Math 30810: Honors Algebra III
Problem Set 5

Do the following problems:

1. Let G be a group such that $x^2 = 1$ for all $x \in G$. Prove that G is abelian.

2. Let $H \subset \mathbb{Q}$ be a finitely generated subgroup. Prove that $H \cong \mathbb{Z}$.

3. Let G be an abelian group and let $g_1, \ldots, g_k \in G$ be elements. Prove that there exists a unique homomorphism $\phi\colon \mathbb{Z}^k \to G$ such that ϕ takes i^{th} basis element of $\mathbb{Z}^k$ (i.e. the k-tuple of integers with a 1 in position i and zeros elsewhere) to g_i for all $1 \leq i \leq k$.

4. A group homomorphism $\phi\colon G \to Q$ is said to *split* if there exists another homomorphism $\psi\colon Q \to G$ such that $\phi\colon \psi = \text{id}$.

   (a) Prove that $\phi\colon G \to Q$ is split, then ϕ is surjective.

   (b) Give an example of a non-split surjective group homomorphism.

   (c) Prove that if G is an abelian group and $\phi\colon G \to Q$ is a split homomorphism, then $G \cong \ker(\phi) \oplus Q$.

   (d) Prove that if G is a finitely generated abelian group and $\phi\colon G \to Q$ is a surjective homomorphism such that Q is a subgroup of $\mathbb{Q}$, then ϕ is split (hint: this requires the classification of finitely generated abelian groups together with problem (4)).

5. Say that an element g of an abelian group G is *unimodular* if there exists a basis $\{x_1, \ldots, x_n\}$ for G such that $x_1 = g$.

   (a) Consider a unimodular element $g \in \mathbb{Z}^n$. Write $g = (a_1, \ldots, a_n)$ with $a_i \in Z$. Prove that $\gcd(a_1, \ldots, a_n) = 1$.

   (b) Consider an element $g \in \mathbb{Z}^n$. Write $g = (a_1, \ldots, a_n)$ with $a_i \in \mathbb{Z}$. Say that $g' \in \mathbb{Z}^n$ is the result of performing an *elementary operation* on g if g' is obtained by doing one of the following things:

      (i) Permuting the entries of g.

      (ii) Multiplying one of the entries of g by -1.

      (iii) For some $1 \leq i, j \leq n$ with $i \neq j$ and some $k \in \mathbb{Z}$, replacing a_j with $a_j + ka_i$ and fixing every other entry.

      Prove that if $g' \in \mathbb{Z}$ is obtained by performing an elementary operation to $g \in \mathbb{Z}^n$ and g' is unimodular, then g is unimodular.

   (c) Consider an element $g = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ such that $\gcd(a_1, \ldots, a_n) = 1$. Prove that g is unimodular. (hint: prove that you can perform a sequence of elementary operations to g to transform it into $(1, 0, \ldots, 0)$, which is clearly unimodular. for this, you'll want to first multiply the entries by

-1 to make them all nonnegative, then permute the so that $a_1 \leq a_i$ for all $2 \leq i \leq n$, and then add multiples of a_1 to a_i to make it so that $0 \leq a_i < a_1$).