{\alpha\to\infty} -e^{-\alpha x} - \lim_{\alpha\to\infty} 1\right)$$
$$= \frac{1}{x}$$

   and second we show that

$$\lim_{t\to\infty}\frac{t^{n-1}}{e^{t/2}} = 0, \quad n\in\mathbb{N}$$

   Via L'Hopital we have

$$\lim_{t\to\infty}\frac{t^{n-1}}{e^{t/2}} = \lim_{t\to\infty}\frac{(n-1)t^{n-1}}{(1/2)e^{t/2}}$$

   Note that

$$\frac{d}{dt^n}t^{n-1} = 0$$

   and so via induction, we have

$$\lim_{t\to\infty}\frac{t^{n-1}}{e^{t/2}} = \lim_{t\to\infty}\frac{0}{(1/2)^n e^{(1/2)t}} = 0$$

Now we will begin to prove our desired result. First, we know via the second fact we proved that if $\varepsilon = 1$, then there is an M such that for all $t \geq M$, we have

$$\left| \frac{t^{n-1}}{e^{(1/2)t}} \right| < \varepsilon$$

and so for all $t \geq M$, $0 \leq t^{n-1} \leq e^{(1/2)t}$. This implies that $0 \leq e^{-t}t^{n-1} \leq e^{-(1/2)t}$ and by the first fact we proved we know that $\int_0^\infty e^{-(1/2)t}\,dt$ converges. Then by comparison theorem,

$$\int_0^\infty e^{-t}t^{n-1}\,dt \tag{1}$$

converges for all natural numbers n

Now, let $x \geq 1$. For $t \geq 0$ we have

$$0 \leq e^{-t}t^{x-1} \leq e^{-t}t^{\text{floor}(x)}$$

By equation (1), we then have that

$$\int_0^\infty e^{-t}t^{\text{floor}(x)}\,dt$$

converges and so by comparison theorem we have that

$$\int_0^\infty e^{-t}t^{x-1}\,dt \tag{2}$$

converges for $x \geq 1$.

Now, let $0 < x < 1$. We know that

$$\frac{1}{e^{(1/2)t}} \leq \frac{t^{x-1}}{e^{(1/2)t}} \leq \frac{t}{e^{(1/2)t}}$$

and from the second fact we proved we know that

$$\lim_{t\to\infty} \frac{t}{e^{(1/2)t}} = \lim_{t\to\infty} \frac{1}{e^{(1/2)t}} = 0$$

Thus, via squeeze theorem we can then state

$$\frac{t^{x-1}}{e^{(1/2)t}} = 0$$

This implies that

$$0 \leq e^{-t}t^{x-1} \leq e^{(1/2)t}$$

and so by comparison theorem we have

$$\int_0^\infty e^{-t}t^{x-1}\,dt \tag{3}$$

converges for $0 < x < 1$. Thus, by equations 1, 2, and 3, we are done.

(b) Prove that for all $x > 0$ in the domain of Γ, $\Gamma(x+1) = x\Gamma(x)$.

Solution: We have

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty e^{-t}t^x\,dt \\
&= \lim_{\alpha\to\infty} \int_0^\alpha e^{-t}t^x\,dt \\
&= \lim_{\alpha\to\infty} \left(-e^{-t}t^x \Big|_0^\alpha + x\int_0^\alpha e^{-t}t^{x-1}\,dt \right), \qquad \text{via integration by parts} \\
&= x\int_0^\alpha e^{-t}t^{x-1}\,dt \\
&= x\Gamma(x)
\end{aligned}$$

(c) Prove that $\Gamma(n) = (n-1)!$ for all natural numbers n (thus the gamma function is a continuous function that extends the factorial function to (almost) all reals).

Solution: We proceed via induction. For $x = 1$, we have via part b

$$\Gamma(2) = 1 \cdot \Gamma(1) = \int_0^\infty e^{-t}\,dt = 1 = 0!$$

Now suppose that $\Gamma(x) = (x-1)!$ for values up to n. Note that via part b we have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!$$

and so we are done.

3. Explain *precisely* what is wrong with the following "proof" that $0 = 1$:

Evaluating $\int \frac{dx}{x}$ using integration by parts, taking $u = 1/x$ and $dv = dx$ (so $du = -dx/x^2$ and $v = x$), yields

$$\int \frac{dx}{x} = \left(\frac{1}{x}\right)x - \int x\left(\frac{-1}{x^2}\right)dx = 1 + \int \frac{dx}{x}.$$

Subtracting $\int \frac{dx}{x}$ from both sides yields $0 = 1$.

No credit for just mumbling something vague about the constant of integration — pinpoint *exactly* what is wrong, and say what the argument actually proves.

Solution:

The error arises in the decision to cancel out the $\int \frac{1}{x}\,dx$ term. The term $\int \frac{1}{x}\,dx$ should not be understood as a function, but rather as a family of functions that only differ up to a local constant. To clarify, let us derive the integration by parts formula for indefinite integrals step-by-step

$$\begin{aligned}
&(f(x)g(x))' = f'(x)g(x) + g'(x)f(x)\\
&\Rightarrow \int (f(x)g(x))'\,dx = \int f'(x)g(x)\,dx + \int g'(x)f(x)\,dx\\
&\Rightarrow f(x)g(x) + C = \int f'(x)g(x)\,dx + \int g'(x)f(x)\,dx\\
&\Rightarrow \int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx + C
\end{aligned}$$

and so the correct statement should be

$$\int \frac{dx}{x} = \left(\frac{1}{x}\right)x - \int x\left(\frac{-1}{x^2}\right)dx + C = 1 + \int \frac{dx}{x} + C$$

where C is some constant. But if we let $C' = C + 1$ then we have

$$\int \frac{dx}{x} = \left(\frac{1}{x}\right)x - \int x\left(\frac{-1}{x^2}\right)dx = \int \frac{dx}{x} + C'$$

which shows that the term $\int \frac{dx}{x}$ represents a family of functions that differ by a local constant. To summarize, indefinite integration is not the same as calculating a definite integral.

What was shown in the original "proof" is that $0 = 1 + C'$, where C' is some constant. This is a true statement, since $0 = 1 + (-1)$. In other words, $0 = 1$ modulo constant.

4. Some problems that are best suited to integration by parts. Do any two of these:

   (a) $\int x^2 \sin x \; dx$

   Solution: We have

$$u = x^2, \quad \frac{du}{dx} = 2x, \quad \frac{dv}{dx} = \sin(x), \quad v = -\cos(x)$$

   and so our integration by parts is

$$\int x^2 \sin x \; dx = -x^2\cos(x) + 2\int x\cos(x)\,dx$$

   Now consider the integral $\int x\cos(x)\,dx$. Here, we let

$$u = x, \quad \frac{du}{dx} = 1, \quad \frac{dv}{dx} = \cos(x), \quad v = \sin(x)$$

   and so we have

$$\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx = x\sin(x) + \cos(x)$$

   Thus, we have that the answer is

$$2x\sin(x) + \left(2 - x^2\right)\cos(x)$$

(b) $\int x(\log x)^2 \ dx$

Solution: Let

$$u = (\ln(x))^2, \quad \frac{du}{dx} = \frac{2\ln(x)}{x}, \quad \frac{dv}{dx} = x, \quad v = \frac{x^2}{2}$$

Then our integral becomes

$$\int x(\log x)^2 \ dx = \frac{(x\ln(x))^2}{2} - \int x\ln(x)\,dx$$

Now let us calculate $\int x\ln(x)\,dx$. For this, we let

$$u = \ln(x), \frac{du}{dx} = \frac{1}{x}, \quad \frac{dv}{dx} = x, \quad v = \frac{1}{2}x^2$$

and so we have

$$\int x\ln(x)\,dx = \frac{x^2\ln(x)}{2} - \frac{1}{2}\int x\,dx = \frac{x^2\ln(x)}{2} - \frac{1}{4}x^2$$

Pluggint this all toget gives us the final answer as

$$\frac{x^2\left(2\ln^2(x) - 2\ln(x) + 1\right)}{4}$$

(c) $\int \sec^3 x \ dx$.

Solution: We let

$$u = \sec(x), \quad \frac{du}{dx} = \sec(x)\tan(x), \quad \frac{dv}{dx} = \sec^2(x), \quad v = \tan(x)$$

Then our integral becomes

$$\begin{aligned}\int \sec^3 x \ dx &= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx \\ &= \sec(x)\tan(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx, \quad \text{as } \tan^2(x) = \sec^2(x) - 1\end{aligned}$$

However, note that

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)|$$

and so we have

$$2\int \sec^3(x)\,dx = \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|$$

which implies

$$\int \sec^3(x)\,dx = \frac{1}{2}(\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|)$$

5. None of $\log(\log x)$, $1/(\log x)$, $x^2e^{-x^2}$ or e^{-x^2} have elementary primitives. However, we can still say things about their primitives. Do any two of these:

(a) Express $\int \log(\log x)dx$ in terms of $\int dx/\log x$.

Solution: We use integration by parts. Let $u = \ln(\ln(x))$ and $\frac{dv}{dx} = 1$. Then $\frac{du}{dx} = \frac{1}{x\ln(x)}$ and $v = x$. Then we have

$$\int \ln(\ln(x))\,dx = x\ln(\ln(x)) - \int \frac{x}{x\ln(x)}\,dx$$

Simplifying the $\int \frac{x}{x\ln(x)}\,dx$ gives our desired result.

(b) Express $\int x^2e^{-x^2}\,dx$ in terms of $\int e^{-x^2}\,dx$.

Solution: We use integration by parts. Let

$$u = x, \quad \frac{du}{dx} = 1, \quad \frac{dv}{dx} = xe^{-x^2}, \quad v = -\frac{e^{-x^2}}{2}$$

Then we have

$$\int x^2e^{-x^2}\,dx = -\frac{xe^{-x^2}}{2} + \frac{1}{2}\int e^{-x^2}\,dx$$

where we solved $\int xe^{-x^2}\,dx$ with the substitution $w = -x^2$ and $\frac{dw}{dx} = -2x$.

(c) Find a reduction formula for $\int(\log x)^n\,dx$, and use it to calculate $\int(\log x)^3\,dx$.

Solution: We use integration by parts. We let

$$u = (\ln(x))^n, \quad \frac{du}{dx} = \frac{n(\ln(x))^{n-1}}{x}, \quad dv = 1, \quad v = x$$

Then we have

$$\begin{aligned}\int(\ln(x))^n\,dx &= (\ln(x))^n x \cdot x - \int \frac{n(\ln(x))^{n-1}}{x}\cdot x\,dx \\ &= x(\ln(x))^n - n\int(\ln(x))^{n-1}\,dx\end{aligned}$$

Now using this we can write

$$\int(\ln(x))^3\,dx = x(\ln(x))^3 - 3\int(\ln(x))^2\,dx$$
$$\int(\ln(x))^2\,dx = x(\ln(x))^2 - 2\int\ln(x)\,dx$$
$$\int\ln(x)\,dx = x\ln(x) - x$$

Plugging this all in accordingly, we get

$$\int(\ln(x))^3\,dx = x\left(\ln^3(x) - 3\ln^2(x) + 6\ln(x) - 6\right)$$

6. Remember that there are no silver-bullet rules for substitution. Just try to substitute for an expression that appears frequently or prominently. If two different troublesome expressions appear, try to express them both in terms of some new expression. Do any two of these:

   (a) $\int \frac{dx}{\sqrt{1+e^x}}$
   Solution: Let $u = e^x + 1$ and $\frac{du}{dx} = e^x$. Then our integral becomes

$$\int \frac{dx}{\sqrt{1+e^x}} = \int \frac{du}{(u-1)\sqrt{u}}$$

   Now let $v = \sqrt{u}$, so $\frac{dv}{du} = \frac{1}{2\sqrt{u}}$. Thus our integral becomes

$$\int \frac{du}{(u-1)\sqrt{u}} = 2\int \frac{dv}{v^2-1}$$

   Via Problem 1, part c we have that this equals

$$2\int \frac{dv}{v^2-1} = \ln\left|\frac{v-1}{v+1}\right|$$
$$= \ln\left|\frac{\sqrt{u}-1}{\sqrt{u}+1}\right|$$
$$\ln\left|\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\right|$$

   (b) $\int \frac{4^x+1}{2^x+1}\,dx$
   Solution: First we rewrite the integral

$$\int \frac{4^x+1}{2^x+1}\,dx = \int \ln(2)2^x \cdot \frac{2^{2x}+1}{\ln(2)2^x(2^x+1)}\,dx$$

   Now, let $u = 2^x + 1$. Then $\frac{du}{dx} = \ln(2)2^x$. Thus our integral becomes

$$\int \ln(2)2^x \cdot \frac{2^{2x}+1}{\ln(2)2^x(2^x+1)}\,dx = \frac{1}{\ln(2)}\int \frac{(u-1)^2+1}{(u-1)u}du$$

   Note that we can simplify the integral in the following way

$$\int \frac{(u-1)^2+1}{(u-1)u}du = \int \left(\frac{2-u}{(u-1)u}+1\right)du$$
$$= \int du - \int \frac{u-2}{(u-1)u}$$
$$= u - \int \left(\frac{2}{u} - \frac{1}{u-1}\right)du$$
$$= u - 2\ln(u) + \ln(u-1)$$

   Plugging this all back into our original integral and reverting the substitution gives us

$$\int \frac{4^x+1}{2^x+1}\,dx = \frac{1}{\ln(2)}\left(2^x - 2\ln(2^x+1)\right) + x$$

(c) $\int \frac{1}{x^2}\sqrt{\frac{x-1}{x+1}}\; dx$

Solution: We perform the substitution

$$u = \sqrt{x-1}, \qquad \frac{du}{dx} = \frac{1}{2\sqrt{x-1}}$$

Thus, our integral becomes

$$\int \frac{1}{x^2}\sqrt{\frac{x-1}{x+1}}\; dx = 2\int \frac{u^2}{(u^2+1)^2\sqrt{u^2+2}}\, du$$

Now let us solve this integral. We perform another substitution

$$v = \arctan\left(\frac{u}{\sqrt{2}}\right) \frac{dv}{du} = \frac{1}{\sqrt{2}\sec^2(v)}$$

and we can write

$$\begin{aligned}\int \frac{u^2}{(u^2+1)^2\sqrt{u^2+2}} &= \int \frac{2^{(3/2)}\sec^2(v)\tan^2(v)}{(2\tan^2(v)+1)^2\sqrt{2\tan^2(v)+2}}dv \\ &= 2\int \frac{\sec(v)\tan^2(v)}{2\tan^2(v)+1}dv, \qquad \text{as } 2\tan^2(v)+2 = 2\sec^2(v)\end{aligned}$$

We use the trig identities

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \sec(x) = \frac{1}{\cos(x)}, \quad \cos^2(x) = 1-\sin^2(x)$$

and write the integral as

$$\int \frac{\sec(v)\tan^2(v)}{2\tan^2(v)+1}dv = \int \cos(v)\cdot\frac{\sin^2(v)}{(\sin^2(v)+1)^2}dv$$

Now we perform the substitution

$$w = \sin(v), \qquad \frac{dw}{dv} = \cos(v)$$

and so we have

$$\begin{aligned}\int \cos(v)\cdot\frac{\sin^2(v)}{(\sin^2(v)+1)^2}dv &= \int \frac{w^2}{(w^2+1)^2}dw \\ &= \int \left(\frac{w^2+1}{(w^2+1)^2} - \frac{1}{(w^2+1)^2}\right) dw \\ &= \int \frac{1}{w^2+1}dw - \int \frac{1}{(w^2+1)^2}dw\end{aligned}$$

Note that

$$\int \frac{1}{w^2+1}\, dw = \arctan(w)$$

and via integration by parts we have

$$\int \frac{1}{(w^2+1)^w}\,dw = \frac{w}{2(w^2+1)} + \frac{1}{2}\int \frac{1}{w^2+1}\,dw$$

Now we can undo the substitution. We have

$$2\int \frac{\sec(v)\tan^2(v)}{2\tan^2(v)+1}dv = \arctan(\sin(v)) - \frac{\sin(v)}{\sin^2(v)+1}$$

and similarly we have

$$2\int \frac{u^2}{(u^2+1)^2\sqrt{u^2+2}}\,du = 2\arctan\left(\frac{u}{\sqrt{2}\sqrt{(u^2/2)+1}}\right) - \frac{\sqrt{2}u}{\sqrt{(u^2/2)+1}\left(\frac{u^2}{2((u^2/2)+1)+1}\right)}$$

after performing the final substitution and simplifying, we have our final answer to be

$$2\arctan\left(\frac{\sqrt{x-1}}{\sqrt{x+1}}\right) - \frac{\sqrt{x-1}\sqrt{x+1}}{x}$$

7. Some problems involving substitutions such as $x = \sin u$, $x = \cos u$: (As well as knowing $\int \sec dx$, it *might* be helpful here to know

$$\int \csc x\ dx = -\log|\csc x + \cot x|,$$

which can also be verified easily by differentiation.) Do any two of these:

(a) $\int \frac{dx}{\sqrt{1-x^2}}$

Solution: Let $x = \sin(u)$, so $\frac{dx}{du} = \cos(u)$. Then our integral becomes

$$\begin{aligned}\int \frac{dx}{\sqrt{1-x^2}} &= \int \frac{\cos(u)}{\sqrt{1-\sin^2(u)}}\,du \\ &= \int \frac{\cos(u)}{\cos(u)}\,du \\ &= \int du \\ &= u \\ &= \arcsin(x)\end{aligned}$$

(b) $\int \frac{dx}{x\sqrt{x^2-1}}$

Solution: Let $u = \sqrt{x^2-1}$. Then $\frac{du}{dx} = \frac{x}{\sqrt{x^2-1}}$. Thus our integral becomes

$$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{1}{1+u^2}\,du$$

Now let $u = \tan(v)$. Then $\frac{du}{dv} = \sec^2(v)$. Thus we have

$$\begin{aligned}
\int \frac{1}{1+v^2}\,dv &= \int \frac{\sec^2(v)}{1+\tan^2(v)}\,dv \\
&= \int \frac{1+\tan^2(v)}{1+\tan^2(v)}\,dv, \qquad \text{as } \sec^2(x) = 1+\tan^2(x) \\
&= \int dv \\
&= v \\
&= \arctan(u) \\
&= \arctan\left(\sqrt{x^2-1}\right)
\end{aligned}$$

(c) $\int x^3\sqrt{1-x^2}\,dx$. This will also involve the integration of powers of sin and cos.
Solution: Let $x = \sin(u)$, then $\frac{dx}{du} = \cos(u)$. Then our integral becomes

$$\int x^3\sqrt{1-x^2}\,dx = \int \sin^3(u)\cos^2(u)\,du = \int \sin^3(u)\,du - \int \sin^5(u)\,du$$

Let $v = \cos(u)$. Then $\frac{dv}{du} = -\sin(u)$. We have

$$\begin{aligned}
\int \sin^3(u)\,du &= \int \sin(u)(1-\cos^2(u))\,du \\
&= -\int (1-v^2)\,dv \\
&= \int (v^2-1)\,dv \\
&= \frac{v^3}{3} - v \\
&= \frac{1}{3}\cos^3(u) - \cos(u)
\end{aligned}$$

and similarly we have

$$\begin{aligned}
\int \sin^5(u)\,du &= \int \sin^3(u)\sin^2(u)\,du \\
&= \int \sin(u)(1-\cos^2(u))^2\,du \\
&= -\int (1-v^2)^2\,dv \\
&= -\int (1-2v^2+v^4)\,dv \\
&= \int (-1+2v^2-v^4)\,dv \\
&= -v + \frac{2}{3}v^3 - \frac{1}{5}v^5 \\
&= -\cos(u) + \frac{2}{3}\cos^3(u) - \frac{1}{5}\cos^5(u)
\end{aligned}$$

Subtracting these two gives us

$$\int \sin^3(u)\cos^2(u)\,du = -\frac{1}{3}\cos^3(u) + \frac{1}{5}\cos^5(u)$$

Thus we have

$$\int \frac{dx}{\sqrt{1-x^2}} = \frac{1}{15}(3x^4 - x^2 - 2)\sqrt{1-x^2}$$