Math 10860: Honors Calculus II, Spring 2021
Homework 2 Solutions

1. Decide which of the following functions are integrable on $[0,2]$, and calculate the integral when the function is integrable. You can use that $\int_a^b x\, dx$ exists and equals $(b^2 - a^2)/2$; but don't assume anything else other than the definition of the integral, and the basic facts that we have proven in class or in the notes.

   (a) $f(x) = \begin{cases} x & \text{if } 0 \le x < 1, \\ x - 2 & \text{if } 1 \le x \le 2. \end{cases}$

   Solution: This function is integrable. Note that we can "split up" the integral in the following way:

$$\begin{aligned} \int_0^2 f(x)\, dx &= \int_0^1 x\, dx + \int_1^2 (x-2)\, dx \\ &= \int_0^1 x\, dx + \int_1^2 x\, dx - \int_1^2 2\, dx \\ &= \int_0^2 x\, dx - \int_1^2 2\, dx \\ &= \frac{2^2}{2} - 2(2-1) \\ &= 0 \end{aligned}$$

   (b) $f(x) = x + [x]$ (recall $[x]$ is the largest integer that is less than or equal to x).

   Solution: This function is integrable. We can rewrite the function $[x]$ on $[0,2]$ as

$$[x] = \begin{cases} 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$$

   and so we can rewrite $f(x)$ on $[0,2]$ as

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ x+1, & 1 \le x < 2 \\ 4, & x = 2 \end{cases}$$

We now compute the integral as follows:

$$\begin{aligned}
\int_0^2 f &= \int_0^1 x\,dx + \int_1^2 (x+1)\,dx \\
&= \int_0^1 x\,dx + \int_1^2 x\,dx + \int_1^2 1\,dx \\
&= \int_0^2 x\,dx + \int_1^2 q\,dx \\
&= \frac{2^2}{2} + (2-1) \\
&= 3
\end{aligned}$$

(c) $f(x) = \begin{cases} 1 & \text{if } x \text{ is of the form } a + b\sqrt{2} \text{ for rational } a, b, \\ 0 & \text{otherwise.} \end{cases}$

Solution: This function is not integrable. Recall that rational multiples of $\sqrt{2}$ are dense in $\mathbb{R}$, and so numbers of the form $a+b\sqrt{2}$, where $a, b \in \mathbb{Q}$, are dense too. Thus in any interval there is at least one such number, and so all upper Darboux sums are 1. Thus, the infimum over the upper Darboux sums is 1.

Now we show that the supremum of the lower Darboux sums is 0. Notice that the rational multiples of $\sqrt{5}$ are dense in the reals. We will show that the only rational multiple of $\sqrt{5}$ that is of form $a + b\sqrt{2}$, where a and b are rational, is 0. Suppose that $c\sqrt{5} = a + b\sqrt{2}$ for $a, b, c \in \mathbb{Q}$. This gives us

$$5c^2 = a^2 + b^2 + 2ab\sqrt{2}$$

after squaring both sides. Because $\sqrt{2}$ is irrational, then either a or b must be 0. Suppose that $b = 0$. This gives us $a = c\sqrt{2}$, but since $\sqrt{5}$ is irrational, this can only hold if $a = c = 0$. Now, suppose that $a = 0$. This implies that $c\sqrt{5} = b\sqrt{2}$, which implies that $c\sqrt{10} = b$. But since $\sqrt{10}$ is irrational, this implies that $b = c = 0$.

Now, notice that if we remove one element of a dense subset of $\mathbb{R}$, the subset is still dense in the reals. As a result, the set of non-zero multiples of $\sqrt{5}$ is dense in $\mathbb{R}$. Because no number in this set is of form $a + b\sqrt{2}$, where a and b are rational, then we can use this dense set to show that the supremum of the lower Darboux sums must be 0.

2. Let $f : [-b, b] \to \mathbb{R}$ be a function that is integrable on the interval $[0, b]$, and that is an odd function ($f(-x) = -f(x)$). Show that $\int_{-b}^{b} f$ exists, and that it equals 0.

   Comment: Note the similarity to question 1 of the first homework. There I was looking for an informal explanation, based on area considerations. Here I'm looking for a *formal* proof, from the definition of the integral.

Solution: Note that because f is odd and bounded on $[0, b]$, it must be bounded on $[-b, 0]$ as well.

Let $I := \int_0^b f(x)\,dx$ for ease of notation. We will show that $\int_{-b}^0 f(x)dx = -I$. Now, fix some $\varepsilon > 0$. There exists a partition $P_1 = \{0 = t_0, t_1, ..., t_n = b\}$ on $[0, b]$ with $I - \varepsilon < L(f, P_1) \le I \le U(f, P_1) < I + \varepsilon$.

Now consider a partition $P_2 = \{-b = -t_n, -t_{n-1}, ..., -t_0 = 0\}$ on $[-b, 0]$. We have that

$$L(f, P_2) = \sum_{i=1}^{n} \inf\{f(x) \mid \ x \in [-t_i, -t_{i-1}]\}(t_i - t_{i-1})$$

Recall that for a bounded non-empty set A, $-\sup A = \inf -A$. If we let $A = \{f(x) \mid x \in [t_{i-1}, t_i]\}$, then because $f(x)$ is odd it follows that $-A = \{f(x) \mid \ x \in [-t_i, -t_{i-1}]\}$. As a result, we can state that

$$\inf\{f(x) \mid \ x \in [-t_i, -t_{i-1}]\} = -\sup\{f(x) \mid \ x \in [t_{i-1}, t_i]\}$$

As a result, we can state that

$$L(f, P_2) = -\sum_{i=1}^{n} \sup\{f(x) \mid \ x \in [t_{i-1}, t_i]\}(t_i - t_{i-1}) = -U(f, P_1)$$

Analogously, we can show that

$$L(f, P_1) = -U(f, P_2)$$

and by using the previously mentioned inequalities, we can now state that

$$-I - \varepsilon \le L(f, P_2) \le I \le U(f, P_2) < -I + \varepsilon$$

This establishes that $\int_{-b}^0 f(x)\,dx$ exists and equals $-I$. From here the desired result follows.

3. Let A be a bounded, non-empty set of real numbers, and let $|A| = \{|a| \mid a \in A\}$. Prove that
$$\sup |A| - \inf |A| \le \sup A - \inf A.$$

Solution: We prove this by cases.

Case 1. Suppose that $\inf A \ge 0$. This implies that $A = |A|$ and so $\sup |A| - \inf |A| = \sup A - \inf A$.

Case 2. Suppose that $\sup A \le 0$. This implies that $-A = |A|$. As a result, we have that $\sup |A| = \sup -A = -\inf A$, and similarly we have that $\inf |A| = -\sup A$. Thus, we have that $\sup |A| - \inf |A| = -\inf A + \sup A = \sup A - \inf A$.

Case 3. Suppose that $\inf A \le 0$ and $\sup A \ge 0$. Note that $\sup A \le \max\{\sup A, -\inf A\}$, $-\inf A \ge 0$, and $\inf |A| \ge 0$. As a result, we can state that $\sup |A| - \inf |A| \le \max\{\sup A, -\inf A\} \le \sup A - \inf A$.

4. The goal of this multi-part question is to establish some properties of integrability that we discussed in class, but did not prove.

   (a) Prove that if f is integrable on $[a, b]$ then so is $|f|$

   Comment: You will most likely need to use the result of the last question.

   Solution: Since f is integrable on $[a, b]$, there is a partition $P = \{t_0, t_1, ..., t_n\}$ of the interval such that $U(f, P) - L(f, P) < \varepsilon$. We can rewrite this as

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=0}^{n} \sup f(x)(t_i - t_{i-1}) - \sum_{i=0}^{n} \inf f(x)(t_i - t_{i-1}) \\ &= \sum_{i=0}^{n} (\sup f(x) - \inf f(x))(t_i - t_{i-1}) \end{aligned}$$

   For ease of notation, we are letting $\sup\{f(x) \mid \ x \in [t_{i-1}, t_i]\}$ be denoted as $\sup f(x)$, and similarly for the infimum. Using Question 3, we can then state that

$$\sum_{i=0}^{n} (\sup |f(x)| - \inf |f(x))|(t_i - t_{i-1}) \leq \sum_{i=0}^{n} (\sup f(x) - \inf f(x))(t_i - t_{i-1})$$

   and so we have that $U(|f|, P) - L(|f|, P) < \varepsilon$. Thus, $|f|$ is integrable on $[a, b]$.

   (b) Deduce from the result of part (a) that if f is integrable on $[a, b]$ then so are both of

   - $\max\{f, 0\}$ (the function which at input x takes the value $f(x)$ if $f(x) \geq 0$, and takes value 0 otherwise) and
   - $\min\{f, 0\}$.

   Comment: This should follow *very quickly*, and without any real technical work, from the result of the last part, if you also use some of the basic properties of the integral that have previously established.

   Solution: We can write the function $f^+ = \max\{f, 0\}$ as

$$f^+ = \frac{1}{2}(f + |f|)$$

   and since f^+ is the linear combinationof integrable functions, as seen in part a, then f^+ is integrable too. Similarly, note that the function $f^- = -\min\{f, 0\}$ can we written as

$$f^- = \frac{1}{2}(|f| - f)$$

   and since f^- is the linear combination of integrable functions, f^- is integrable. Thus, $-f^- = \min\{f, 0\}$ is integrable as well.

   (c) The *positive part of* f is the function $f^+ = \max\{f, 0\}$. Informally, think of the positive part of f as being obtained from f by pushing all parts of the graph of f that lie below the x-axis, up to the x-axis. The *negative part of* f is the function

$f^- = -\min\{f, 0\}$. Note that $f = f^+ - f^-$ is a representation of f as a linear combination of non-negative functions.

Deduce from the previous parts of this question that f is integrable on $[a, b]$ if and only if f^+ and f^- are both integrable on $[a, b]$.

Comment: As with the last part, this should be quick.

Solution: Suppose that f is integrable. By part a, this implies that $|f|$ is integrable. Because we can write the positive and negative parts of the function as

$$f^+ = \frac{1}{2}\left(f + |f|\right), \quad \text{and} \quad f^- = \frac{1}{2}\left(|f| - f\right)$$

we then see that f^+ and f^- are the linear combinations of integrable functions, and hence they are integrable as well.

Now suppose that f^+ and f^- are integrable. Because we can rewrite f as $f = f^+ - f^-$, we then see that f is the linear combination of integrable functions. Thus, f is integrable too.

5. Prove the *triangle inequality* for integrals: if f is integrable on $[a, b]$ then

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

Solution: By the properties of absolute value, we have that

$$-|f(x)| \leq f(x) \leq |f(x)|$$

By Question 4, part a, since f is integrable, so too is $|f|$. Thus our inequality implies

$$-\int_a^b |f(x)|\, dx \leq \int_a^b f(x)\, dx \leq \int_a^b |f(x)|\, dx$$

This directly implies that

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx$$

6. The goal of this question is to establish that if f and g are integrable on $[a, b]$, then so is fg.

(a) Suppose that f and g are both non-negative on $[a, b]$. Let $P = \{t_0, \ldots, t_n\}$ be a partition of $[a, b]$. Define

$$m_k^f = \inf\{f(x) \mid t_{k-1} \leq x \leq t_k\} \quad \text{and} \quad m_k^g = \inf\{g(x) \mid t_{k-1} \leq x \leq t_k\}$$

and

$$m_k = \inf\{f(x)g(x) \mid t_{k-1} \leq x \leq t_k\}$$

and

$$M_k^f = \sup\{f(x) \mid \ t_{k-1} \le x \le t_k\} \quad \text{and} \quad M_k^g = \sup\{g(x) \mid \ t_{k-1} \le x \le t_k\}$$

and

$$M_k = \sup\{f(x)g(x) \mid \ t_{k-1} \le x \le t_k\}$$

Prove that

$$M_k \le M_k^f M_k^g \quad \text{and} \quad m_i^f m_i^g \le m_i.$$

Solution: We have that

$$0 \le m_k^f \le f(x) \le M_k^f \quad \text{and} \quad 0 \le m_k^g \le g(x) \le M_k^g$$

These inequalities imply that

$$0 \le m_k^f m_k^g \le f(x)g(x) \le M_k^f M_k^g$$

From the definition of infimum and supremum, it directly follows that

$$m_k \ge m_k^f m_k^g \quad \text{and} \quad M_k \le M_k^f M_k^g$$

(b) By using the trick

$$M_k^f M_k^g - m_k^f m_k^g = M_k^f M_k^g - m_k^f M_k^g + m_k^f M_k^g - m_k^f m_k^g,$$

together with the result of part (a), show that fg is integrable.

Comment: For this part it might be helpful to remember that f and g are bounded.

Solution: Because $f(x)$ and $g(x)$ are integrable on $[a, b]$, both f and g are bounded on this interval. Let M^f be the upper bound for f on this interval, and let M^g be the bound for g on the interval. Define the common bound as $M := \max\{M^f, M^g\}$. Also, because f and g are integrable, we know that for $\varepsilon > 0$, there exists a partition $P = \{t_0, t_1, ..., t_n\}$ of $[a, b]$ such that

$$U(f, P) - L(f, P) < \frac{\varepsilon}{2M} \quad \text{and} \quad U(g, P) - L(g, P) < \frac{\varepsilon}{2M}$$

.

We can use this, along with the inequalities established in part a, to state

$$
\begin{aligned}
U(fg,P)-L(fg,P) &= \sum_{k=1}^{n}(M_k-m_k)(t_k-t_{k-1}) \\
&\le \sum_{k=1}^{n}(M_k^f M_k^g - m_k^f m_k^g)(t_k-t_{k-1}) \\
&= \sum_{k=1}^{n}(M_k^f M_k^g - m_k^f M_k^g + m_k^f M_k^g - m_k^f m_k^g)(t_k-t_{k-1}) \\
&= \sum_{k=1}^{n}(M_k^f M_k^g - m_k^f M_k^g)(t_k-t_{k-1}) + \sum_{k=1}^{n}(m_k^f M_k^g - m_k^f m_k^g)(t_k-t_{k-1}) \\
&= M_k^g\sum_{k=1}^{n}(M_k^f - m_k^f)(t_k-t_{k-1}) + m_k^f\sum_{k=1}^{n}(M_k^g - m_k^g)(t_k-t_{k-1}) \\
&\le M\sum_{k=1}^{n}(M_k^f - m_k^f)(t_k-t_{k-1}) + M\sum_{k=1}^{n}(M_k^g - m_k^g)(t_k-t_{k-1}) \\
&= M\left(\sum_{k=1}^{n}(M_k^f - m_k^f)(t_k-t_{k-1}) + \sum_{k=1}^{n}(M_k^g - m_k^g)(t_k-t_{k-1})\right) \\
&= M\left(U(f,P)-L(f,P)+U(g,P)-L(g,P)\right) \\
&< M\left(\frac{\varepsilon}{2M}+\frac{\varepsilon}{2M}\right) \\
&= \varepsilon
\end{aligned}
$$

and so fg is integrable on $[a,b]$.

(c) Use the result of Question 4, part (c) (together with some basic properties of the integral) to show that if f and g are both arbitrary (not necessarily non-negative) integrable functions on $[a,b]$, then fg is integrable on $[a,b]$.

Solution: We will show that, for an arbitrary function f, both f^+ and f^- must be nonnegative. First consider $f^+(x) = \max\{f(x), 0\}$. If the value of $f(x)$ at x is greater than or equal to 0, then $f^+(x) \ge 0$. Otherwise, then $f^+(x) = 0$. Either way, $f^+ \ge 0$ in the entire domain. Now consider $f^-(x)$. If $f(x)$ at x is greater than or equal to 0, then $f^-(x) = 0$. Otherwise, $f^-(x) = -f(x) > 0$. Either way, $f^- \ge 0$ on the domain. As a result, we can state then that for arbitrary function f and g, the functions f^+g^+, f^+g^-, f^-g^+, and f^-g^- must be integrable by part b and what we showed above, as we showed that f^+, f^-, g^+. and g^- are nonnegative above, and they must be integrable by Question 4, part c.

Now, let f and g be arbitrary functions that are integrable on $[a,b]$. Recall from Question 4, part c that $f = f^+ - f^-$. Thus, we have that

$$fg = (f^+ - f^-)(g^+ - g^-) = f^+g^+ - f^+g^- - f^-g^+ + f^-g^-$$

Using what we stated above along with the result from Question 4, part c, we see that fg is the linear combination of integrable functions, and so fg is integrable as well.

7. Suppose that f is integrable on $[0, x]$ for all $x \geq 0$ and that $\lim_{x\to\infty} f(x) = a$. Find (with proof)

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x f(t)\ dt.$$

Comment: Draw a picture to get an intuition for what the limit should be.

Solution: By definition of the limit, for $\varepsilon > 0$, there exists an N such that for $t \geq N$

$$|f(t) - a| < \varepsilon$$

This implies that

$$\left| \int_N^{N+N'} f(t)\, dt - N'a \right| < N'\varepsilon$$

and so we can state that

$$\left| \frac{1}{N+N'} \int_N^{N+N'} f(t)\, dt - \frac{N'}{N+N'} a \right| < \frac{N'}{N+N'} \varepsilon < \varepsilon$$

Now, let us choose an N' such that

$$\left| \frac{N'}{N+N'} a - a \right| < \varepsilon \quad \text{and} \quad \left| \frac{1}{N+N'} \int_0^N f(t)\, dt \right| < \varepsilon$$

Combining these with the previous inequality via triangle inequality gives us

$$\left| \frac{1}{N+N'} \int_0^{N+N'} f(t)\, dt - a \right| < 3\varepsilon$$

Since N' is much larger than N, we can rewrite this as

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x f(t)\, dt = a$$

- **An extra credit problem**: Suppose that n real numbers sum to 1. What's the smallest possible value for the sum of their squares? Justify!

Solution: We use the Cauchy-Schwarz inequality here. Recall that the Cauchy-Schwarz inequality states that

$$\left(\sum_{i=1}^n x_i y_i \right)^2 \leq \left(\sum_{i=1}^n x_i^2 \right) \left(\sum_{i=1}^n y_i^2 \right)$$

Let $y_i = 1$ for all i. Our inequality becomes

$$\left(\sum_{i=1}^n x_i \right)^2 \leq \left(\sum_{i=1}^n x_i^2 \right) n$$

By hypothesis, we have that $x_1 + x_2 + \cdots + x_n = 1$ and so our inequality simplifies to

$$1 \leq \left(\sum_{i=1}^{n} x_i^2 \right) n$$

which implies that

$$\frac{1}{n} \leq \left(\sum_{i=1}^{n} x_i^2 \right)$$