Math 10860: Honors Calculus II, Spring 2021
Homework 1

1. Directly from the definitions, prove that $\int_0^b x^3\ dx = b^4/4$. You can use the formula $\sum_{k=1}^n n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$.

   Solution: Let $f : [0, b] \to \mathbb{R}$ be given by $f(x) = x^3$. Let $P_n = \{0, \frac{b}{n}, 2 \cdot \frac{b}{n}, \cdots, n \cdot \frac{b}{n}\}$ be a partition of $[0, b]$. Since f is increasing on $[0, b]$, we have $m_i = \frac{(i-1)^3 b^3}{n^3}$ and $M_i = \frac{i^3 b^3}{n^3}$, where m_i, M_i are defined as usual. So the upper and lower sums are

$$\begin{aligned}
L(f, P_n) &= \sum_{i=1}^n \frac{(i-1)^3 b^3}{n^3} \cdot \frac{b}{n} \\
&= \frac{b^4}{n^4} \sum_{i=1}^n (i-1)^3 \\
&= \frac{b^4}{n^4} \cdot \frac{(n-1)^2 n^2}{4} \\
&= \frac{b^4}{4} \cdot \frac{(n-1)^2}{n^2}, \\
U(f, P_n) &= \sum_{i=1}^n \frac{i^3 b^3}{n^3} \cdot \frac{b}{n} \\
&= \frac{b^4}{n^4} \sum_{i=1}^n i^3 \\
&= \frac{b^4}{n^4} \cdot \frac{n^2(n+1)^2}{4} \\
&= \frac{b^4}{4} \cdot \frac{(n+1)^2}{n^2}.
\end{aligned}$$

   From these equations it is clear that $L(f, P_n) \leq \frac{b^4}{4} \leq U(f, P_n)$ for all n. We can also see that

$$\begin{aligned}
U(f, P_n) - L(f, P_n) &= \frac{b^4}{4} \cdot \frac{4n}{n^2} \\
&= \frac{b^4}{n},
\end{aligned}$$

   so we can make this difference arbitrarily small by choosing n large enough, so f is integrable on $[0, b]$. We must have $L(f, P_n) \leq \int_0^b x^3\ dx \leq U(f, P_n)$ for each n, and since the above difference can be made as small as desired, there is only one number with that property, which is $\frac{b^4}{4}$, so we must have $\int_0^b x^3\ dx = \frac{b^4}{4}$.

2. Without doing any serious computations, evaluate the following integrals. You can be informal here; I'm not looking for a watertight ε-δ justification, but rather an explanation that shows me that you know what is going on with the integral, and its interpretation as an area. We have not proved the fundamental theorem of calculus, so you can't use it.

   (a) $\int_{-1}^{1} x^3\sqrt{1-x^2}\ dx$

   (b) $\int_{-1}^{1} (x^5+3)\sqrt{1-x^2}\ dx$.

   Solution:

   (a) The integrand is an odd function, so the negative values on the interval $[-1,0]$ exactly cancel the positive values on the interval $[0,1]$. Thus $\int_{-1}^{1} x^3\sqrt{1-x^2}\ dx = 0$.

   (b) This integral is equal to $\int_{-1}^{1} x^5\sqrt{1-x^2}\ dx + 3\int_{-1}^{1}\sqrt{1-x^2}\ dx$. By the same reasoning as part (a), the first integral is 0. The second is 3 times the area of a half circle of radius 1, so the whole integral is equal to $\frac{3\pi}{2}$.

3. Let $f, g\colon [a,b] \to \mathbb{R}$ both be bounded, and let m, m^f and m^g be given by

- $m = \inf\{f(x) + g(x) \mid x \in [a,b]\}$
- $m^f = \inf\{f(x) \mid x \in [a,b]\}$
- $m^g = \inf\{g(x) \mid x \in [a,b]\}$

(a) Show that $m^f + m^g \leq m$.

(b) Show, by way of an example, that it is possible to have $m^f + m^g < m$.

Solution:

(a) For all $x \in [a,b]$, we have $m^f \leq f(x)$ and $m^g \leq g(x)$, so $m^f + m^g \leq f(x) + g(x)$. So $m^f + m^g$ is a lower bound for $f + g$, but m is the greatest lower bound. Thus $m^f + m^g \leq m$.

(b) Let $f, g : [0,1] \to \mathbb{R}$ be given by $f(x) = 1 - x$ and $g(x) = x$. Then $m^f = m^g = 0$, so $m^f + m^g = 0$, but $f(x) + g(x) = 1$ for all $x \in [0,1]$, so $m = 1$.

4. (a) Which functions $f\colon [a,b] \to \mathbb{R}$ have the property that *every* lower sum $L(f,P)$ equals *every* upper sum $U(f,Q)$?

   (b) Which functions $f\colon [a,b] \to \mathbb{R}$ have the property that there is *some* lower sum $L(f,P)$ that equals *some* upper sum $U(f,Q)$?

   (c) Which *continuous* functions $f\colon [a,b] \to \mathbb{R}$ have the property that all lower sums $L(f,P)$ are equal?

Solution: It is clear that constant functions satisfy each of these properties, and in fact they are the only functions that satisfy any of them.

(a) Consider the partition $P = Q = \{a,b\}$. Since $L(f,P) = U(f,Q)$, we have $m = \inf\{f(x) : x \in [a,b]\} = \sup\{f(x) : x \in [a,b]\} = M$. Now let $x \in [a,b]$. Then $m \leq f(x) \leq M$ by definition of m and M, so $f(x) = m = M$, and since x is arbitrary and $m = M$ is fixed, f must be constant.

(b) Let $S = P \cup Q$. Then $L(f,P) \leq L(f,S) \leq U(f,S) \leq U(f,Q)$, and since $L(f,P) = U(f,Q)$, we have $L(f,S) = U(f,S)$. Let $S = \{a = t_0, t_1, \cdots, t_{n-1}, t_n = b\}$. Let m_i, M_i be defined as usual. Then $\sum_{i=1}^n m_i(t_i - t_{i-1}) = \sum_{i=1}^n M_i(t_i - t_{i-1})$, so $\sum_{i=1}^n (M_i - m_i)(t_i - t_{i-1}) = 0$, and since $t_i - t_{i-1} > 0$ and $M_i - m_i \geq 0$ for all i, we must have $M_i = m_i$ for all i. The same argument as (a) gives that f is constant on each interval $[t_{i-1}, t_i]$, but since each adjacent pair of intervals shares an endpoint, f must be constant everywhere.

(c) Assume for a contradiction that a non constant continuous function has all lower sums $L(f,P)$ equal. Let m be the minimum of f on $[a,b]$ (a minimum does in fact exist since f is continuous on a closed interval). Since f is not constant and m is the minimum, there exists $x_0 \in [a,b]$ such that $f(x_0) > m$. Since f is continuous, there is an interval $[a_0, b_0]$ around x_0 completely contained in $[a,b]$ with $a_0 > a$ and $b_0 < b$ such that $f(x) > m$ for all $x \in [a_0, b_0]$. Consider the partitions $P = \{a,b\}$ and $Q = \{a, a_0, b_0, b\}$. Let m_i be defined as usual for Q, so the above gives that $m_2 > m$. We have $L(f,Q) = m_1(a_0 - a) + m_2(b_0 - a_0) + m_3(b - b_0) > m(a_0 - a) + m(b_0 - a_0) + m(b - b_0) = m(b-a) = L(f,P)$, contradicting that all lower sums are equal.

5. (a) Suppose f is bounded and integrable on $[a, b]$, and that m is a lower bound for f on $[a, b]$ and M an upper bound. Show that

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

(b) With the same hypotheses as for the last part, show that there exists a number μ, satisfying $m \leq \mu \leq M$, such that

$$\int_a^b f(x)\, dx = \mu(b-a).$$

(c) Show that if f is integrable on $[a, b]$, and if $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f \geq 0$.

(d) Prove that if f and g are both integrable on $[a, b]$, and if $f(x) \geq g(x)$ for all $x \in [a, b]$, then $\int_a^b f \geq \int_a^b g$.

Solution:

(a) Let $m^* = \inf\{f(x) : x \in [a, b]\}$ and $M^* = \sup\{f(x) : x \in [a, b]\}$, so $m \leq m^*$ and $M \geq M^*$. Consider the partition $P = \{a, b\}$. We have $L(f, P) = m^*(b-a) \geq m(b-a)$ and $U(f, P) = M^*(b-a) \leq M(b-a)$. We have $L(f, P) \leq \int_a^b f \leq U(f, P)$, so $m(b-a) \leq \int_a^b f \leq M(b-a)$.

(b) Let $\mu = \frac{1}{b-a} \int_a^b f$, so $\mu(b-a) = \int_a^b f$. Since $b-a = 0$, we have $m \leq \mu \leq M$ if and only if $m(b-a) \leq \mu(b-a) \leq M(b-a)$, i.e. $m(b-a) \leq \int_a^b f \leq M(b-a)$, which we showed in part (a).

(c) Consider the partition $P = \{a, b\}$. Since $f(x) \geq 0$ for all $x \in [a, b]$, we have $L(f, P) \geq 0(b-a) = 0$, and since $\int_a^b f \geq L(f, P)$, we have $\int_a^b f \geq 0$.

(d) Let $h : [a, b] \to \mathbb{R}$ be given by $h(x) = f(x) - g(x)$, so $h(x) \geq 0$ for all $x \in [a, b]$. As a difference of integrable functions, h is integrable on $[a, b]$, so we can apply (c) to get $\int_a^b h = \int_a^b (f-g) = \int_a^b f - \int_a^b g \geq 0$, so $\int_a^b f \geq \int_a^b g$.

6. Suppose that f is weakly increasing (a.k.a non-decreasing) on $[a, b]$. The aim of this question is to show that f is integrable on $[a, b]$ *without making any assumption on the continuity or otherwise of f.*

(a) Prove that f is bounded on $[a, b]$.

(b) If $P = \{t_0 < t_1 < \cdots < t_n\}$ is a partition of $[a, b]$, what are $L(f, P)$ and $U(f, P)$?

(c) Suppose that P_n is the equipartition of $[a, b]$ into n subintervals, i.e.

$$P = \{t_0 < t_1 < \cdots < t_n\} \quad \text{with} \quad t_1 - t_0 = t_2 - t_1 = t_3 - t_2 = \cdots = t_n - t_{n-1}.$$

Calculate $U(f, P) - L(f, P)$ as a short, explicit expression, involving n, a and b, that doesn't involve a summation.

(d) Prove that f is integrable on $[a, b]$.

(e) Give an example of a bounded weakly increasing function on $[0, 1]$ which is discontinuous at infinitely many points (such a function is still integrable, by the last part of the question).

Solution:

(a) We have $f(a) \leq f(x) \leq f(b)$ for all $x \in [a, b]$ so f is bounded below by $f(a)$ and above by $f(b)$.

(b) We have $\inf\{f(x) : x \in [t_{i-1}, t_i]\} = f(t_{i-1})$ and $\sup\{f(x) : x \in [t_{i-1}, t_i]\} = f(t_i)$, so

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1}), \\ U(f, P) &= \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1}). \end{aligned}$$

(c) For each i, we have $t_i - t_{i-1} = \frac{b-a}{n}$, so

$$\begin{aligned} U(f, P) - L(f, P) &= \frac{b-a}{n} \sum_{i=1}^{n} (f(t_i) - f(t_{i-1})) \\ &= \frac{b-a}{n} (f(t_1) - f(t_0) + f(t_2) - f(t_1) + \cdots + f(t_n) - f(t_{n-1}) \\ &= \frac{b-a}{n} (f(t_n) - f(t_0)) \\ &= \frac{(b-a)(f(b) - f(a))}{n}. \end{aligned}$$

(d) Let $\varepsilon > 0$ be arbitrary. Assume $f(b) > f(a)$, because otherwise f is constant and the result is trivial. Let $n > \frac{(b-a)(f(b)-f(a))}{\varepsilon}$ be an integer. Then

$$\begin{aligned} U(f,P) - L(f,P) &= \frac{(b-a)(f(b)-f(a))}{n} \\ &< \frac{(b-a)(f(b)-f(a))}{\frac{(b-a)(f(b)-f(a))}{\varepsilon}} \\ &= \varepsilon. \end{aligned}$$

(e) Let $f : [0,1] \to \mathbb{R}$ be given by $f(x) = \begin{cases} \frac{1}{\lfloor \frac{1}{x} \rfloor}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. This function is discontinuous at $\frac{1}{n}$ for each $n \in \mathbb{N}$, and it is weakly increasing and bounded.

7. Recall the "stars over Babylon" function $s\colon [0,1] \to \mathbb{R}$ defined by

$$s(x) = \begin{cases} 0 & \text{if } x = 0, 1, \text{ or if } x \text{ is irrational,} \\ 1/q & \text{if } x \in \mathbb{Q} \text{ and } x = p/q \text{ in lowest terms.} \end{cases}$$

Is s integrable on $[0,1]$? If it is, calculate its integral. *Carefully* justify your answer!

Solution: s is integrable on $[0,1]$, with $\int_0^1 s = 0$. To see this, let $\varepsilon > 0$ be arbitrary and let $N > \frac{4}{3\varepsilon}$. Let S be the set of rational numbers $\frac{p}{q}$ in $[0,1]$ with $\gcd(p,q) = 1$ and with $q \leq N$. In particular, S is finite. Let $P = \{0 = t_0, t_1, \cdots, t_n = b\}$ be a partition of $[0,1]$ with $t_i - t_{i-1} < \frac{\varepsilon}{4|S|}$ for each i. There are at most $2|S|$ intervals $[t_{i-1}, t_i]$ that include a member of S ($2|S|$ because it is possible that each element of S is at the endpoint between two intervals, and that no interval has two elements of S in it). These intervals contribute at most $2|S| \cdot \frac{\varepsilon}{4|S|} \cdot \frac{1}{2} = \frac{\varepsilon}{4}$ to $U(s,P)$. For the rest of the intervals, we have $0 \leq f(x) \leq \frac{1}{N}$ for each x in the interval, so these contribute at most $\frac{1}{N} < \frac{3\varepsilon}{4}$ to the sum. Thus $U(s,P) < \frac{\varepsilon}{4} + \frac{3\varepsilon}{4} = \varepsilon$. It is clear that $L(s,P) = 0$ since the irrationals are dense. Thus $U(s,P) - L(s,P) < \varepsilon$, so s is integrable on $[0,1]$, and since $U(s,P)$ can be made arbitrarily close to 0, and $L(s,P) = 0$, we have $\int_0^1 s = 0$.