Math 60330: Basic Geometry and Topology
Problem Set 8

1. (a) Let $U \subset \mathbb{R}^n$ be an open subset and let $f\colon U \to \mathbb{R}$ be a smooth function. Prove that

$$\operatorname{grad}(f)(p) = \frac{\partial f}{\partial x_1}(p)\frac{\partial}{\partial x_1} + \cdots + \frac{\partial f}{\partial x_n}(p)\frac{\partial}{\partial x_n},$$

where $\operatorname{grad}(f)$ is the vector field on U defined in class.

(b) Regard S^n as a submanifold of $\mathbb{R}^{n+1}$ in the usual way, and write $f\colon S^n \to \mathbb{R}$ via the formula

$$f(x_1, \ldots, x_{n+1}) = x_1^2 + 2x_2^2 + \cdots + (n+1)x_{n+1}^2.$$

Calculate $\operatorname{grad}(f)$ (warning: you can't just regard f as a function on $\mathbb{R}^{n+1}$ and take its gradient – the result will not even be a vector field on S^n).

2. Let M be a smooth manifold and let ν be a vector field on M. Recall from the previous homework that given a function $f\colon M \to \mathbb{R}$ and a tangent vector $\vec{v} \in T_pM$, we can define $\nabla_{\vec{v}}(f) \in \mathbb{R}$. Given a smooth function $f\colon M \to \mathbb{R}$, we define $\nu(f)\colon M \to \mathbb{R}$ via the formula

$$\nu(f)(p) = \nabla_{\nu(p)}(f) \qquad (p \in M).$$

(a) Let $\phi_t\colon M \to M$ be the flow generated by ν. Prove that

$$\nu(f)(p) = \frac{(f \circ \phi_t)(p)}{\partial t}|_{t=0}$$

for all $p \in M$.

(b) Given vector fields ν_1 and ν_2 on M, prove that there exists a unique vector field $[\nu_1, \nu_2]$ on M such that

$$[\nu_1, \nu_2](f) = \nu_1(\nu_2(f)) - \nu_2(\nu_1(f))$$

for all smooth functions $f\colon M \to \mathbb{R}$.

(c) Prove that if ν_1 and ν_2 and ν_3 are smooth vector fields on M, then

$$[\nu_1, [\nu_2, \nu_3]] + [\nu_2, [\nu_3, \nu_1]] + [\nu_3, [\nu_1, \nu_2]] = 0.$$

3. Let G be a Lie group. A vector field ν on G is said to be *left invariant* if for all $g \in G$, we have $(m_g)_*(\nu) = \nu$, where $m_g\colon G \to G$ is the diffeomorphism that multiplies by g on the left. The set of all left-invariant vector fields on G is a vector space called the *Lie algebra* of G.

(a) Letting e be the identity element of G, construct a vector space isomorphism between the Lie algebra of G and T_eG.

(b) Prove that if ν_1 and ν_2 are left-invariant vector fields on G, then $[\nu_1, \nu_2]$ is also a left invariant vector fields on G.

(c) Look up the definition of a Lie algebra (say on wikipedia) and verify that with the aforementioned bracket operation the Lie algebra of G is indeed a Lie algebra.

(d) Prove that the Lie algbera of $\mathrm{GL}_n(\mathbb{R})$ is precisely the set of $n \times n$ real matrices with the bracket

$$[A, B] = AB - BA.$$