Math 60440: Basic Topology II
Problem Set 2

1. Let X be an n-dimensional CW-complex. Let the interiors of the n-cells be $\{U_\alpha\}_{\alpha \in I}$. For each $\alpha \in I$, pick a point $p_\alpha \in U_\alpha$. Prove that there exists a deformation retract from $X \setminus \{p_\alpha \mid \alpha \in I\}$ to X^{n-1}.

2. Let X be a CW-complex.

   (a) If X has only finitely many cells, then prove that X is compact.

   (b) Let $C \subset X$ be a compact subset (not necessarily a subcomplex). Prove that C only intersects finitely many cells of X, and in particular lies in $X^{(n)}$ for some n.

3. Prove that all the homotopy groups of S^∞ are trivial (hint: the previous step will help here!).

4. Given positive integers v and e and f satisfiying $v - e + f = 2$, construct a CW complex structure on S^2 with v zero-cells, e one-cells, and f two-cells. We remark that later in the course we will prove that $v - e + f = 2$ for all CW complex structures on S^2.

5. Let $\{p_1, \ldots, p_n\}$ be distinct points on S^2. Let X be the topological space obtained from S^2 by identifying all the p_i to a single point. Construct an explicit CW-complex structure on X.

6. Let $f\colon \widetilde{X} \to X$ be a covering space. Assume that X is endowed with the structure of a CW complex. Prove that $\widetilde{X}$ can be endowed with the structure of a CW complex such that f takes the interiors of k-cells in $\widetilde{X}$ homeomorphically to the interiors of k-cells in X. Hint: start by letting $\widetilde{X}^{(0)} = f^{-1}(X^{(0)})$. Next, construct $X^{(1)}$ by letting the 1-cells of $\widetilde{X}$ be all the paths in $\widetilde{X}$ obtained by lifting 1-cells of X, using the path lifting property of covering spaces. After this, construct $X^{(2)}$, then $X^{(3)}$, etc. At each stage, you will use the lifting criterion (in terms of the fundamental groups!) to figure out the attaching maps for the various cells.