Math 60440: Basic Topology II
Problem Set 1

1. Let (X, x_0) be a based space. Recall that when we defined the multiplication on $\pi_n(X, x_0)$, we stacked things using the first coordinate of maps $(I^n, \partial I^n) \to (X, x_0)$. Prove that using any of the other coordinates instead would result in the same group structure.

2. Let (X, x_0) be a path-connected based space. Prove that $\pi_k(X, x_0)$ is the trivial group if and only if every map $S^k \to X$ extends to a map of the disc D^{k+1}. Notice that these maps make no reference to the basepoint (hint: last semester you should have proved this for π_1, and you should meditate on that proof).

3. A space X is an *H-space* if there exists a point $e \in X$ (the *identity*) and a continuous map $m\colon X \times X \to X$ such that $m(x, e) = m(e, x) = x$ for all $x \in X$. For example, X might be a Lie group and m might be the multiplication. Let X be an H-space with identity e and let $f, g\colon (I^n, \partial I^n) \to (X, e)$ be continuous maps. Define $h\colon (I^n, \partial I^n) \to (X, e)$ via the formula $h(x) = m(f(x), g(x))$. Prove that h is homotopic to the map used to define $f \cdot g$ in the definition of $\pi_n(X, e)$.

4. In class, we constructed a set map $\pi_n(S^n, *) \to \mathbb{R}$ by pulling back a volume form and integrating (using Stokes Theorem to prove that this does not depend on the homotopy class of a map). Prove that this is a group homomorphism.

5. (a) Let $f\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a based covering map. Prove that the induced map $f_*\colon \pi_n(\widetilde{X}, \widetilde{x}_0) \to \pi_n(X, x_0)$ is an isomorphism for $n \geq 2$.

   (b) Let $X = S^2 \vee S^1$, with wedge point $*$. Prove that $\pi_2(X, *)$ is not finitely generated. Hint: construct a homomorphism

$$\rho\colon \pi_2(X, *) \to \bigoplus_{i \in \mathbb{Z}} \mathbb{R}$$

   whose image contains $\oplus_i \mathbb{Z}$ by looking at the universal cover and using the degree map we constructed in class (which in class we only showed had image in $\mathbb{R}$ containing $\mathbb{Z}$).