Math 10860: Honors Calculus II, Spring 2021
Homework 5

1. Compute $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$.

   Solution: By repeated use of L'Hopital's rule, we have

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right) &= \lim_{x\to 0}\left(\frac{\sin x - x}{x\sin x}\right)\\
&= \lim_{x\to 0}\left(\frac{\cos x - 1}{\sin x + x\cos x}\right)\\
&= \lim_{x\to 0}\left(\frac{-\sin x}{\cos x + \cos x - x\sin x}\right)\\
&= \frac{-\sin 0}{2\cos 0 - 0\sin 0}\\
&= 0.
\end{aligned}$$

2. Differentiate these functions: (Convention: a^{b^c} always means $a^{(b^c)}$.)

   (a) $f(x) = e^{e^{e^{e^x}}}$

   (b) $f(x) = e^{\left(\int_0^x e^{-t^2}\,dt\right)}$

   (c) $f(x) = (\log x)^{\log x}$

   Solution:

   (a) $f'(x) = e^{e^{e^{e^x}}} e^{e^{e^x}} e^{e^x} e^x$.

   (b) $f'(x) = e^{\left(\int_0^x e^{-t^2}\,dt\right)} \frac{d}{dx} \int_0^x e^{-t^2} dt = e^{\left(\int_0^x e^{-t^2}\,dt\right)} e^{-x^2}$.

   (c) $f'(x) = \frac{d}{dx} e^{(\log x)(\log \log x)} = e^{(\log x)(\log \log x)} \left(\frac{\log \log x}{x} + \frac{\log x}{x \log x} \right) = (\log x)^{\log x} \left(\frac{1 + \log \log x}{x} \right)$.

3. The logarithmic derivative of f is the expression f'/f. It's called "logarithmic derivative" because it is the derivative of $\log \circ f$. It is often easier to compute the derivative of log of a function than it is to compute the derivative of the function directly, because taking logs turns products into (simpler to differentiate) sums, and turns powers into (simpler to differentiate) products. The derivative of the original function can then be recovered by multiplying by the original function.

   Compute the logarithmic derivatives of these functions:

   (a) $f(x) = x^x$

   (b) $f(x) = \frac{(3-x)^{1/3}x^2}{(1-x)(3+x)^{2/3}}$

   (c) $f(x) = \frac{e^x - e^{-x}}{e^{2x}(1+x^3)}$

   Solution:

   (a) $\frac{f'(x)}{f(x)} = \frac{d}{dx}\log(x^x) = \frac{d}{dx} x \log x = x\frac{1}{x} + \log x = 1 + \log x.$

   (b) $\frac{f'(x)}{f(x)} = \frac{d}{dx}\log\left(\frac{(3-x)^{1/3}x^2}{(1-x)(3+x)^{2/3}}\right) = \frac{d}{dx}\left(\frac{1}{3}\log(3-x) + 2\log x - \log(1-x) - \frac{2}{3}\log(3+x)\right) = -\frac{1}{3(3-x)} + \frac{2}{x} + \frac{1}{1-x} - \frac{2}{3(3+x)}.$

   (c) $\frac{f'(x)}{f(x)} = \frac{d}{dx}\log\left(\frac{e^x - e^{-x}}{e^{2x}(1+x^3)}\right) = \frac{d}{dx}\left(\log(e^x - e^{-}x) - 2x - \log(1+x^3)\right) = \frac{e^x + e^{-}x}{e^x - e^{-}x} - 2 - \frac{3x^2}{1+x^3}.$

4. Compute these limits:

   (a) $\lim_{x\to 0} \frac{e^x-1-x-x^2/2}{x^2}$
   (b) $\lim_{x\to\infty} \frac{x}{(\log x)^n}$, where n is a natural number.
   (c) $\lim_{x\to 0^+} \frac{x}{(\log x)^n}$, where n is a natural number.
   (d) $\lim_{x\to 0^+} x^x$

   Solution:

   (a) By repeated application of L'Hopital's rule, we have

$$\begin{aligned}\lim_{x\to 0}\frac{e^x-1-x-x^2/2}{x^2} &= \lim_{x\to 0}\frac{e^x-1-x}{2x}\\ &= \lim_{x\to 0}\frac{e^x-1}{2}\\ &= \frac{e^0-1}{2}\\ &= 0.\end{aligned}$$

   (b) The limit is ∞, which I will prove by induction on n. When $n=1$, we have

$$\begin{aligned}\lim_{x\to\infty}\frac{x}{\log x} &= \lim_{x\to\infty}\frac{1}{\frac{1}{x}}\\ &= \lim_{x\to\infty} x\\ &= \infty.\end{aligned}$$

   Now assume that $\lim_{x\to\infty}\frac{x}{(\log x)^n} = \infty$ for some $n\in\mathbb{N}$. Then L'Hopital's rule gives

$$\begin{aligned}\lim_{x\to\infty}\frac{x}{(\log x)^{n+1}} &= \lim_{x\to\infty}\frac{1}{\frac{(n+1)(\log x)^n}{x}}\\ &= \frac{1}{n+1}\lim_{x\to\infty}\frac{x}{(\log x)^n}.\end{aligned}$$

   The last limit is ∞ by the induction hypothesis, so we conclude that $\lim_{x\to\infty}\frac{x}{(\log x)^n} = \infty$ for all $n\in\mathbb{N}$.

   (c) The limit is 0. We have $\lim_{x\to 0^+} f(x) = \lim_{x\to\infty} f(\frac{1}{x})$ for a function f as long as one of these limits exists. Thus

$$\begin{aligned}\lim_{x\to 0^+}\frac{x}{(\log x)^n} &= \lim_{x\to\infty}\frac{\frac{1}{x}}{(\log(\frac{1}{x}))^n}\\ &= (-1)^n\lim_{x\to\infty}\frac{1}{x(\log x)^n}.\end{aligned}$$

   Since $\lim_{x\to\infty} x(\log x)^n = \infty$, we have

$$(-1)^n\lim_{x\to\infty}\frac{1}{x(\log x)^n} = 0.$$

(d) We have $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\log x}$. Since e^x is continuous, this equals $e^{\lim_{x\to 0^+} x\log x}$. Next $\lim_{x\to 0^+} x\log x = \lim_{x\to 0^+} \frac{\log x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$. We conclude $\lim_{x\to 0^+} x^x = e^0 = 1$.

5. Which number is bigger: e^π or π^e? (Rigorously justify your answer!)

Solution: I claim that $e^\pi > \pi^e$. Consider the function $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = \frac{\log x}{x}$. We have $f'(x) = \frac{1-\log x}{x^2}$. This is 0 if and only if $\log x = 1$ if and only if $x = e$. For $x < e$, $1 - \log x > 0$, so $f'(x) > 0$, and for $x > e$, $1 - \log x < 0$, so $f'(x) < 0$. Thus The global maximum for f occurs at $x = e$. So $\frac{\log x}{x} < \frac{\log e}{e} = \frac{1}{e}$ for all positive $x \neq e$. Then $e \log x < x$, so $\log(x^e) < x$, so $x^e < e^x$. In particular, letting $x = \pi$, we have $e^\pi > \pi^e$.

6. Prove that $F(x) = \int_2^x \frac{dt}{\log t}$ is not a bounded function on $[2, \infty)$.

Meta-question: Why am I asking this question? There is an important mathematical concept, one that you've been familiar with for many years, and one that most non-mathematics are familiar with, that this integral is intimately related to. What is the concept, and what is the connection?

Solution: For all $t \geq 2$, we have $t > \log t$. To see this, consider the function $f : [2, \infty) \to \mathbb{R}$ given by $f(t) = t - \log t$. Then $f'(t) = 1 - \frac{1}{t} > 0$ for all $t \geq 2$, so f is always increasing. $f(2) = 2 - \log 2 > 0$, so indeed $f(t) > 0$ for all $t \geq 2$, so $t > \log t > 0$ for all $t \geq 2$. Then $\frac{1}{t} < \frac{1}{\log t}$, so

$$\begin{aligned} F(x) &= \int_2^x \frac{dt}{\log t} \\ &> \int_2^x \frac{dt}{t} \\ &= \log x - \log 2, \end{aligned}$$

and since $\log x$ is unbounded on $[2, \infty)$, so is F.

7. This question guides you to an alternate expression for e.

(a) Find $\lim_{y\to 0} \frac{\log(1+y)}{y}$.

(b) Find $\lim_{x\to\infty} x\log(1+1/x)$.

(c) Prove that

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

(d) Go though the same process to argue that for all real a

$$e^a = \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x.$$

Specifically:

- First, argue $\lim_{y\to 0} \frac{\log(1+ay)}{y} = a$.
- Next, argue $\lim_{x\to\infty} x\log(1+a/x) = a$.
- Finally, argue $e^a = \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$.

Solution:

(a) By L'Hopital's rule, $\lim_{y\to 0} \frac{\log(1+y)}{y} = \lim_{y\to 0} \frac{\frac{1}{1+y}}{1} = 1$.

(b) Arguing as in 4c, we have $\lim_{x\to\infty} x\log(1+\frac{1}{x}) = \lim_{y\to 0^+} \frac{1}{y}\log(1+\frac{1}{\frac{1}{y}}) = \lim_{y\to 0^+} \frac{\log(1+y)}{y} = 1$.

(c) This is equivalent to proving that $1 = \log\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right)$. Since the logarithm function is continuous, $\log\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right) = \lim_{x\to\infty}\log\left(\left(1+\frac{1}{x}\right)^x\right) = \lim_{x\to\infty} x\log\left(1+\frac{1}{x}\right) = 1$, so we are done.

(d) By L'Hopital's rule, $\lim_{y\to 0} \frac{\log(1+ay)}{y} = \lim_{y\to 0} \frac{\frac{a}{1+ay}}{1} = a$.

Again arguing as in 4c, we have $\lim_{x\to\infty} x\log(1+\frac{a}{x}) = \lim_{y\to 0^+} \frac{1}{y}\log(1+\frac{a}{\frac{1}{y}}) = \lim_{y\to 0^+} \frac{\log(1+ay)}{y} = a$.

Proving the final statement is equivalent to proving $a = \log\left(\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x\right)$.
Again by continuity of the logarithm function, $\log\left(\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x\right) = \lim_{x\to\infty}\log\left(\left(1+\frac{a}{x}\right)^x\right) = \lim_{x\to\infty} x\log\left(1+\frac{a}{x}\right) = a$, so we are done.

8. *Newton's law of cooling* says that an object cools at a rate proportional to the difference between its temperature and the temperature of the surrounding medium. Suppose that an object has temperature T_0 at time $t = 0$, and that the temperature of the surrounding medium remains at a constant M throughout time.

   Find the temperature of the object at time t (in terms of T_0, M, and k, the implicit constant of proportionality in Newton's law).

Solution: We are given that $T'(t) = k(T(t) - M)$ for some constant k. Let $F(t) = T(t) - M$. Then $F'(t) = (T(t) - M)' = T'(t)$, so the above is equivalent to $F'(t) = kF(t)$. We have previously shown that this implies that $F(t) = ce^{kt}$ for a constant c. Then $T(t) = M + ce^{kt}$. We know $T(0) = T_0$, so $T_0 = M + ce^{k \cdot 0} = M + c$, so $c = T_0 - M$. We conclude that the temperature at time t is given by

$$T(t) = M + (T_0 - M)e^{kt}.$$