A geometrically-minded introduction to smooth manifolds

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CHAPTER 1

Smooth manifolds

This chapter defines smooth manifolds and gives some basic examples. We also discuss smooth partitions of unity.

1.1. The definition

We start with the definition of a manifold (not yet smooth).

DEFINITION. A *manifold* of dimension n is a second countable Hausdorff space M^n such that for every $p \in M^n$ there exists an open set $U \subset M^n$ containing p and a homeomorphism $\phi : U \to V$, where $V \subset \mathbb{R}^n$ is open. The map $\phi : U \to V$ is a *chart* around p. We will often call V a *local coordinate system* around p and identify it via ϕ^{-1} with a subset of M^n. □

REMARK. We require M^n to be Hausdorff and second countable to avoid various pathologies, some of which are discussed in the exercises. The existence of charts is the real fundamental defining property of a manifold. □

Our goal is to learn how to do calculus on manifolds. The idea is that notions like derivatives are *local*: they only depend on the behavior of functions in small neighborhoods of a point. We can thus use charts and local coordinate systems to identify small pieces of our manifold with open sets in $\mathbb{R}^n$ and thereby apply calculus in $\mathbb{R}^n$ to our manifolds. As a test case, we would like to say what it means for a function on a manifold to be smooth. On an open subset of $\mathbb{R}^n$, the correct definition is as follows.

DEFINITION. Let $U \subset \mathbb{R}^n$ be open. A function $f\colon U \to \mathbb{R}^m$ is *smooth* if it satisfies the following condition. Let $f = (f_1, \ldots, f_m)$ be the component functions of f, so $f_i\colon U \to \mathbb{R}$ is a function for all $1 \leq i \leq m$. We then require that the mixed partial deriviatives of all orders exist for f_i for all $1 \leq i \leq m$. □

If we tried to use charts to use this to say what it means for a function $f\colon M^n \to \mathbb{R}^m$ to be smooth, then we would immediately run into problems: different charts might give completely unrelated notions of smoothness. To fix this, we will have to carefully choose our charts.

DEFINITION. Given two charts $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ on a manifold M^n, the *transition function* from U_1 to U_2 is the function $\tau_{21} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ defined via the formula $\tau_{21} = \phi_2 \circ (\phi_1|_{\phi_1(U_1 \cap U_2)})^{-1}$. Here observe that $\phi_1(U_1 \cap U_2)$ is an open subset of $V_1 \subset \mathbb{R}^n$ and $\phi_2(U_1 \cap U_2)$ is an open subset of $V_2 \subset \mathbb{R}^n$. □

DEFINITION. A *smooth atlas* for a manifold M^n is a set $\mathcal{A} = \{\phi_i : U_i \to V_i\}_{i \in I}$ of charts on M^n with the following properties.

- The U_i cover M^n, i.e. $M^n = \cup_{i \in I} U_i$.

- For all $i, j \in I$, the transition function from U_1 to U_2 is smooth in the sense of Definition 1.1. Of course, this only has content if $U_i \cap U_j \neq \emptyset$.

Two smooth atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are *compatible* if $\mathcal{A}_1 \cup \mathcal{A}_2$ is also a smooth atlas. This defines an equivalence relation on smooth atlases. A *smooth manifold* is a manifold equipped with an equivalence class of smooth atlases. □

REMARK. We will give examples of manifolds by describing an atlas for them. However, this atlas is not a fundamental property of the manifold, and when we subsequently make use of charts for the manifold we will allow ourselves to use charts from any equivalent atlas. The first place where this freedom will play an important role is when we define what it means for a function between two smooth manifolds to be smooth. □

1.2. Basic examples

Here are a number of examples.

EXAMPLE. If $U \subset \mathbb{R}^n$ is an open set, then U is naturally a smooth manifold with the smooth atlas $\mathcal{A}$ consisting of a single chart $\phi : U \to V$ with $V = U$ and $\phi = \text{id}$. These can be complicated and wild; for instance, U might be the complement of a Cantor set embedded in $\mathbb{R}^n$. □

EXAMPLE. An important special case of an open subset of Euclidean space is the general linear group $\text{GL}_n(\mathbb{R})$. The set $\text{Mat}(n, n)$ of $n \times n$ real matrices can be identified with $\mathbb{R}^{n^2}$ in the obvious way, and $\text{GL}_n(\mathbb{R})$ is the complement of the closed subset where the determinant vanishes. This is an example of a *Lie group*, that is, a smooth manifold which is also a group and for which the group operations are continuous (and, in fact, smooth). □

EXAMPLE. More generally, if M^n is a smooth manifold with smooth atlas $\mathcal{A} = \{\phi_i : U_i \to V_i\}_{i \in I}$ and $U \subset M^n$ is an open subset, then U is naturally a smooth manifold with smooth atlas $\{\phi_i|_{U \cap U_i} : U_i \cap U \to \phi_i(U \cap U_i)\}_{i \in I}$. □

EXAMPLE. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$, i.e.

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

Then S^n can be endowed with the following smooth atlas. For $1 \leq i \leq n+1$, define

$$U_{x_i > 0} = \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_i > 0\}$$

and

$$U_{x_i < 0} = \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_i < 0\}.$$

Let $V \subset \mathbb{R}^n$ be the open unit disc. Define $\phi_{x_i > 0} : U_{x_i > 0} \to V$ via the formula

$$\phi_{x_i > 0}(x_1, \ldots, x_{n+1}) = (x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}) \in V;$$

here $\widehat{x_i}$ indicates that this single coordinate should be omitted. Define $\phi_{x_i < 0} : U_{x_i < 0} \to V$ similarly. We claim that

$$\mathcal{A} = \{\phi_{x_i > 0} : U_{x_i > 0} \to V\}_{i=1}^{n+1} \cup \{\phi_{x_i < 0} : U_{x_i < 0} \to V\}_{i=1}^{n+1}$$

is a smooth atlas. Since the $U_{x_i > 0}$ and $U_{x_i < 0}$ clearly cover S^n, it is enough to check that the transition functions are smooth. As an illustration of this, we will verify that for $1 \leq i < j \leq n+1$ the transition function τ from $U_{x_i > 0}$ to $U_{x_j > 0}$ is smooth (all the other needed verifications are similar, and this will allow us to avoid introducing some terrible notation for the various special cases). Define

$V_{ij} = \phi_i(U_{x_i>0} \cap U_{x_j>0})$ and $V_{ji} = \phi_j(U_{x_i>0} \cap U_{x_j>0})$, so V_{ij} consists of points $(y_1, \ldots, y_n) \in V$ such that $y_{j-1} > 0$ and V_{ji} consists of points $(y_1, \ldots, y_n) \in V$ such that $y_i > 0$. The transition function $\tau_{ji} : V_{ij} \to V_{ji}$ is then given by the formula

$$\begin{aligned}\tau_{ji}(y_1, \ldots, y_n) &= \phi_{x_j>0}(\phi^{-1}_{x_i>0}(y_1, \ldots, y_n)) \\ &= \phi_{x_j>0}(y_1, \ldots, y_{i-1}, \sqrt{1 - y_1^2 - \cdots - y_n^2}, y_i, \ldots, y_n) \\ &= (y_1, \ldots, y_{i-1}, \sqrt{1 - y_1^2 - \cdots - y_n^2}, y_i, \cdots, \widehat{y_{j-1}}, \ldots, y_n).\end{aligned}$$

This is clearly a smooth function. □

Example. Here is another smooth atlas for S^n. Let $U_1 = S^n \setminus \{(0,0,1)\}$ and $U_{-1} = S^n \setminus \{(0,0,-1)\}$. Identifying $\mathbb{R}^n$ with the subspace of $\mathbb{R}^{n+1}$ consisting of points whose last coordinate is 0, define a function $\phi_1 : U_1 \to \mathbb{R}^n$ by letting $\phi_1(p)$ be the unique intersection point of the line joining $p \in U_1 \subset S^n \subset \mathbb{R}^{n+1}$ and $(0,0,1)$ with the plane $\mathbb{R}^n$. It is clear that ϕ_1 is a homeomorphism. Similarly, define $\phi_{-1} : U_{-1} \to \mathbb{R}^n$ by letting $\phi_{-1}(p)$ be the unique intersection point of the line joining $p \in U_{-1} \subset S^n \subset \mathbb{R}^{n+1}$ and $(0,0,-1)$ with the plane $\mathbb{R}^n$. Again, ϕ_{-1} is a homeomorphism. In the exercises, you will show that the set $\{\phi_1 : U_1 \to \mathbb{R}^n, \phi_{-1} : U_{-1} \to \mathbb{R}^n\}$ is a smooth atlas for S^n which is equivalent to the smooth atlas for S^n given in the previous example. □

Example. Define $\mathbb{RP}^n$ to be *real projective space*, i.e. the quotient $S^n/\sim$, where $\sim$ identifies antipodal points (that is, $x \sim -x$ for all $x \in S^n$). For $1 \leq i \leq n+1$, define $U_i \subset \mathbb{RP}^n$ to be the image of $U_{x_i>0} \subset S^n$ under the quotient map $S^n \to \mathbb{RP}^n$. Since $U_{x_i>0}$ does not contain any antipodal points, the map $U_{x_i>0} \to U_i$ is a homeomorphism. Clearly the U_i cover $\mathbb{RP}^n$. Letting V be the unit disc in $\mathbb{R}^n$, we can define homeomorphisms $\phi_i : U_i \to V$ as the composition

$$U_i \cong U_{x_i>0} \xrightarrow{\phi_{x_i>0}} V.$$

The set $\mathcal{A} = \{\phi_i : U_i \to V\}_{i=1}^{n+1}$ then forms a smooth atlas for $\mathbb{RP}^n$; the fact that the transition maps for the sphere are smooth implies that the transition maps for $\mathcal{A}$ are. □

Example. For $j = 1, 2$, let $M_j^{n_j}$ be a smooth n_j-dimensional manifold with smooth atlas $\{\phi_i^j : U_i^j \to V_i^j\}_{i \in I_j}$. Then $M_1^{n_1} \times M_2^{n_2}$ is a smooth (n_1+n_2)-dimensional manifold with smooth atlas $\{\phi_i^1 \times \phi_{i'}^2 : U_i^1 \times U_{i'}^2 \to V_i^1 \times V_{i'}^2\}_{(i,i') \in I_1 \times I_2}$. An important special case of a product is the *n-torus*, i.e. the product $S^1 \times \cdots \times S^1$ of n copies of S^1. □

For our final family of examples of smooth manifolds, we need the following definition.

Definition. Let $X \subset \mathbb{R}^n$ be an arbitrary set and let $f : X \to \mathbb{R}^m$ be a function. We say that f is *smooth at a point* $p \in X$ if there exists an open set $U \subset \mathbb{R}^n$ with $p \in U$ and a smooth function $g \colon U \to \mathbb{R}^m$ such that $g|_{U \cap X} = f|_{U \cap X}$. We will say that f is *smooth* if f is smooth at all points $p \in X$. If $Y \subset \mathbb{R}^m$ is the image of f, then we will say that $f \colon X \to Y$ is a *diffeomorphism* if it is a homeomorphism and both $f : X \to Y$ and $f^{-1} : Y \to X$ are smooth. □

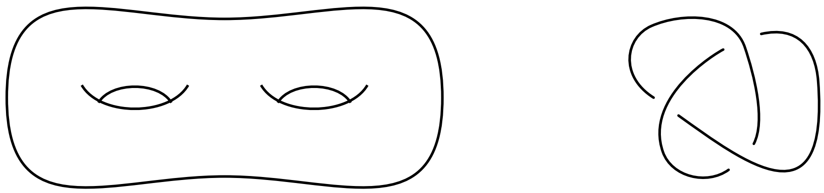

FIGURE 1.1. On the left is a genus 2 surface (a "donut with two holes"), which is a 2-dimensional smooth submanifold of $\mathbb{R}^3$. On the right is a trefoil knot, which is a 1-dimensional smooth submanifold of $\mathbb{R}^3$.

EXAMPLE. An *n-dimensional smooth submanifold of* $\mathbb{R}^m$ is a subset $M^n \subset \mathbb{R}^m$ such that for each point $p \in M^n$, there exists a chart $\phi : U \to V$ around p such that ϕ is a diffeomorphism. Here we emphasize that we are using the definition of diffeomorphism discussed in the previous definition. The collection of all such charts forms a smooth atlas on M^n; the fact that we require the charts to be diffeomorphisms makes the fact that the transition functions are smooth automatic. It is easy to draw many interesting examples of smooth submanifolds of $\mathbb{R}^3$; see, for example, the genus 2 surface and the knotted circle in Figure 1.1. □

REMARK. The charts in the first smooth atlas on S^n we gave above are diffeomorphisms, so we were really making use of the fact that S^n is an n-dimensional smooth submanifold of $\mathbb{R}^{n+1}$. □

REMARK. In fact, all smooth manifolds can be realized as smooth submanifolds of $\mathbb{R}^m$ for some $m \gg 0$ (in other words, all smooth manifolds can be "embedded" in $\mathbb{R}^m$). We will prove this for compact smooth manifolds in Theorem 3.5 below. □

1.3. Smooth functions

As we said before defining them, one of the reasons for introducing smooth atlases is to allow us to talk about smooth functions on a manifold. The appropriate definition is as follows.

DEFINITION. Let M^n be a smooth n-manifold and let $f : M^n \to \mathbb{R}$ be a function. We say that f is *smooth at a point* $p \in M^n$ if the following condition holds.

- Let $\phi : U \to V$ be a chart such that $p \in U$. Then the function $f \circ \phi^{-1} : V \to \mathbb{R}$ is smooth at $\phi(p)$. Here V is an open subset of $\mathbb{R}^n$, so smoothness is as defined in Definition 1.1.

We say that f is *smooth* if it is smooth at all points $p \in M^n$. We will denote the set of all smooth functions on M^n by $C^\infty(M^n, \mathbb{R})$. □

LEMMA 1.1. *The notion of $f : M^n \to \mathbb{R}$ being smooth at a point $p \in M^n$ is well-defined, i.e. it does not depend on the choice of chart $\phi : U \to V$ such that $p \in U$.*

PROOF. Let $\phi_1 : U_1 \to V_1$ be another chart such that $p \in U_1$. We must prove that $f \circ \phi^{-1} : V \to \mathbb{R}$ is smooth at $\phi(p)$ if and only if $f \circ \phi_1^{-1} : V_1 \to \mathbb{R}$ is smooth at $\phi_1(p)$. Let $\tau : \phi(U \cap U_1) \to \phi_1(U \cap U_1)$ be the transition map between our two charts, so $\tau = \phi_1 \circ (\phi|_{U \cap U_1})^{-1}$. On $\phi(U \cap U_1)$, we have

$$f \circ \phi^{-1} = f \circ \phi_1^{-1} \circ \phi_1 \circ \phi^{-1} = f \circ \phi_1^{-1} \circ \tau.$$

Since τ is smooth, the function $f \circ \phi^{-1}$ is smooth at $\phi(p)$ if and only if the function $f \circ \phi_1^{-1}$ is smooth at $\phi_1(p)$, as desired. $\square$

DEFINITION. If $f : M^n \to \mathbb{R}$ is a smooth function on M^n and $\phi : U \to V$ is a chart on M^n, then the smooth function $f \circ \phi^{-1} : V \to \mathbb{R}$ will be called the *expression for f in the local coordinates V*. $\square$

REMARK. If M^n is a smooth submanifold of $\mathbb{R}^m$, then we now have two different definitions of what it means for a function $f : M^n \to \mathbb{R}$ to be smooth:

(1) The definition we just gave, and
(2) The definition given right before the definition of a smooth submanifold of $\mathbb{R}^m$, i.e. a function $f : M^n \to \mathbb{R}$ that can locally be extended to a smooth function on an open subset of $\mathbb{R}^m$.

It is clear that the second definition implies the first. This allows us to write down many examples of smooth functions. For example, regarding S^n as a smooth submanifold of $\mathbb{R}^{n+1}$, the function $f : S^n \to \mathbb{R}$ defined via the formula

$$f(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} i x_i^{2i+1}$$

is smooth. It is more difficult to see that the first definition implies the second. We will prove this in Chapter 3; see Lemma 3.4. $\square$

Defining what it means for a map between arbitrary manifolds to be smooth is a little complicated. Consider the following example.

EXAMPLE. Define a map $f : \mathbb{R} \to S^1$ via the formula $f(t) = (\cos(t), \sin(t)) \in S^1 \subset \mathbb{R}^2$. We clearly want f to be smooth. Recall that $\mathbb{R}$ is endowed with the smooth atlas with a single chart, namely the identity map $\mathbb{R} \to \mathbb{R}$. The image of this chart under f is not contained in any single chart for S^1, so we cannot define smoothness for f locally using this smooth atlas. $\square$

The problem with the above example is that we really need to use "smaller" charts on $\mathbb{R}$. We now adapt the following convention to circumvent this.

CONVENTION. If M^n is a smooth manifold with smooth atlas $\mathcal{A}$, then we will automatically enlarge $\mathcal{A}$ to the maximal atlas compatible with $\mathcal{A}$ (remember our equivalence relation on smooth atlases!). In particular, if $\phi : U \to V$ is a chart for M^n, then so is $\phi|_{U'} : U' \to \phi(U')$ for any open set $U' \subset U$. $\square$

With this convention, we make the following definition.

DEFINITION. Let $f : M_1^{n_1} \to M_2^{n_2}$ be a map between smooth manifolds. We say that f is *smooth at a point* $p \in M_1^{n_1}$ if there exist charts $\phi_1 : U_1 \to V_1$ for $M_1^{n_1}$ and $\phi_2 : U_2 \to V_2$ for $M_2^{n_2}$ with the following properties.

- $p \in U_1$.
- $f(U_1) \subset U_2$.
- The composition

$$V_1 \xrightarrow{\phi_1^{-1}} U_1 \xrightarrow{f} U_2 \xrightarrow{\phi_2} V_2$$

is smooth at $\phi_1(p)$; this makes sense since V_1 and V_2 are open subsets of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively.

We say that f is *smooth* if it is smooth at all points $p \in M_1^{n_1}$. We will denote the set of all smooth functions from $M_1^{n_1}$ to $M_2^{n_2}$ by $C^\infty(M_1^{n_1}, M_2^{n_2})$. A *diffeomorphism* is a smooth bijection whose inverse is also smooth. $\square$

Just like for real-valued smooth functions, this does not depend on the choice of charts.

Definition. If $f : M_1^{n_1} \to M_2^{n_2}$ is a smooth function between smooth manifolds, $\phi_1 : U_1 \to V_1$ is a chart for $M_1^{n_1}$, and $\phi_2 : U_2 \to V_2$ is a chart for $M_2^{n_2}$ such that $f(U_1) \subset U_2$, then the smooth function $V_1 \to V_2$ obtained as the composition

$$V_1 \xrightarrow{\phi_1^{-1}} U_1 \xrightarrow{f} U_2 \xrightarrow{\phi_2} V_2$$

will be called the *expression for f in the local coordinates V_1 and V_2*. $\square$

Example. It is immediate that the function $f : \mathbb{R} \to S^1$ discussed above defined via the formula $f(t) = (\cos(t), \sin(t)) \in S^1 \subset \mathbb{R}^2$ is smooth. $\square$

Remark. Just as before, if M_1 and M_2 are smooth submanifolds of Euclidean space this definition agrees with the definition given just before the definition of smooth submanifolds. This allows us to write down many interesting examples of smooth maps. For example, regarding S^1 as a smooth submanifold of $\mathbb{R}^2$ we can define a smooth map $f : S^1 \to S^1$ via the formula $f(x_1, x_2) = (x_1^2 - x_2^2, 2x_1x_2)$. $\square$

1.4. Partitions of unity

We now introduce an important technical device. In calculus, we learned how to construct many interesting functions on open subsets of $\mathbb{R}^n$. To use these functions to prove theorems about manifolds, we need a tool for assembling local information into global information. This tool is called a *smooth partition of unity*, which we now define. Recall that if $f : M^n \to \mathbb{R}$ is a function, then the *support* of f, denoted $\mathrm{Supp}(f)$, is the closure of the set $\{x \in M^n \mid f(x) \neq 0\}$.

Definition. Let M^n be a smooth manifold and let $\{U_i\}_{i \in I}$ be an open cover of M^n. A *smooth partition of unity* subordinate to $\{U_i\}_{i=1}^k$ is a collection of smooth functions $\{f_i : M^n \to \mathbb{R}\}_{i \in I}$ satisfying the following properties.

- We have $0 \leq f_i(x) \leq 1$ for all $1 \leq i \leq k$ and $x \in M^n$.
- We have $\mathrm{Supp}(f_i) \subset U_i$ for all $1 \leq i \leq k$.
- For all $p \in M^n$, there exists an open neighborhood W of p such that the set $\{i \in I \mid W \cap \mathrm{Supp}(f_i) \neq \emptyset\}$ is finite.
- For all $p \in M^n$, we have $\sum_{i \in I} f_i(p) = 1$. This sum makes sense since the previous condition ensures that only finitely many terms in it are nonzero. $\square$

Theorem 1.2 (Existence of partitions of unity). *Let M^n be a smooth manifold and let $\{U_i\}_{i \in I}$ be an open cover of M^n. Then there exists a smooth partition of unity subordinate to $\{U_i\}_{i \in I}$.*

For the proof of Theorem 1.2, we need the following lemma.

Lemma 1.3 (Bump functions, weak). *Let M^n be a smooth manifold, let $p \in M^n$ be a point, and let $U \subset M^n$ be a neighborhood of p. Then there exists a smooth* *function $f : M^n \to \mathbb{R}$ such that $0 \le f(x) \le 1$ for all $x \in M^n$, such that f equals 1 in some neighborhood of p, and such that* $\mathrm{Supp}(f) \subset U$.

PROOF. We will construct f in a sequence of steps.

STEP 1. *There exists a smooth function $g : \mathbb{R} \to \mathbb{R}$ such that $0 \le g(x) \le 1$ for all $x \in \mathbb{R}$, such that $g(x) = 1$ when $|x| \le 1$, and such that* $\mathrm{Supp}(g) \subset (-3, 3)$.

Define $g_1 : \mathbb{R} \to \mathbb{R}$ via the formula

$$g_1(x) = \begin{cases} 0 & \text{if } x \le 0, \\ e^{-1/x} & \text{if } x > 0. \end{cases} \qquad (x \in \mathbb{R}).$$

The function g_1 is a smooth function such that $g_1(x) \ge 0$ for all $x \in \mathbb{R}$, such that $g_1(x) = 0$ when $x \le 0$, and such that $g_1(x) > 0$ when $x > 0$. Next, define $g_2 : \mathbb{R} \to \mathbb{R}$ via the formula

$$g_2(x) = \frac{g_1(x)}{g_1(x) + g_1(1-x)},$$

so g_2 is a smooth function such that $0 \le g_2(x) \le 1$ for all $x \in \mathbb{R}$, such that $g_2(x) = 0$ when $x \le 0$, and such that $g_2(x) = 1$ when $x \ge 1$. Finally, define g via the formula

$$g(x) = g_1(2+x)g_1(2-x).$$

Clearly g satisfies the desired conditions.

STEP 2. *Let $C_0 = \{x \in \mathbb{R}^n \mid \|x\| \le 1\}$ and $U_0 = \{x \in \mathbb{R}^n \mid \|x\| < 2\}$. Then there exists a smooth function $h : \mathbb{R}^n \to \mathbb{R}$ such that $0 \le h(x) \le 1$ for all $x \in \mathbb{R}^n$, such that $h|_{C_0} = 1$, and such that* $\mathrm{Supp}(h) \subset U_0$.

Let g be as in Step 1. Define h via the formula

$$h(x_1, \ldots, x_n) = g(x_1^2 + \cdots + x_n^2).$$

Clearly h satisfies the desired conditions.

STEP 3. *There exists a smooth function f as in the statement of the lemma.*

Let C_0 and U_0 and h be as in Step 2. We can then find an open set $U' \subset U$ such that $p \in U'$ and a diffeomorphism $\phi : U' \to V$, where V is either an open subset of $\mathbb{R}^n$ containing U_0 or an open subset of $\mathbb{H}^n$ containing $U_0 \cap \mathbb{H}^n$ and $\phi(p) = 0$. The function $f : M^n \to \mathbb{R}$ can then be defined via the formula

$$f(x) = \begin{cases} g(\phi(x)) & \text{if } x \in U', \\ 0 & \text{otherwise.} \end{cases} \qquad (x \in M^n).$$

Clearly f satisfies the conditions of the lemma. □

PROOF OF THEOREM 1.2. Since M^n is paracompact and locally compact, we can find open covers $\{U'_j\}_{j \in J}$ and $\{U''_j\}_{j \in J}$ of M^n with the following properties.

- The cover $\{U'_j\}_{j \in J}$ refines the cover $\{U_i\}_{i \in I}$, i.e. for all $j \in J$ there exists some $i_j \in I$ such that the closure of U'_j is contained in U_{i_j}.
- The cover $\{U'_j\}_{j \in J}$ is locally finite, i.e. for all $p \in M^n$ there exists some open neighborhood W of p such that $\{j \in J \mid W \cap U'_j \ne \emptyset\}$ is finite.
- The closure of U''_j is a compact subset of U'_j for all $j \in J$.

For each $p \in M^n$, choose j_p such that $p \in U''_{j_p}$ and use Lemma 1.3 to find a smooth function $g_p : M^n \to \mathbb{R}$ such that $0 \leq g_p(x) \leq 1$ for all $x \in M^n$, such that $\mathrm{Supp}(g_p) \subset U'_{j_p}$, and such that g_p equals 1 in some neighborhood V_p of p. Since the closure of U''_j in U'_j is compact for all $j \in J$, we can find a set $\{p_k\}_{k \in K}$ of points of M^n such that the set $\{V_{p_k} \mid k \in K,\ j_{p_k} = j\}$ is a finite cover of U''_j for all $j \in J$. For all $j \in J$, define $h_j : M^n \to \mathbb{R}$ to be the sum of all the g_{p_k} such that $j_{p_k} = j$ (a finite sum), so h_j is a smooth function such that $h_j(x) \geq 0$ for all $x \in M^n$, such that $h_j(x) > 0$ for all $x \in U''_j$, and such that $\mathrm{Supp}(h_j) \subset U'_j$. Finally, for all $i \in I$, define $f_i : M^n \to \mathbb{R}$ via the formula

$$f_i(x) = \frac{\sum_{i_j = i} h_j(x)}{\sum_{j \in J} h_j(x)} \qquad (x \in M^n).$$

These are not finite sums, but because the cover $\{U'_j\}_{j \in J}$ is locally finite and $\mathrm{Supp}(h_j) \subset U'_j$ for all $j \in J$, only finitely many terms in each are nonzero for any choice of $x \in M^n$ and the numerator and denominator are smooth functions. Also, the denominator is nonzero since $h_j(x) > 0$ for all $x \in U''_j$ and the set $\{U''_j\}_{j \in J}$ is a cover.

By construction, we have $\mathrm{Supp}(f_i) \subset U_i$. Moreover, for all $x \in M^n$ the fact that the cover $\{U'_j\}_{j \in J}$ is locally finite and $\mathrm{Supp}(h_j) \subset U'_j$ for all $j \in J$ implies that there exists some open neighborhood W of x such that the set $\{i \in I \mid W \cap \mathrm{Supp}(f_i) = \emptyset\}$ is finite. Finally, for all $x \in M^n$ we have

$$\sum_{i \in I} f_i(x) = \frac{\sum_{i \in I} \sum_{i_j = i} h_j(x)}{\sum_{j \in J} h_j(x)} = \frac{\sum_{j \in J} h_j(x)}{\sum_{j \in J} h_j(x)} = 1,$$

as desired. □

As a first illustration of how Theorem 1.2 can be used, we prove the following lemma.

Lemma 1.4 (Bump functions, strong). *Let M^n be a smooth manifold, let $C \subset M^n$ be a closed set, and let $U \subset M^n$ be an open set such that $C \subset U$. Then there exists a smooth function $f : M^n \to \mathbb{R}$ such that $0 \leq f(x) \leq 1$ for all $x \in M^n$, such that $f(x) = 1$ for all $x \in C$, and such that* $\mathrm{Supp}(f) \subset U$.

Proof. Set $U' = M^n \setminus C$. The set $\{U, U'\}$ is then an open cover of M^n. Using Theorem 1.2, we can find smooth functions $f : M^n \to \mathbb{R}$ and $g : M^n \to \mathbb{R}$ such that $0 \leq f(x), g(x) \leq 1$ for all $x \in M^n$, such that $\mathrm{Supp}(f) \subset U$ and $\mathrm{Supp}(g) \subset U'$, and such that $f + g = 1$. The function f then satisfies the conditions of the lemma. □

This has the following useful consequence. Just like for functions on Euclidean space, if C is an arbitrary subset of a smooth manifold M_1 and $f : C \to M_2$ is a function to another smooth manifold, then f is said to be smooth if there exists an open set $U \subset M_1$ containing C and a smooth function $g : U \to M_2$ such that $g|_C = f$.

Lemma 1.5 (Extending smooth functions). *Let M be a smooth manifold, let $C \subset M$ be a closed set, and let $U \subset M$ be an open set such that $C \subset U$. Let $f : C \to \mathbb{R}$ be a smooth function. Then there exists a smooth function $g : M \to \mathbb{R}$ such that $g|_C = f$ and such that* $\mathrm{Supp}(g) \subset U$.

Proof. By definition, there exists an open set $U' \subset M$ containing C and a smooth function $g_1 : U' \to \mathbb{R}$ such that $g_1|_C = f$. Shrinking U' if necessary, we can assume that $U' \subset U$. Use Lemma 1.4 to construct a smooth function $h : M \to \mathbb{R}$ such that $0 \leq h(x) \leq 1$ for all $x \in M$, such that $h(x) = 1$ for all $x \in C$, and such that $\text{Supp}(h) \subset U'$. Define $g : M \to \mathbb{R}$ via the formula

$$g(x) = \begin{cases} h(x)g_1(x) & \text{if } x \in U', \\ 0 & \text{otherwise.} \end{cases} \qquad (x \in M).$$

Clearly g satisfies the conclusions of the lemma. □

1.5. Approximating continuous functions, I

As another illustration of how partitions of unity can be used, we will prove the following.

Theorem 1.6. *Let M^n be a smooth manifold and let $f : M^n \to \mathbb{R}^m$ be a continuous function. Then for all $\epsilon > 0$ there exists a smooth function $g : M^n \to \mathbb{R}^m$ such that $\|f(x) - g(x)\| < \epsilon$ for all $x \in M^n$.*

Remark. If M^n is not compact, then it is often useful to require that $\|f(x) - g(x)\| < \epsilon(x)$ for all $x \in M^n$, where $\epsilon : M^n \to \mathbb{R}$ is a fixed function such that $\epsilon(x) > 0$ for all $x \in M^n$. The proof is exactly the same. □

For the proof of Theorem 1.6, we need the following lemma.

Lemma 1.7. *Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}^m$ be a continuous function such that $\text{Supp}(f) \subset U$. Then for all $\epsilon > 0$ there exists a smooth function $g : U \to \mathbb{R}^m$ such that $\text{Supp}(g) \subset U$ and such that $\|f(x) - g(x)\| < \epsilon$ for all $x \in M^n$.*

Proof. The Stone-Weierstrass theorem says that we can find a smooth function $g_1 : U \to \mathbb{R}^m$ such that $\|f(x) - g_1(x)\| < \epsilon$ for all $x \in U$ (in fact, it says that we can take g_1 to be a function whose coordinate functions are polynomials). Let $C = \text{Supp}(f)$, so C is a closed subset of U. Using Lemma 1.4, we can find a smooth function $\beta : U \to \mathbb{R}$ such that $0 \leq \beta(x) \leq 1$ for all $x \in U$, such that $\beta|_C = 1$, and such that $\text{Supp}(\beta) \subset U$. Define $g : U \to \mathbb{R}^m$ via the formula $g(x) = \beta(x) \cdot g_1(x)$. Since $\text{Supp}(\beta) \subset U$, we also have $\text{Supp}(g) \subset U$. Also, we clearly have $\|f(x) - g(x)\| < \epsilon$ for all $x \in C$. For $x \in U \setminus C$, we have $f(x) = 0$, so $\|g_1(x)\| < \epsilon$ and hence

$$\|f(x) - g(x)\| = \|\beta(x) \cdot g_1(x)\| \leq \|g_1(x)\| < \epsilon,$$

as desired. □

Proof of Theorem 1.6. In the exercises, you will construct a smooth atlas $\mathcal{A} = \{\phi_i : U_i \to V_i\}_{i \in I}$ for M^n and a large integer K such that for all $p \in M^n$, there exists a neighborhood W of p with $|\{i \in I \mid U_i \cap W \neq \emptyset\}| < K$. We remark that this is trivial if M^n is compact. Using Theorem 1.2, we can find a smooth partition of unity $\{\nu_i : U_i \to \mathbb{R}\}_{i \in I}$ subordinate to $\{U_i\}_{i \in I}$. Define $f_i : M^n \to \mathbb{R}^m$ via the formula $f_i(x) = \nu_i(x) \cdot f(x)$. We thus have

$$\sum_{i \in I} f_i(x) = (\sum_{i \in I} \nu_i(x)) \cdot f(x) = f(x) \qquad (x \in M^n).$$

These sums makes sense since only finitely many terms in them are nonzero for any fixed $x \in M^n$. Moreover, $\text{Supp}(f_i) \subset U_i$. Define $\widehat{f_i} : V_i \to \mathbb{R}^m$ to be the expression for f_i in the local coordinates V_i, so $\widehat{f}_i = f \circ \phi_i^{-1}$. Applying Lemma 1.7, we can find a smooth function $\widehat{g}_i : V_i \to \mathbb{R}^m$ such that $\mathrm{Supp}(\widehat{g}_i) \subset V_i$ and such that $\|\widehat{f}_i(x) - \widehat{g}_i(x)\| < \epsilon/K$ for all $x \in V_i$. Define $g_i : M^n \to \mathbb{R}^m$ via the formula

$$g_i(x) = \begin{cases} \widehat{g}_i(\phi_i(x)) & \text{if } x \in U_i, \\ 0 & \text{otherwise} \end{cases} \qquad (x \in M^n).$$

Since $\mathrm{Supp}(\widehat{g}_i) \subset V_i$, this is a smooth function on M^n satisfying $\mathrm{Supp}(g_i) \subset U_i$. Moreover, $\|f_i(x) - g_i(x)\| < \epsilon/K$ for all $x \in M^n$. Define $g : M^n \to \mathbb{R}^m$ via the formula

$$g(x) = \sum_{i \in I} g_i(x) \qquad (x \in M^n);$$

this makes sense because $\mathrm{Supp}(g_i) \subset U_i$, and hence only finitely many terms in this sum are nonzero for any fixed $x \in M^n$. The function g is a smooth function and

$$\|f(x) - g(x)\| = \|\sum_{i \in I}(f_i(x) - g_i(x))\| \leq \sum_{i \in I} \|f_i(x) - g_i(x)\| < K(\epsilon/K) = \epsilon,$$

as desired. □

The following "relative" version of Theorem 1.6 will also be useful.

THEOREM 1.8. *Let M^n be a smooth manifold and let $f : M^n \to \mathbb{R}^m$ be a continuous function. Assume that $f|_U$ is smooth for some open set U. Then for all $\epsilon > 0$ and all closed sets $C \subset M^n$ with $C \subset U$, there exists a smooth function $g : M^n \to \mathbb{R}^m$ such that $\|f(x) - g(x)\| < \epsilon$ for all $x \in M^n$ and such that $g|_C = f|_C$.*

PROOF. The proof is very similar to the proof of Theorem 1.6, so we only describe how it differs. The key is to choose the smooth atlas $\mathcal{A} = \{\phi_i : U_i \to V_i\}_{i \in I}$ for M^n at the beginning of the proof such that if $U_i \cap C \neq \emptyset$ for some $i \in I$, then $U_i \subset U$. For $i \in I$ with $U_i \subset U$, we can then take our "approximating functions" $\widehat{g}_i$ to simply equal $\widehat{f}_i$, and thus $g_i = f_i$. These choices ensure that the function $g : M^n \to \mathbb{R}^m$ constructed in the proof of Theorem 1.6 satisfies $g|_C = f|_C$, as desired. □

CHAPTER 2

Tangent vectors

In this chapter, we will define tangent vectors on a smooth manifold and describe how to use them to differentiate smooth functions. We will then discuss vector fields and show how then can be integrated to flows. Finally, as an application we will prove that if M is a smooth manifold and $p, q \in M$ are points, then there exists a diffeomorphism $f : M \to M$ such that $f(p) = q$.

2.1. Tangent spaces on Euclidean space

We begin by defining tangent vectors on Euclidean space.

Definition. Let $U \subset \mathbb{R}^n$ be open and let $p \in U$. The *tangent space* to U at p, denoted T_pU, is the vector space $\mathbb{R}^n$. One should view elements of T_pU as being vectors or arrows whose initial point is at p. □

Remark. The tangent space T_pU is a vector space over $\mathbb{R}$. For distinct $p, q \in U$, the vector spaces T_pU and T_qU are isomorphic vector spaces, but they should not be thought of as being the *same* vector space; for instance, it does not make sense to add a vector in T_pU to a vector in T_qU. □

This tangent space has the following standard basis.

Definition. Let $U \subset \mathbb{R}^n$ be open and let $p \in U$. Let the coordinate functions of $\mathbb{R}^n$ be $x_1, \ldots, x_n$. The *standard basis* for $T_pU = \mathbb{R}^n$ is $\{\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p\}$, where $\left(\frac{\partial}{\partial x_i}\right)_p \in \mathbb{R}^n$ is the vector with a 1 in position i and 0's elsewhere. □

We now discuss the derivative of a function between open subsets of Euclidean space.

Definition. Let $f\colon U \to V$ be a smooth map from an open set $U \subset \mathbb{R}^n$ to an open set $V \subset \mathbb{R}^m$. For $p \in U$, the *derivative* of f at p, denoted D_pf, is the linear map

$$D_pf\colon T_pU \to T_{f(p)}V$$

defined as follows. Let $x_1, \ldots, x_n$ be the coordinate functions on U and let $y_1, \ldots, y_m$ be the coordinate functions on V. Also, let $f = (f_1, \ldots, f_m)$ be the components of f, so $f_i\colon U \to \mathbb{R}$ is a smooth function. Then D_pf is defined via the formula

$$(D_pf)(\left(\frac{\partial}{\partial x_i}\right)_p) = \sum_{j=1}^{m} \frac{\partial f_j}{\partial x_i}(p) \cdot \left(\frac{\partial}{\partial y_j}\right)_{f(p)} \qquad (1 \leq i \leq n).$$

In other words, with respect to the bases $\{\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p\}$ for T_pU and $\{\left(\frac{\partial}{\partial y_1}\right)_{f(p)}, \ldots, \left(\frac{\partial}{\partial y_m}\right)_{f(p)}\}$ for $T_{f(p)}V$, the linear map D_pf is represented by the $m \times n$ matrix whose (i,j)-entry is $\frac{\partial f_i}{\partial x_j}$. □

Example. Define $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ via the formula

$$f(x_1, x_2) = (x_1^2 - 3x_2^3, x_1x_2, x_2 + 3) \in \mathbb{R}^3 \qquad ((x_1, x_2) \in \mathbb{R}^2).$$

Then for $p = (p_1, p_2) \in \mathbb{R}^2$ and

$$\vec{v} = v_1 \left(\frac{\partial}{\partial x_1}\right)_p + v_2 \left(\frac{\partial}{\partial x_2}\right)_p \in T_p\mathbb{R}^2$$

we have

$$\begin{aligned}(D_pf)(\vec{v}) =&(2p_1v_1 - 9p_2^2v_2) \cdot \left(\frac{\partial}{\partial y_1}\right)_{f(p)} \\ &+ (p_2v_1 + p_1v_2) \cdot \left(\frac{\partial}{\partial y_2}\right)_{f(p)} + v_2 \cdot \left(\frac{\partial}{\partial y_3}\right)_{f(p)};\end{aligned}$$

here $y_1, \ldots, y_3$ are the coordinate functions on $\mathbb{R}^3$. The matrix representing the linear map D_pf is

$$\begin{pmatrix} 2p_1 & -9p_2^2 \\ p_2 & p_1 \\ 0 & 1 \end{pmatrix}.$$

□

One of the most important property of derivatives is the *chain rule*. Let $f : V_1 \to V_2$ and $g : V_2 \to V_3$ be smooth maps, where $V_1 \subset \mathbb{R}^n$ and $V_2 \subset \mathbb{R}^m$ and $V_3 \subset \mathbb{R}^\ell$ are open. We then have the composition $g \circ f : V_1 \to V_3$. For $p \in V_1$, we have linear maps

$$D_pf : T_pV_1 \to T_{f(p)}V_2$$

and

$$D_{f(p)}g : T_{f(p)}V_2 \to T_{g(f(p))}V_3$$

and

$$D_p(g \circ f) : T_pV_1 \to T_{g(f(p))}V_3.$$

The chain rule can be stated as follows.

Theorem 2.1 (Chain Rule, Euclidean space). *Let $V_1 \subset \mathbb{R}^n$ and $V_2 \subset \mathbb{R}^m$ and $V_3 \subset \mathbb{R}^\ell$ be open sets and let $f : V_1 \to V_2$ and $g : V_2 \to V_3$ be smooth maps. Then for all $p \in V_1$ we have*

$$D_p(g \circ f) = (D_{f(p)}g) \circ (D_pf).$$

Example. Define $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ via the formula

$$f(x_1, x_2) = (x_1^2 - 3x_2^3, x_1x_2, x_2 + 3) \in \mathbb{R}^3 \qquad ((x_1, x_2) \in \mathbb{R}^2)$$

and $g : \mathbb{R}^2 \to \mathbb{R}^1$ via the formula

$$g(y_1, y_2, y_3) = (y_1 + 2y_2^2 + 3y_3^3).$$

As we calculated in the previous example, for $p \in \mathbb{R}^2$ written as $p = (p_1, p_2)$ the linear map $D_p f : T_p\mathbb{R}^2 \to T_{f(p)}\mathbb{R}^3$ is represented by the matrix

$$\begin{pmatrix} 2p_1 & -9p_2^2 \\ p_2 & p_1 \\ 0 & 1 \end{pmatrix}.$$

For $q \in \mathbb{R}^3$ written as $q = (q_1, q_2, q_3)$, the linear map $D_q g : T_q\mathbb{R}^3 \to T_{g(q)}\mathbb{R}^1$ is represented by the matrix

$$\begin{pmatrix} 1 & 4q_2 & 9q_3^2 \end{pmatrix}.$$

Let's now check the chain rule. The composition $g \circ f : \mathbb{R}^2 \to \mathbb{R}^1$ is given via the formula

$$(g \circ f)(p_1, p_2) = ((p_1^2 - 3p_2^3) + 2(p_1p_2)^2 + 3(p_2 + 3)^3) \in \mathbb{R}^1.$$

The derivative $D_p(g \circ f)$ of this at $p = (p_1, p_2)$ is represented by the matrix

$$\begin{pmatrix} 2p_1 + 4(p_1p_2)p_2 & -9p_2^2 + 4(p_1p_2)p_1 + 9(p_2+3)^2 \end{pmatrix}.$$

Plugging the equations of $f(p)$ into the above formula for $D_q g : T_q\mathbb{R}^3 \to T_{g(q)}\mathbb{R}^1$, the linear map $D_{f(p)}g : T_{f(p)}\mathbb{R}^3 \to T_{g(f(p))}\mathbb{R}^1$ is represented by the matrix

$$\begin{pmatrix} 1 & 4(p_1p_2) & 9(p_2+3)^2 \end{pmatrix}$$

The chain rule then asserts that

$$\begin{aligned} &\begin{pmatrix} 2p_1 + 4(p_1p_2)p_2 & -9p_2^2 + 4(p_1p_2)p_1 + 9(p_2+3)^2 \end{pmatrix} \\ &\quad = \begin{pmatrix} 1 & 4(p_1p_2) & 9(p_2+3)^2 \end{pmatrix} \cdot \begin{pmatrix} 2p_1 & -9p_2^2 \\ p_2 & p_1 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

which is easily verified. □

2.2. Tangent spaces

Let M^n be a smooth n-manifold and let $p \in M^n$. Our goal is to construct an n-dimensional vector space T_pM^n called the tangent space to M^n at p. If $\phi : U \to V$ is a chart around p, then vectors in T_pM^n should be represented by elements of $T_{\phi(p)}V = \mathbb{R}^n$. To make a definition that does not depend on any particular choice of chart, we introduce the following equivalence relation.

Definition. Let M^n be a smooth n-manifold, let $p \in M^n$, and let $\{\phi_i : U_i \to V_i\}_{i \in I}$ be the set of charts around p. For $i, j \in I$, let $\tau_{ji} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ be the transition function from U_i to U_j. Finally, let $\mathcal{X}(M^n, p)$ be the set of pairs $(i, \vec{v})$, where $i \in I$ and $\vec{v} \in T_{\phi_i(p)}V_i$. Define $\sim$ to be the relation on $\mathcal{X}(M^n, p)$ where where $(i, \vec{v}) \sim (j, \vec{w})$ when $(D_{\phi_i(p)}\tau_{ji})(\vec{v}) = \vec{w}$. □

Lemma 2.2. *The relation $\sim$ defined in the previous definition is an equivalence relation on $\mathcal{X}(M^n, p)$.*

Proof. We must check reflexivity, symmetry, and transitivity.

For $(i, \vec{v}) \in \mathcal{X}(M^n, p)$, we have $(i, \vec{v}) \sim (i, \vec{v})$ since the relevant transition function $\tau_{ii} : \phi_i(U_i \cap U_i) \to \phi_i(U_i \cap U_i)$ is the identity.

If $(i, \vec{v}), (j, \vec{w}) \in \mathcal{X}(M^n, p)$ satisfy $(i, \vec{v}) \sim (j, \vec{w})$, then by definition we have $(D_{\phi_i(p)}\tau_{ji})(\vec{v}) = \vec{w}$. From its definition, we see that $\tau_{ij} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is the inverse of $\tau_{ji} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$. From Theorem 2.1 (the Chain

Rule), we have $(D_{\phi_j(p)}\tau_{ij}) \circ (D_{\phi_i(p)}\tau_{ji}) = \mathrm{id}$, so $(D_{\phi_j(p)}\tau_{ij})(\vec{w}) = \vec{v}$ and hence $(j, \vec{w}) \sim (i, \vec{v})$.

If $(i, \vec{v}), (j, \vec{w}), (k, \vec{u}) \in \mathcal{X}(M^n, p)$ satisfy $(i, \vec{v}) \sim (j, \vec{w})$ and $(j, \vec{w}) \sim (k, \vec{u})$, then by definition we have $(D_{\phi_i(p)}\tau_{ji})(\vec{v}) = \vec{w}$ and $(D_{\phi_j(p)}\tau_{kj})(\vec{w}) = \vec{u}$. From its definition, we see that on $\phi_i(U_i \cap U_j \cap U_k)$ we have $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$. Again using Theorem 2.1 (the Chain Rule), we see that $D_{\phi_i(p)}\tau_{ki} = (D_{\phi_j(p)}\tau_{kj}) \circ (D_{\phi_i(p)}\tau_{ji})$, so $(D_{\phi_i(p)}\tau_{ki})(\vec{v}) = \vec{u}$ and hence $(i, \vec{v}) \sim (k, \vec{u})$. $\square$

This allows us to make the following definition.

Definition. Let M^n be a smooth manifold and let $p \in M^n$. Let $\{\phi_i : U_i \to V_i\}_{i \in I}$ be the set of charts around p. The *tangent space to M^n at p*, denoted T_pM^n, is the set of equivalence classes of elements of $\mathcal{X}(M^n, p)$ under the equivalence relation given by Lemma 2.2. $\square$

Lemma 2.3. *Let M^n be a smooth manifold and let $p \in M^n$. Then the tangent space T_pM^n is an n-dimensional vector space and for all charts $\phi\colon U \to V$ the natural map $T_{\phi(p)}V \to T_pM^n$ is an isomorphism.*

Proof. This follows from the fact that the derivatives used to define the equivalence relation are vector space isomorphisms, so the vector space structures on the various $T_{\phi_i(p)}V_i$ used to define T_pM^n descend to a vector space structure on T_pM^n. $\square$

Convention. The notation $\mathcal{X}(M^n, p)$ that we used when defining T_pM^n will not be used again. In the future, instead of talking about elements of T_pM^n being equivalence classes of pairs $(i, \vec{v})$, we will simply say that a given element of T_pM^n is represented by some $\vec{v} \in T_{\phi_i(p)}V_i$. $\square$

Convention. Consider a smooth manifold M^n, a point $p \in M^n$, and a chart $\phi\colon U \to V$ with $p \in U$. We have the standard basis $\{\left(\frac{\partial}{\partial x_1}\right)_{\phi(p)}, \ldots, \left(\frac{\partial}{\partial x_n}\right)_{\phi(p)}\}$ for $T_{\phi(p)}V = \mathbb{R}^n$. Each $\left(\frac{\partial}{\partial x_i}\right)_{\phi(p)}$ represents a vector in T_pM^n which we will write $\left(\frac{\partial}{\partial x_i}\right)_p$; the resulting basis for T_pM^n will be called the the standard basis for T_pM^n with respect to the chart $\phi\colon U \to V$. $\square$

2.3. Derivatives

Let $f : M_1^{n_1} \to M_2^{n_2}$ be a smooth map between smooth manifolds and let $p \in M_1^{n_1}$. We now show how to construct the derivative $D_pf : T_pM_1^{n_1} \to T_{f(p)}M_2^{n_2}$, which is a linear map between these vector spaces. Let $\phi_1 : U_1 \to V_1$ be a chart around p and let $\phi_2 : U_2 \to V_2$ be a chart around $\phi(p)$ such that $f(U_1) \subset U_2$. We thus have identifications $T_pM_1^{n_1} = T_{\phi_1(p)}V_1$ and $T_{f(p)}M_2^{n_2} = T_{\phi_2(f(p))}V_2$. We define $D_pf : T_pM_1^{n_1} \to T_{f(p)}M_2^{n_2}$ to be composition

$$T_pM_1^{n_1} \xrightarrow{=} T_{\phi_1(p)}V_1 \xrightarrow{D_{\phi_1(p)}(\phi_2 \circ f \circ \phi_1^{-1})} T_{\phi_2(f(p))}V_2 \xrightarrow{=} T_{f(p)}M_2^{n_2}.$$

Lemma 2.4. *This does not depend on the choice of charts.*

Proof. This is in the exercises; it provides good practice in the various identifications we have made. $\square$

Theorem 2.1 (the Chain Rule in Euclidean space) immediately implies the following version of the chain rule.

THEOREM 2.5 (Manifold Chain Rule). *Let $f : M_1^{n_1} \to M_2^{n_2}$ and $g : M_2^{n_2} \to M_3^{n_3}$ be smooth maps between smooth manifolds. Then for all $p \in M_1^{n_1}$ we have*

$$D_p(g \circ f) = (D_{f(p)}g) \circ (D_p f).$$

2.4. The tangent bundle

Let M^n be a smooth manifold. We now explain how to assemble all the tangent spaces of M^n into a single object TM^n called the *tangent bundle*. As a set, it is easy to define:

$$TM^n = \{(p, \vec{v}) \mid p \in M^n \text{ and } \vec{v} \in T_pM^n\}.$$

A subtle point in this is that the set in which the second coordinate $\vec{v}$ of $(p, \vec{v})$ lies depends on the first coordinate p, so this is *not* a product. For all $p \in M^n$, we will identify T_pM^n with the subset $\{(p, \vec{v}) \mid \vec{v} \in T_pM^n\}$ of TM^n. Under this identification, we have

$$TM^n = \bigsqcup_{p \in M^n} T_pM^n.$$

We now define a topology on TM^n as follows. Let $\phi\colon U \to V$ be a chart on M^n. Define TU to be the subset

$$\{(p, \vec{v}) \mid p \in U \text{ and } \vec{v} \in T_pM^n\}$$

of TM^n. For $p \in U$, our definition of T_pM^n identifies it with $T_{\phi(p)}V = \mathbb{R}^n$. Define a map $T\phi\colon TU \to V \times \mathbb{R}^n$ via the formula

$$T\phi(p, \vec{v}) = (\phi(p), \vec{v}).$$

We want to construct a topology on TM^n such that if TU is given the subspace topology, then $T\phi$ is a homeomorphism. Define

$$\mathcal{U} = \{(T\phi)^{-1}(W) \mid \phi\colon U \to V \text{ a chart on } M^n \text{ and } W \subset V \times \mathbb{R}^n \text{ is open}\}.$$

It is easy to see that $\mathcal{U}$ is a basis for a topology, and that under this topology the induced topology on the subsets TU is such that $T\phi$ is a homeomorphism. We endow TM^n with this topology.

Now, the set $V \times \mathbb{R}^n$ is an open subset of $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$. The maps $T\phi$ are thus charts, so TM^n is a $2n$-dimensional manifold. We now prove that it is in fact a smooth manifold.

LEMMA 2.6. *Let M^n be a smooth manifold. Then the set*

$$\mathcal{A} = \{T\phi\colon TU \to V \times \mathbb{R}^n \mid \phi\colon U \to V \text{ a chart on } M^n\}$$

is a smooth atlas on TM^n.

PROOF. Consider two charts $\phi_1\colon U_1 \to V_1$ and $\phi_2\colon U_2 \to V_2$ on M^n. Let $\tau_{12}\colon \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ be the transition map from ϕ_1 to ϕ_2. By definition, τ_{12} is smooth. The transition map on TM^n from $T\phi_1\colon TU_1 \to V_1 \times \mathbb{R}^n$ to $T\phi_2\colon TU_2 \to V_2 \times \mathbb{R}^n$ is the map

$$T\tau_{12}\colon \phi_1(U_1 \cap U_2) \times \mathbb{R}^n \longrightarrow \phi_2(U_1 \cap U_2) \times \mathbb{R}^n$$

defined via the formula

$$T\tau_{12}(q, \vec{v}) = (\tau_{12}(q), D_q\tau_{12}(\vec{v})) \in \phi_2(U_1 \cap U_2) \times \mathbb{R}^n.$$

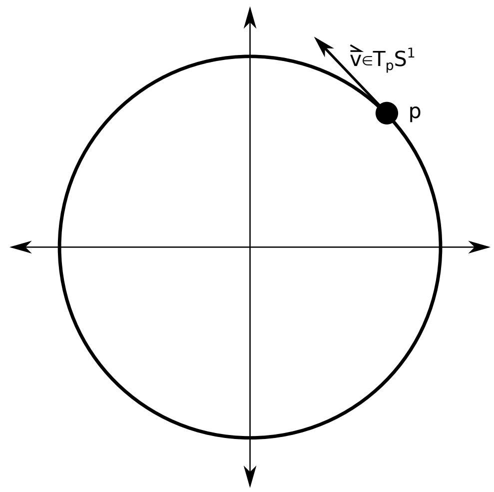

FIGURE 2.1. A vector $\vec{v} \in T_p S^1$ is orthogonal to the line from 0 to p.

This is clearly a smooth map, as desired. □

2.5. Visualizing the tangent bundle

Our construction of the tangent bundle was very abstract. In the case of smooth submanifolds of $\mathbb{R}^m$, there is a simpler construction which is a great aid to visualization. Consider a smooth submanifold $M^n \subset \mathbb{R}^m$. For $p \in M^n$, we can regard $T_p M^n$ as a subspace of $T_p\mathbb{R}^m = \mathbb{R}^m$ in the following way. By definition, there is a diffeomorphism $\phi : U \to V$, where $U \subset M^n$ is an open neighborhood of p and $V \subset \mathbb{R}^n$ is an open set. The inverse ϕ^{-1} can be regarded as a smooth map from V to $\mathbb{R}^m$, and thus it has a derivative

$$D_{\phi(p)}\phi^{-1} : T_{\phi(p)}V \to T_p\mathbb{R}^m = \mathbb{R}^m.$$

The image of this derivative can be identified with the tangent space $T_p M^n$; it is easy to see that it does not depend on the choice of diffeomorphism $\phi : U \to V$.

Using this, we can regard the tangent bundle TM^n as the subspace

$$\{(p, \vec{v}) \in T\mathbb{R}^m \mid p \in M^n,\ \vec{v} \in T_p M^n \subset T_p\mathbb{R}^m\} \subset T\mathbb{R}^m = \mathbb{R}^m \times \mathbb{R}^m.$$

This results in the familar picture of tangent vectors to M^n as being arrows in $\mathbb{R}^m$ that "point in the direction of the tangent plane to M^n".

EXAMPLE. For $S^n \subset \mathbb{R}^{n+1}$, you will prove in the exercises that

$$TS^n = \{(p, \vec{v}) \in T\mathbb{R}^{n+1} \mid \|p\| = 1 \text{ and } \vec{v} \text{ is orthogonal to the line from 0 to } p\}.$$

See Figure 2.1. □

The derivative map can also be understood from this perspective. Let $M_1^{n_1} \subset \mathbb{R}^{m_1}$ and $M_2^{n_2} \subset \mathbb{R}^{m_2}$ be smooth submanifolds of Euclidean space and let $f : M_1^{n_1} \to M_2^{n_2}$ be a smooth map. Fix some $p \in M_1^{n_1}$. As we will see in Lemma 3.4 of Chapter 3, there exists an open set $U \subset \mathbb{R}^{m_1}$ containing p and a smooth map $g : U \to \mathbb{R}^{m_2}$ such that $g|_{M_1^{n_1}} = f$. The map g induces a derivative map $D_p g : T_p U \to T_p\mathbb{R}^{m_2}$ in the sense of multivariable calculus. The derivative $D_p f : T_p M_1^{n_1} \to T_{f(p)} M_2^{n_2}$ is then just the restriction of $D_p g$ to $T_p M_1^{n_1} \subset T_p U$; this image of this restriction lies in $T_{f(p)} M_2^{n_2} \subset T_{f(p)} \mathbb{R}^{m_2}$.

Often the smooth map $f : M_1^{n_1} \to M_2^{n_2}$ is given by a formula which can be extended to an open set U (often all of $\mathbb{R}^{m_1}$, or at least $\mathbb{R}^{m_1}$ minus some isolated points where the formula has a singularity). Using this formula, it is easy to use the above recipe to work out the effect of $D_p f$.

CHAPTER 3

The structure of smooth maps

3.1. Local diffeomorphisms

The first property of smooth maps we will study is as follows.

DEFINITION. Let $f : M_1 \to M_2$ be a smooth map between smooth manifolds and let $p \in M_1$. The map f is a *local diffeomorphism at* p if there exists an open neighborhood U_1 of p such that $U_2 := f(U_1)$ is an open subset of M_2 and $f|_{U_1} : U_1 \to U_2$ is a diffeomorphism. The map f is a *local diffeomorphism* if it is a local diffeomorphisms at all points. □

REMARK. This implies that M_1 and M_2 have the same dimension. □

EXAMPLE. Let $f : \mathbb{R} \to S^1$ be the smooth map defined via the formula $f(t) = (\cos(t), \sin(t)) \in S^1 \subset \mathbb{R}^2$. Then f is a local diffeomorphism. Since f is not injective, f is not itself a diffeomorphism. □

EXAMPLE. Recall that $\mathbb{RP}^n$ is the quotient space of S^n via the equivalence relation $\sim$ that identifies antipodal points $x \in S^n$ and $-x \in S^n$. The projection map $f : S^n \to \mathbb{RP}^n$ is a smooth map which is a local diffeomorphism. □

The following is an easy criterion for recognizing a local diffeomorphism. As we will see, it is essentially a restatement of the implicit function theorem.

THEOREM 3.1 (Implicit Function Theorem). *Let $f : M_1 \to M_2$ be a smooth map between smooth manifolds and let $p \in \mathrm{Int}(M_1)$. Then f is a local diffeomorphism at $p \in M_1$ if and only if the linear map $D_p f : T_p M_1 \to T_{f(p)} M_2$ is an isomorphism.*

PROOF. Assume first that f is a local diffeomorphism at $p \in M_1$ and let $U_1 \subset \mathrm{Int}(M_1)$ be an open neighborhood of p such that $U_2 := f(U_1)$ is open and $f|_{U_1} : U_1 \to U_2$ is a diffeomorphism. Replacing U_1 with a smaller open subset if necessary, we can find charts $\phi_1 : U_1 \to V_1$ for M_1 and $\phi_2 : U_2 \to V_2$ for M_2. Let $F : V_1 \to V_2$ be the expression for f in these local coordinates, i.e. the composition

$$V_1 \xrightarrow{\phi_1^{-1}} U_1 \xrightarrow{f} U_2 \xrightarrow{\phi_2} V_2.$$

Setting $q = \phi_1(p)$, we have identifications $T_q V_1 \cong T_p M_1$ and $T_{F(q)} V_2 = T_{f(q)} M_2$, and it is enough to prove that $D_q F : T_q V_1 \to T_{F(q)} V_2$ is an isomorphism. Since F is a diffeomorphism, it has an inverse $G : V_2 \to V_1$. Applying the chain rule (Theorem 2.1) to $\mathrm{id}_{V_1} = G \circ F$, we see that

$$\mathrm{id} = D_q \mathrm{id}_{V_1} = (D_{F(q)} G) \circ (D_p F).$$

Similarly, we have

$$\mathrm{id} = D_{F(q)} \mathrm{id}_{V_2} = (D_p F) \circ (D_{F(q)} G).$$

We conclude that $D_p F$ is an isomorphism, as desired.

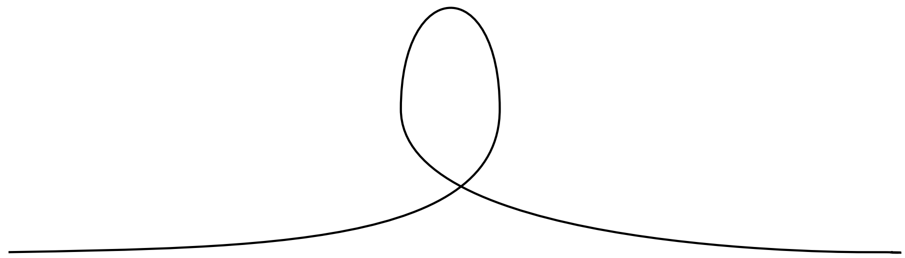

FIGURE 3.1. An immersion $f : \mathbb{R} \to \mathbb{R}^2$ that is not an embedding.

Now assume conversely that the linear map $D_p f : T_p M_1 \to T_{f(p)} M_2$ is an isomorphism. Choose charts $\phi_1 : U_1 \to V_1$ for M_1 and $\phi_2 : U_2 \to V_2$ for M_2 such that $p \in U_1$ and $f(U_1) \subset U_2$ and $U_1 \subset \mathrm{Int}(M_1)$ and $U_2 \subset \mathrm{Int}(M_2)$. Let $F : V_1 \to V_2$ be the expression for f in these local coordinates, i.e. the composition

$$V_1 \xrightarrow{\phi_1^{-1}} U_1 \xrightarrow{f} U_2 \xrightarrow{\phi_2} V_2.$$

Setting $q = \phi_1(p)$, our assumptions imply that $D_q F : T_q V_1 \to T_{F(q)} V_2$ is an isomorphism. Since V_1 and V_2 are open subsets of Euclidean space, we can now apply the ordinary inverse function theorem to deduce that F is a local diffeomorphism at q. This implies that f is a local diffeomorphism at p, as desired. $\square$

3.2. Immersions

We now turn to the following property.

DEFINITION. Let $f : M_1 \to M_2$ be a smooth map between smooth manifolds and let $p \in M_1$. The map f is an *immersion at* p if the derivative $D_p f : T_p M_1 \to T_{f(p)} M_2$ is an injective linear map. The map f is an *immersion* if it is an immersion at all points. $\square$

REMARK. This implies that the dimension of M_2 is at least the dimension of M_1. $\square$

EXAMPLE. If $f : M_1 \to M_2$ is a local diffeomorphism at p, then f is an immersion at p. $\square$

EXAMPLE. Consider the smooth map $f : \mathbb{R} \to \mathbb{R}^2$ whose image is as in Figure 3.1. Then f is an immersion but is not an embedding. $\square$

EXAMPLE. If M_1 and M_2 are smooth manifolds and $x \in M_2$, then the map $f : M_1 \to M_1 \times M_2$ defined via the formula $f(p) = (p, x)$ is an immersion. $\square$

The following theorem says that all immersions look locally like the final example above.

THEOREM 3.2 (Local Immersion Theorem). *Let $f : M_1^{n_1} \to M_2^{n_2}$ be a smooth map between smooth manifolds that is an immersion at $p \in \mathrm{Int}(M_1^{n_1})$. There then exists an open neighborhood $U_1 \subset M_1^{n_1}$ of p and an open subset $U_2 \subset M_2^{n_2}$ satisfying $f(U_1) \subset U_2$ such that the following hold. There exists an open subset $W \subset \mathbb{R}^{n_2 - n_1}$, a point $w \in W$, and a diffeomorphism $\psi : U_2 \to U_1 \times W$ such that the composition*

$$U_1 \xrightarrow{f} U_2 \xrightarrow{\psi} U_1 \times W$$

takes $u \in U_1$ *to* $(u, w) \in U_1 \times W$.

PROOF. Choose charts $\phi_1 : U_1 \to V_1$ for M_1 and $\phi_2 : U_2 \to V_2$ for M_2 such that $p \in U_1$ and $f(U_1) \subset U_2$ and $U_1 \subset \mathrm{Int}(M_1)$ and $U_2 \subset \mathrm{Int}(M_2)$. Let $F : V_1 \to V_2$ be the expression for f in these local coordinates, i.e. the composition

$$V_1 \xrightarrow{\phi_1^{-1}} U_1 \xrightarrow{f} U_2 \xrightarrow{\phi_2} V_2.$$

Set $q = \phi_1(p)$. The map F is an immersion at q, and it is enough to prove the theorem for this immersion.

By assumption, the map $D_q F : T_q V_1 \to T_{F(q)} V_2$ is an injection. Let

$$X \subset T_{F(q)} V_2 = \mathbb{R}^{n_2}$$

be a vector subspace such that

$$T_{F(q)} V_2 = \mathrm{Im}(D_q F) \oplus X.$$

We thus have $X \cong \mathbb{R}^{n_2 - n_1}$. Define $G : V_1 \times X \to \mathbb{R}^{n_2}$ via the formula

$$G(p, x) = F(q) + x.$$

We have $T_{(q,0)}(V_1 \times X) = (T_q V_1) \oplus X$ and by construction the derivative $D_{(q,0)} G : T_{(q,0)}(V_1 \times X) \to T_{F(q)} V_2$ is an isomorphism. Theorem 3.1 (the Implicit Function Theorem) thus implies that G is a local diffeomorphism at $(q, 0)$. This implies that we can find open subsets $V_1' \times W \subset V_1 \times X$ and $V_2' \subset V_2$ such that $(q, 0) \in V_1' \times W$ and $G(V_1' \times W) = V_2'$ and such that G restricts to a diffeomorphism between $V_1' \times W$ and V_2'. The composition

$$V_1' \xrightarrow{F} V_2' \xrightarrow{G^{-1}} V_1' \times W$$

then takes $v \in V_1'$ to $(v, 0) \in V_1' \times W$, as desired. $\square$

3.3. Embeddings

We now discuss embeddings of manifolds.

DEFINITION. A smooth map $f \colon M_1^{n_1} \to M_2^{n_2}$ between smooth manifolds is an *embedding* if it satisfies the following two properties.

- Letting M_3 be the image of f, the map f is a homeomorphism between $M_1^{n_1}$ and M_3.
- Let $g \colon M_3 \to M_1^{n_1}$ be the inverse of f. Then g is smooth (see the remark below). $\square$

REMARK. A priori the set M_3 is merely a subset of $M_2^{n_2}$, so the assertion that g is smooth means smooth in the sense of Definition 1.2, i.e. that for all $p \in M_3$ there exists an open set $U \subset M_2^{n_2}$ containing p and a smooth function $G \colon U \to M_2^{n_2}$ such that $G|_{U \cap M_3} = g|_{U \cap M_3}$. $\square$

The following lemma gives an infinitesimal criterion for a map to be an embedding.

LEMMA 3.3. *Let* $f \colon M_1^{n_1} \to M_2^{n_2}$ *be a smooth map between smooth manifolds that is a homeomorphism onto its image. Then* f *is an embedding if and only if* f *is an immersion.*

PROOF. Assume first that f is an embedding. Consider $p \in M_1^{n_1}$. We want to prove that the derivative map $D_p f\colon T_p M_1^{n_1} \to T_{f(p)} M_2^{n_2}$ is injective. By definition, there exists an open set $U \subset M_2^{n_2}$ containing p and a smooth map $G\colon U \to M_1^{n_1}$ such that $G \circ f = \mathrm{id}$. Using the chain rule (Theorem 2.5), we see that

$$\mathrm{id} = (D_{f(p)} G) \circ (D_p f).$$

This immediately implies that $D_p f$ is injective, as desired.

Now assume that f is an immersion. Let M_3 be the image of f and let $g\colon M_3 \to M_1^{n_1}$ be the inverse of f. We want to prove that g is smooth. Consider a point $q \in M_3$ and let $p \in M_1^{n_1}$ be such that $f(p) = q$. Since f is an immersion, we can apply the Local Immersion Theorem (Theorem 3.2) to see that there exists a neighborhood $U_1 \subset M_1^{n_1}$ of p and an open subset $U_2 \subset M_2^{n_2}$ satisfying $f(U_1) \subset U_2$ with the following property. There exists an open subset $W \subset \mathbb{R}^{n_2 - n_1}$, a point $w \in W$, and a diffeomorphism $\psi : U_2 \to U_1 \times W$ such that the composition

$$U_1 \xrightarrow{f} U_2 \xrightarrow{\psi} U_1 \times W$$

takes $u \in U_1$ to $(u, w) \in U_1 \times W$. The composition

$$U_2 \xrightarrow{\psi} U_1 \times W \xrightarrow{\text{proj}} U_1$$

is then a smooth map $G\colon U_2 \to U_1$ such that $G|_{U_2 \cap M_3} = g|_{U_2 \cap M_3}$, as desired. □

This brings us to the following definition.

DEFINITION. A *smooth submanifold* of a smooth manifold $M_2^{n_2}$ is the image of a smooth embedding $f\colon M_1^{n_1} \to M_2^{n_2}$. This image has the structure of a smooth n_1-dimensional manifold that can be identified with $M_1^{n_1}$. □

REMARK. If $M_2^{n_2}$ is Euclidean space, then this reduces to the definition of a smooth submanifold of Euclidean space as defined in Example 1.2. □

If $M_1^{n_1}$ is a smooth submanifold of a smooth manifold $M_2^{n_2}$, then we now have to different definitions of what it means for a function $f\colon M_1^{n_1} \to \mathbb{R}$ to be smooth:

(1) The definition in terms of charts for $M_1^{n_1}$, and
(2) The definition where we require f to locally extend to a smooth function on an open subset of $M_2^{n_2}$.

This same issue has already arose for smooth submanifolds of Euclidean space; see Remark 1.3. As we promised in that remark, we now prove that these two definitions are equivalent.

LEMMA 3.4. *Let $M_1^{n_1}$ be a smooth submanifold of a smooth manifold $M_2^{n_2}$ and let $f\colon M_1^{n_1} \to \mathbb{R}$ be a function. Then the above two definitions of what it means for f to be smooth are equivalent.*

PROOF. It is clear that the second definition implies the first, so we must only prove that the first implies the second. Assume that f is smooth in terms of the charts on $M_1^{n_1}$ and consider $p \in M_1^{n_1}$. Let $f\colon M_1^{n_1} \to M_2^{n_2}$ be the embedding. The local immersion theorem (Theorem 3.2) implies that there exists a neighborhood $U_1 \subset M_1^{n_1}$ of p and an open subset $U_2 \subset M_2^{n_2}$ satisfying $f(U_1) \subset U_2$ with the following property. There exists an open subset $W \subset \mathbb{R}^{n_2 - n_1}$, a point $w \in W$, and a diffeomorphism $\psi : U_2 \to U_1 \times W$ such that the composition

$$U_1 \xrightarrow{f} U_2 \xrightarrow{\psi} U_1 \times W$$

takes $u \in U_1$ to $(u, w) \in U_1 \times W$. Let $F\colon U_2 \to \mathbb{R}$ be the composition

$$U_2 \xrightarrow{\psi} U_1 \times \xrightarrow{\text{proj}} U_1 \xrightarrow{f} \mathbb{R}.$$

Then F is smooth and $F|_{M_1^{n_1} \cap U_2} = f|_{M_1^{n_1} \cap U_2}$, as desired. $\square$

3.4. Embedding manifolds into Euclidean space

We now prove that every compact smooth manifold can be realized as a smooth submanifold of Euclidean space.

THEOREM 3.5. *If M^n is a compact smooth manifold, then for some $m \gg 0$ there exists an embedding $f : M^n \to \mathbb{R}^m$.*

REMARK. This is also true for noncompact manifolds manifolds, though the proof is a little more complicated. Whitney proved a difficult theorem that says that we can take $m = 2n$. $\square$

PROOF OF THEOREM 3.5. Since M^n is compact, there exists a finite atlas

$$\mathcal{A} = \{\phi_i : U_i \to V_i\}_{i=1}^k.$$

Choose open subsets $W_i \subset U_i$ such that $\{W_i\}_{i=1}^k$ is still a cover of M^n and such that the closure of W_i in U_i is compact. Using Lemma 1.4, we can find a smooth function $\nu_i : M^n \to \mathbb{R}$ such that $(\nu_i)|_{W_i} = 1$ and $(\nu_i)|_{M^n \setminus U_i} = 0$. Next, define a function $\eta_i : M^n \to \mathbb{R}^n$ via the formula

$$\eta_i(p) = \begin{cases} \nu_i(p) \cdot \phi_i(p) & \text{if } p \in U_i, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly η_i is a smooth function. Finally, define $f : M^n \to \mathbb{R}^{k(n+1)}$ via the formula

$$f(p) = (\nu_1(p), \eta_1(p), \ldots, \nu_k(p), \eta_k(p)).$$

The function f is then a smooth map.

By Lemma 3.3, to prove that f is an embedding it is enough to prove that it is a homeomorphism onto its image and that it is an immersion. Since M^n is compact, to prove that f is a homeomorphism onto its image it is enough to prove that f is injective. Consider points $p_1, p_2 \in M^n$ such that $f(p_1) = f(p_2)$. We thus have in particular that $\nu_i(p_1) = \nu_i(p_2)$ for all $1 \leq i \leq k$. Since the W_i form a cover of M^n and $\nu_i|_{W_i} = 1$, we can find some $1 \leq i \leq k$ such that $\eta_i(p_1) = \eta_i(p_2) = 1$. This implies that $p_1, p_2 \in U_i$ and

$$\phi_i(p_1) = \eta_i(p_1) = \eta_i(p_2) = \phi_i(p_2).$$

Since $\phi_i\colon U_i \to V_i$ is a diffeomorphism, we conclude that $p_1 = p_2$, as desired.

It remains to prove that f is an immersion. Fix a point $p_0 \in M^n$. Pick $1 \leq j \leq k$ such that $p_0 \in W_j$. Let

$$g\colon \mathbb{R}^{k(n+1)} \longrightarrow \mathbb{R}^n$$

be the projection onto the coordinates corresponding to η_j, so $g \circ f = \eta_j$. The chain rule (Theorem 2.5) thus implies that

$$D_{p_0}\eta_j = (D_{f(p_p)}g) \circ (D_{p_0}f). \tag{1}$$

Since $p_0 \in W_j$ and $\eta_j|_{W_j} = \phi_j|_{W_j}$, we see that $D_{p_0}\eta_j = D_{p_0}\phi_j$. Since $\phi_j\colon U_j \to V_j$ is a diffeomorphism, we see that $D_{p_0}\phi_j$ is an isomorphism. The formula in (1) thus implies that $D_{p_0}f$ is an injection, as desired. $\square$

3.5. Submersions

We now turn to the following.

Definition. Let $f : M_1 \to M_2$ be a smooth map between smooth manifolds and let $p \in M_1$. The map f is a *submersion at* p if the derivative $D_p f : T_p M_1 \to T_{f(p)} M_2$ is a surjective linear map. The map f is a *submersion* if it is a submersion at all points. □

Remark. This implies that the dimension of M_1 is at least the dimension of M_2. □

Example. If $f : M_1 \to M_2$ is a local diffeomorphism at p, then f is a submersion at p. □

Example. If M_1 and M_2 are smooth manifolds, then the map $f : M_1 \times M_2 \to M_1$ defined via the formula $f(p_1, p_2) = p_2$ is a submersion. □

The following theorem says that all submersions look locally like the final example above.

Theorem 3.6 (Local Submersion Theorem). *Let $f : M_1^{n_1} \to M_2^{n_2}$ be a smooth map between smooth manifolds that is a submersion at $p \in \mathrm{Int}(M_1^{n_1})$. There then exists an open neighborhood $U_1 \subset M_1^{n_1}$ of p and an open subset $U_2 \subset M_2^{n_2}$ satisfying $f(U_1) \subset U_2$ such that the following hold. There exists an open subset $W \subset \mathbb{R}^{n_1 - n_2}$ and a diffeomorphism $\psi : U_2 \times W \to U_1$ such that the composition*

$$U_2 \times W \xrightarrow{\psi} U_1 \xrightarrow{f} U_2$$

takes $(u, w) \in U_2 \times W$ to $u \in U_2$.

Proof. Choose charts $\phi_1 : U_1 \to V_1$ for M_1 and $\phi_2 : U_2 \to V_2$ for M_2 such that $p \in U_1$ and $f(U_1) \subset U_2$ and $U_1 \subset \mathrm{Int}(M_1)$ and $U_2 \subset \mathrm{Int}(M_2)$. Let $F : V_1 \to V_2$ be the expression for f in these local coordinates, i.e. the composition

$$V_1 \xrightarrow{\phi_1^{-1}} U_1 \xrightarrow{f} U_2 \xrightarrow{\phi_2} V_2.$$

Set $q = \phi_1(p)$. The map F is a submersion at q, and it is enough to prove the theorem for this submersion.

By assumption, the map $D_q F : T_q V_1 \to T_{F(q)} V_2$ is a surjection. Let $X = \ker(D_q F)$, so $X \cong \mathbb{R}^{n_1 - n_2}$. Identifying $T_q V_1$ with $\mathbb{R}^{n_1}$, let $\pi : \mathbb{R}^{n_1} \to X$ be a linear map such that $\pi|_X = \mathrm{id}$. Define $G : V_1 \to V_2 \times X$ via the formula

$$G(v) = (F(v), \pi(v)).$$

We have $T_{(F(q),\pi(q))}(V_2 \times X) = (T_{F(q)} V_2) \times X$ and by construction the derivative $D_q G : T_q V_1 \to T_{(F(q),\pi(q))}(V_2 \times X)$ is an isomorphism. Theorem 3.1 (the Implicit Function Theorem) thus implies that G is a local diffeomorphism at q. This implies that we can find open subset $V_1' \subset V_1$ and $V_2' \times W \subset V_2 \times X$ such that $q \in V_1'$ and $G(V_1') \subset V_2' \times W$ and such that G restricts to a diffeomorphism between V_1' and $V_2' \times W$. The composition

$$V_2' \times W \xrightarrow{G^{-1}} V_1' \longrightarrow V_2'$$

then takes $(v, w) \in V_2' \times W$ to $v \in V_2'$, as desired. □

3.6. Regular values

We now discuss regular values, which are defined as follows.

Definition. Let $f : M_1 \to M_2$ be a smooth map between smooth manifolds and let $q \in M_2$. Then $q \in M_2$ is a *regular value* if f is a submersion at each point of $f^{-1}(q)$. $\square$

Before we discuss some examples, we prove the following theorem.

Theorem 3.7. *Let $f : M_1^{n_1} \to M_2^{n_2}$ be a smooth map between smooth manifolds and let $q \in M_2^{n_2}$ be a regular value such that $f^{-1}(q)$ is nonempty. Then $f^{-1}(q)$ is a smooth $(n_1 - n_2)$-dimensional smooth submanifold of $M_1^{n_1}$.*

Proof. Consider $p \in f^{-1}(q)$. Theorem 3.6 (the Submersion Theorem) implies that there exists an open neighborhood $U_1 \subset M_1^{n_1}$ of p and an open subset $U_2 \subset M_2^{n_2}$ satisfying $f(U_1) \subset U_2$ such that the following hold. There exists an open subset $W \subset \mathbb{R}^{n_1 - n_2}$ and a diffeomorphism $\psi : U_2 \times W \to U_1$ such that the composition

$$U_2 \times W \xrightarrow{\psi} U_1 \xrightarrow{f} U_2$$

takes $(u, w) \in U_2 \times W$ to $u \in U_2$. This implies that ψ^{-1} restricts to a diffeomorphism between $f^{-1}(q) \cap U_1$ and $\{q\} \times W$, i.e. that the point $p \in f^{-1}(q)$ has a neighborhood diffeomorphic to the open subset W of $\mathbb{R}^{n_1 - n_2}$, as desired. $\square$

It turns out that all smooth maps have many smooth values.

Theorem 3.8 (Sard's Theorem). *Let $f : M_1^{n_1} \to M_2^{n_2}$ be a smooth map between smooth manifolds. Then the set of regular values of f is open and dense in $M_2^{n_2}$.*

Remark. In fact, Sard's theorem asserts something stronger: the set of non-regular values forms a set of measure 0. Defining what this means requires a discussion of measure theory, which we prefer to to include. $\square$

Proof of Theorem 3.8. Omitted. The proof in Milnor's "Topology from the differential viewpoint" is very readable. $\square$

We now discuss a large number of illustrations of Theorem 3.7.

Example. Let $f : M_1^{n_1} \to M_2^{n_2}$ be a smooth map such that $n_1 < n_2$. For instance, f might be an embedding of an n-manifold into $\mathbb{R}^m$ for some $m > n$. Then f is clearly not a submersion anywhere, so the only regular values of f are the points *not* in the image of f. For such a point q, we have $f^{-1}(q) = \emptyset$, which is what Theorem 3.7 predicts. Sard's Theorem (Theorem 3.8) implies that such regular values must exist. This implies in particular that there does not exist a smooth surjective map $f : S^1 \to \mathbb{R}^n$ with $n \geq 2$. This is in contrast to the fact that there exist continuous space-filling curves. $\square$

Example. As in Figure 3.2, consider the 2-torus T embedded in $\mathbb{R}^3$ and let $f : T \to \mathbb{R}$ be the "height function", i.e. the function defined by the formula $f(x, y, z) = z$ for all $(x, y, z) \in T$. The only non-regular values of f are then $\{0, 2, 4, 6\}$. For a regular value $x \in \mathbb{R} \setminus \{0, 2, 4, 6\}$, the subset $f^{-1}(x) \subset T$ is a 1-manifold. There are several cases:

- If $x < 0$ or $x > 6$, then $f^{-1}(x) = \emptyset$.
- If $0 < x < 2$ or $4 < x < 6$, then $f^{-1}(x)$ consists of a single circle.

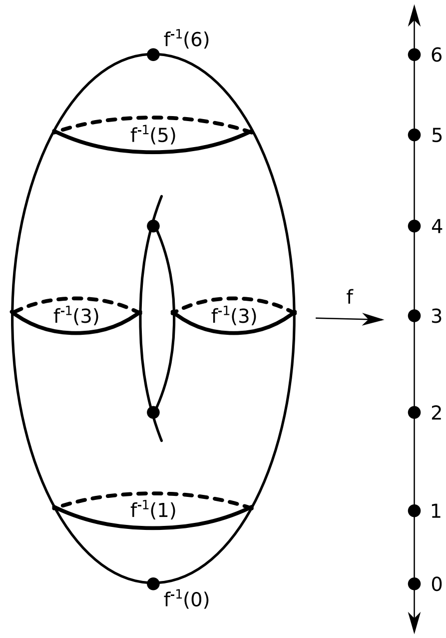

FIGURE 3.2. The torus T in $\mathbb{R}^3$ together with the height function $f : T \to \mathbb{R}$.

- If $2 < x < 4$, then $f^{-1}(x)$ consists of the disjoint union of two circles.

For $x \in \{0, 2, 4, 6\}$, the set $f^{-1}(x)$ is not a 1-manifold. For $x \in \{0, 6\}$, the set $f^{-1}(x)$ consists of a single point (a 0-manifold). For $x \in \{2, 4\}$, the set $f^{-1}(x)$ is not even a manifold (it is a "figure 8"). $\square$

EXAMPLE. Consider the map $f : \mathbb{R}^{n+1} \to \mathbb{R}$ defined via the formula

$$f(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2.$$

The derivative of this at $p = (p_1, \ldots, p_{n+1})$ is the linear map $D_p f : T_p\mathbb{R}^{n+1} \to T_{f(p)}\mathbb{R}^1$ represented by the $1 \times (n+1)$-matrix

$$\begin{pmatrix} 2p_1 & 2p_2 & \cdots & 2p_{n+1} \end{pmatrix}.$$

This is surjective as long as it is nonzero. We conclude that f is a submersion at every point except for $0 \in \mathbb{R}^{n+1}$, and thus that every nonzero point of $\mathbb{R}$ is a regular value. Since $f^{-1}(1) = S^n$, applying Theorem 3.7 furnishes us with another proof that S^n is a smooth n-manifold. $\square$

Many smooth manifolds can be constructed like S^n was above. The following example is a very important case of this.

EXAMPLE. We can identify the set Mat_n of $n \times n$ real matrices with $\mathbb{R}^{n^2}$, and thus endow it with the structure of a smooth manifold. The map $f : \mathrm{Mat}_n \to \mathbb{R}$ defined via $f(A) = \det(A)$ is clearly a smooth map. We claim that f is a submersion at all points $A \in \mathrm{Mat}_n$ such that $f(A) \neq 0$. Indeed, fixing such an A we define a smooth map $g : \mathbb{R} \to \mathrm{Mat}_n$ via the formula $g(t) = tA$. We have

$$f(g(t)) = \det(tA) = t^n \det(A).$$

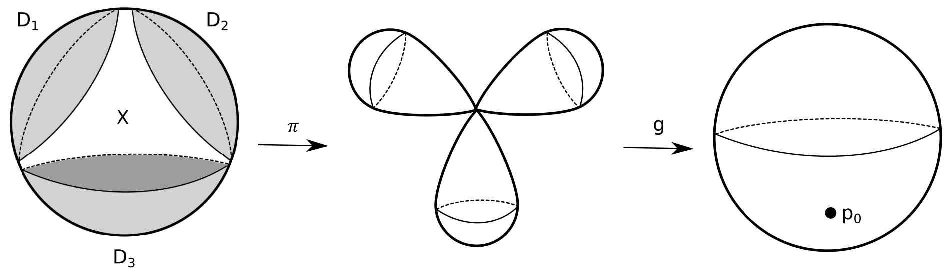

FIGURE 3.3. The function $f : S^2 \to S^2$ equals $g \circ \pi$. It takes X to p_0 and each open disc D_i diffeomorphically to $S^2 \setminus \{p_0\}$.

The ordinary calculus derivative of the map $f \circ g : \mathbb{R} \to \mathbb{R}$ is thus nonzero at $t = 1$, which implies that the derivative map $D_1(f \circ g) : T_1\mathbb{R} \to T_{\det(A)}\mathbb{R}$ is a surjective linear map (it is just multiplication by our nonzero ordinary calculus derivative!). The chain rule (Theorem 2.5) implies that

$$D_1(f \circ g) = (D_A f) \circ (D_1 g).$$

Since $D_1(f \circ g)$ is surjective, we conclude that $D_A f$ is surjective, i.e. that f is a submersion at A, as claimed. The upshot is that all nonzero numbers are regular values of $f : \mathrm{Mat}_n \to \mathbb{R}$. In particular, Theorem 3.7 implies that

$$\mathrm{SL}_n(\mathbb{R}) = f^{-1}(1)$$

is a smooth manifold of dimension $n^2 - 1$. Just like $\mathrm{GL}_n(\mathbb{R})$, this is an example of a Lie group (a group which is also a smooth manifold and for which the group operations are smooth). □

EXAMPLE. As in Figure 3.3, let D_1 and D_2 and D_3 be three disjoint open round discs in S^2 and let $X = S^2 \setminus (D_1 \cup D_2 \cup D_3)$. We construct a function $f : S^2 \to S^2$ as follows.

- Let $S^2 \vee S^2 \vee S^2$ be the result of gluing three copies of S^2 together at a single point (which we will call the "wedge point"). The space $S^2 \vee S^2 \vee S^2$ is not a manifold because the wedge point does not have a neighborhood homeomorphic to an open set in Euclidean space. There is a map $\pi : S^2 \to S^2 \vee S^2 \vee S^2$ obtained by collapsing the subset X to a single point; the map π takes X to the wedge point and each open disc D_i homeomorphically to the result of removing the wedge point from one of the S^2's.
- Fix some basepoint $p_0 \in S^2$. There is a map $g : S^2 \vee S^2 \vee S^2 \to S^2$ that takes each copy of S^2 homeomorphically onto S^2 and takes the wedge point to p_0.
- We define $f = g \circ \pi$.

If one is careful in the above construction, we can ensure that f is a smooth map. The regular values of f are $S^2 \setminus \{p_0\}$. For $x \in S^2 \setminus \{p_0\}$, the set $f^{-1}(x)$ consists of three point, one in each disc D_i. As we expect, this is a 0-manifold. The set $f^{-1}(p_0)$ is X; this is not even a manifold. □

CHAPTER 4

Vector fields

In this chapter, we discuss some basic results about vector fields, including their integral curves and flows.

4.1. Definition and basic examples

Let M^n be a smooth manifold. Intuitively, a smooth vector field on M^n is a smoothly varying choice of vector $T_p M^n$ for each $p \in M^n$. More precisely, a smooth vector field on M^n is a smooth map $\nu\colon M^n \to TM^n$ such that $\nu(p) \in T_p M^n$ for all $p \in M^n$. Let $\mathfrak{X}(M^n)$ be the set of smooth vector fields on M^n. The vector space structures on each $T_p M^n$ together endow $\mathfrak{X}(M^n)$ with the structure of a real vector space (infinite dimensional unless M^n is a compact 0-manifold).

If $\nu \in \mathfrak{X}(M^n)$ and $\phi\colon U \to V$ is a chart on M^n, then for all $p \in U$ we have the standard basis $\{\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p\}$ for $T_p M^n$ with respect to $\phi\colon U \to V$; here recall that $\left(\frac{\partial}{\partial x_i}\right)_p \in T_p M^n$ is the vector represented by the vector $\left(\frac{\partial}{\partial x_i}\right)_{\phi(p)}$ in $T_{\phi(p)} V = \mathbb{R}^n$ with a 1 in position i and 0's elsewhere. We can then write

$$\nu(p) = \nu_1(p) \cdot \left(\frac{\partial}{\partial x_1}\right)_p + \cdots + \nu_n(p) \cdot \left(\frac{\partial}{\partial x_n}\right)_p$$

for some unique $\nu_1(p), \ldots, \nu_n(p) \in \mathbb{R}$. The resulting functions $\nu_i\colon U \to \mathbb{R}$ are smooth functions; indeed, if you unwind the definitions you will see that this is equivalent to the fact that the map $\nu\colon M^n \to TM^n$ is smooth. The expression

$$\nu_1 \cdot \frac{\partial}{\partial x_1} + \cdots + \nu_n \cdot \frac{\partial}{\partial x_n}$$

will be called the *expression for ν with respect to the local coordinates V*.

Remark. If $\phi\colon U \to V$ is a chart on a smooth manifold M^n, then for $1 \leq i \leq n$ we have a vector field $\frac{\partial}{\partial x_i}$ not on all of M^n, but only on the open subset U of M^n. $\square$

It is particularly easy to write down smooth vector fields on smooth submanifolds M^n of $\mathbb{R}^m$. Namely, recall that the embedding of M^n in $\mathbb{R}^m$ identifies each $T_p M^n$ with an n-dimensional subspace of $T\mathbb{R}^m = \mathbb{R}^m$. A smooth vector field on M^n can thus be identified with a smooth map $\nu\colon M^n \to \mathbb{R}^m$ such that $\nu(p) \in T_p M^n \subset \mathbb{R}^m$ for each $p \in M^n$. We warn the reader that this is *different* from the expressions for ν in local coordinates defined above.

Example. Consider an odd-dimensional sphere $S^{2n-1} \subset \mathbb{R}^{2n}$. Recall that

$$TS^{2n-1} = \{(p, \vec{v}) \in T\mathbb{R}^{2n} \mid \|p\| = 1 \text{ and } \vec{v} \text{ is orthogonal to the line from 0 to } p\}.$$

We can then define a smooth vector field on S^{2n-1} via the formula

$$\nu(x_1, \ldots, x_{2n}) = (x_2, -x_1, x_4, -x_3, \ldots, x_{2n}, -x_{2n-1}) \in T_{(x_1,\ldots,x_{2n})}S^{2n-1} \subset \mathbb{R}^m$$

for each $(x_1, \ldots, x_{2n}) \in S^{2n-1}$. The smooth vector field ν has the property that $\nu(p) \neq 0$ for all $p \in S^{2n-1}$. A basic theorem from topology (the "hairy ball theorem") asserts that no such nonvanishing smooth vector field exists on an even-dimensional sphere. $\square$

EXAMPLE. Let M^n be a smooth submanifold of $\mathbb{R}^m$ and let $f\colon M^n \to \mathbb{R}$ be a smooth function. We can then define a smooth vector field $\text{grad}(f)$ on M^n in the following way. Consider $p \in M^n$. We can define a linear map $\eta_p\colon T_pM^n \to \mathbb{R}$ via the formula

$$\eta_p(\vec{v}) = \mathfrak{X}_{\vec{v}}(f).$$

Let $\omega(\cdot, \cdot)$ be the usual inner product on $\mathbb{R}^m$. There then exists a unique vector $\text{grad}(f)(p) \in T_pM^n$ such that

$$\eta_p(\vec{v}) = \omega(\text{grad}(f)(p), \vec{v}) \qquad (\vec{v} \in T_pM^n).$$

It is easy to see that this map $\text{grad}(f)\colon M^n \to TM^n$ is a smooth vector field. $\square$

REMARK. In the construction of $\text{grad}(f)$, we used the embedding of M^n into $\mathbb{R}^m$ to obtain an inner product on each T_pM^n. More generally, a Riemannian metric on M^n is a choice of a nondegenerate symmetric bilinear form on each T_pM^n that varies smoothly in an appropriate sense. Given a Riemannian metric on M^n, we can define a smooth vector field $\text{grad}(f)$ on M^n for any smooth function $F\colon M^n \to \mathbb{R}$ via the above procedure. $\square$

4.2. Integral curves of vector fields

Let M be a smooth manifold and let $\nu \in \mathfrak{X}(M)$. Informally, an integral curve of ν is a smoothly embedded curve that moves in the direction of ν. To make this precise, if $I \subset \mathbb{R}$ is an open interval and $\gamma\colon I \to M$ is a smooth map, then for $t \in I$ we define $\gamma'(t) \in T_{\gamma(t)}M$ to be the image under the map $D_t\gamma\colon T_tI \to T_{\gamma(t)}M$ of the element $1 \in T_tI = \mathbb{R}^n$. The curve γ is an *integral curve of* ν if $\gamma'(t) = \nu(\gamma(t))$ for all $t \in I$.

Our main theorem then is as follows.

THEOREM 4.1 (Existence of integral curves). *Let M be a smooth manifold and let $\nu \in \mathfrak{X}(M)$. Assume that* $\text{Supp}(\nu)$ *is compact. Then for all $p \in M$, there a unique integral curve $\gamma\colon \mathbb{R} \to M$ of ν such that $\gamma(0) = p$.*

REMARK. The hypothesis that $\text{Supp}(\nu)$ is compact holds automatically if M is compact. $\square$

REMARK. The theorem is not necessarily true if $\text{Supp}(\nu)$ is not compact. The problem is that the integral curve might "escape" the manifold in finite time. As an example of what we mean here, let $M = \mathbb{R} \setminus \{2\}$. This has a single chart $\text{id}\mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{2\}$ and we have a vector field $\frac{\partial}{\partial x}$ defined on all of M. We can then define an integral curve $\gamma\colon (-\infty, 2) \to M$ for $\frac{\partial}{\partial x}$ via the formula $\gamma(t) = t$; however, we cannt extend γ past 2 since it runs into the missing point 2. $\square$

The key technical input to the proof is the following lemma.

LEMMA 4.2. *Consider a chain of open sets $V'' \subset V' \subset V \subset \mathbb{R}^n$ such that the closure of V'' is a compact subset of V' and such that the closure of V' is a compact subset of V. Consider $\nu \in \mathfrak{X}(V)$. Then there is an $\epsilon > 0$ such that for all $p \in V''$, there exists an integral curve $\gamma\colon (-\epsilon, \epsilon) \to V$ such that $\gamma(0) = p$ and $\gamma'(t) = \nu(\gamma(t))$ for all $t \in (-\epsilon, \epsilon)$. The integral curve γ is unique in the following sense: if for some $\delta > 0$ there is another integral curve $\lambda\colon (-\delta, \delta) \to V$ with $\lambda(0) = p$, then $\gamma(t) = \lambda(t)$ for all $t \in (-\epsilon, \epsilon) \cap (-\delta, \delta)$.*

PROOF. This is simply a restatement into our language of the usual existence and uniqueness for solutions of systems of ordinary differential equations. □

This lemma provides the local result needed for the following.

LEMMA 4.3. *Let M be a smooth manifold and let $\nu \in \mathfrak{X}(M)$. Assume that* $\mathrm{Supp}(\nu)$ *is a compact subset of* $\mathrm{Int}(M)$. *There then exists some $\epsilon > 0$ such that for all $p \in M$, there exists an integral curve $\gamma\colon (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$. The integral curve γ is unique in the following sense: if for some $\delta > 0$ there is another integral curve $\lambda\colon (-\delta, \delta) \to M$ with $\lambda(0) = p$, then $\gamma(t) = \lambda(t)$ for all $t \in (-\epsilon, \epsilon) \cap (-\delta, \delta)$.*

PROOF. Let $\{U_i\}_{i=1}^k$ and $\{U_i'\}_{i=1}^k$ and $\{U_i''\}_{i=1}^k$ be finite open covers of the compact set $\mathrm{Supp}(\nu)$ such that the following hold for all $1 \leq i \leq k$.

- There exists a chart $\phi_i\colon U_i \to V_i$.
- The set U_i lies in $\mathrm{Int}(M)$.
- The closure of U_i' is a compact subset of U_i.
- The closure of U_i'' is a compact subset of U_i'.

For $1 \leq i \leq k$, we can apply Lemma 4.2 to find some $\epsilon_i > 0$ such that for all $p \in U_i''$, there exists a smooth map $\gamma\colon (-\epsilon_i, \epsilon_i) \to U_i$ with $\gamma(0) = 0$ and $\gamma'(t) = \nu(\gamma(t))$ for all $t \in (-\epsilon_i, \epsilon_i)$. Let $\epsilon > 0$ be the minimum of the ϵ_i. Then the desired curve $\gamma\colon (-\epsilon, \epsilon) \to M$ exists and is unique for all $p \in \mathrm{Supp}(\nu)$. But for $p \notin \mathrm{Supp}(\nu)$ we have $\nu(p) = 0$, and thus the desired curve is the constant curve $\gamma\colon (\epsilon, \epsilon) \to M$ defined by $\gamma(t) = p$ for all t. □

PROOF OF THEOREM 4.1. Let $\epsilon > 0$ be the constant given by Lemma 4.3 and let $p \in M$. For $k \geq 1$, we will prove that there exists a unique smooth function $\gamma_k\colon (-k\epsilon/2, k\epsilon/2) \to M$ such that $\gamma_k(0) = p$ and $\gamma_k'(t) = \nu(\gamma_k(t))$ for all $t \in (-k\epsilon/2, k\epsilon/2)$. Before we do that, observe that the uniqueness of γ_k implies that $\gamma_{k+1}(t) = \gamma_k(t)$ for $t \in (-k\epsilon/2, k\epsilon/2)$, so the desired integral curve $\gamma\colon \mathbb{R} \to M$ can be defined by $\gamma(t) = \gamma_k(t)$, where k is chosen large enough such that $t \in (-k\epsilon/2, k\epsilon/2)$. The uniqueness of our integral curve follows from the uniqueness of the γ_k.

It remains to construct the γ_k. This construction will be inductive. First, we can use Lemma 4.3 to construct and prove unique the desired $\gamma_1\colon (-\epsilon/2, \epsilon/2) \to M$ (in fact, we could ensure that γ_1 was defined on $(-\epsilon, \epsilon)$, but this will simplify our inductive procedure). Now assume that γ_k has been constructed and proven to be unique. Set $q_k = \gamma_k((k-1)\epsilon/2)$ and $r_k = \gamma_k(-(k-1)\epsilon/2)$. Another application of Lemma 4.3 implies that there exists smooth functions $\zeta_k\colon (-\epsilon, \epsilon) \to M$ and $\kappa_k\colon (-\epsilon, \epsilon) \to M$ such that

$$\zeta_k(0) = p_k \quad \text{and} \quad \kappa_k(0) = r_k$$

and such that

$$\zeta_k'(t) = \nu(\zeta_k(t)) \quad \text{and} \quad \kappa_k'(t) = \nu(\kappa_k(t))$$

for all $t \in (-\epsilon, \epsilon)$. The uniqueness statement in Lemma 4.3 implies that

$$\zeta_k(t) = \gamma_k((k-1)\epsilon/2 + t) \quad \text{and} \quad \kappa_k(t) = \gamma_k(-(k-1)\epsilon/2 + t)$$

for all $t \in (-\epsilon/2, \epsilon/2)$. The desired function $\gamma_{k+1} \colon (-(k+1)\epsilon/2, (k+1)\epsilon/2) \to M$ is then defined via the formula

$$\gamma_{k+1}(t) = \begin{cases} \kappa_k(t + (k-1)\epsilon/2) & \text{if } -(k+1)\epsilon/2 < t < -(k-1)\epsilon/2, \\ \gamma_k(t) & \text{if } -k\epsilon/2 < t < k\epsilon/2, \\ \zeta_k(t - (k-1)\epsilon/2) & \text{if } (k-1)\epsilon/2 < t < (k+11)\epsilon/2. \end{cases}$$

Its uniqueness follows from the uniqueness statement in Lemma 4.3. □

4.3. Flows

Let M be a smooth manifold and let $\nu \in \mathfrak{X}(M)$. In this section, we use the results of the previous section to prove an important theorem which says that in most cases ν determines a flow, that is, a family of diffeomorphisms of M that move points in the direction of ν. More precisely, a *flow on M in the direction of ν* consists of smooth maps $f_t \colon M \to M$ for each $t \in \mathbb{R}$ with the following properties.

- For all $t \in \mathbb{R}$, the map f_t is a diffeomorphism.
- Define $F \colon M \times \mathbb{R} \to M$ via the formula $F(p,t) = f_t(p)$. Then F is smooth.
- For all $t, s \in \mathbb{R}$, we have $f_{t+s} = f_t \circ f_s$. In particular, $f_0 = \text{id}$.
- For all $p \in M$, define $\gamma_p \colon \mathbb{R} \to M$ via the formula $\gamma_p(t) = f_t(p)$. Then γ_p is an integral curve for ν starting at p.

Our main theorem is as follows.

THEOREM 4.4 (Existence of flows). *Let M be a smooth manifold and let $\nu \in \mathfrak{X}(M)$ be such that* $\mathrm{Supp}(\nu)$ *is compact. Then there exists a unique flow on M in the direction of ν.*

PROOF. Theorem 4.1 implies that for all $p \in M$, there exists a unique integral curve $\gamma_p \colon \mathbb{R} \to M$ for ν starting at p. From the uniqueness of this integral curve, we see that

$$\gamma_p(s+t) = \gamma_{\gamma_p(s)}(t) \qquad (p \in M, s, t \in \mathbb{R}). \tag{2}$$

Define $F \colon M \times \mathbb{R} \to M$ via the formula $F(p,t) = \gamma_p(t)$. It follows from the smooth dependence on initial conditions of solutions to systems of ordinary differential equations that F is smooth. For $t \in \mathbb{R}$, define $f_t \colon M \to M$ via the formula $f_t(p) = F(p,t)$ for $p \in M$. The equation (2) implies that $f_{s+t} = f_s \circ f_t$ for all $s, t \in \mathbb{R}$. Since $f_0 = \text{id}$ by construction, this implies that $f_{-t} \circ f_t = \text{id}$ for all $t \in \mathbb{R}$, and hence each f_t is a diffeomorphism. The theorem follows. □

4.4. Moving points around via flows

As an application of flows, we will prove the following theorem.

THEOREM 4.5. *Let M^n be a connected smooth manifold and let $p, q \in M^n$. Then there exists a diffeomorphism $f \colon M^n \to M^n$ such that $f(p) = q$.*

PROOF. The proof has two steps.

STEP 1. *Let $V \subset \mathbb{R}^n$ be an open disc with center x_0 and radius $r > 0$. Consider $p, q \in V$. Then there exists a diffeomorphism $g \colon V \to V$ such that $g(p) = q$ and such that for some $\epsilon > 0$ we have $g(x) = x$ for all $x \in V$ satisfying $\|x - x_0\| > r - \epsilon$.*

Without loss of generality, we can assume that $x_0 = 0$. Let $\vec{v} \in \mathbb{R}^n$ be the vector $q - p$ and let $\nu_1 \in \mathfrak{X}(V)$ be the constant vector field $\nu_1(p) = \vec{v}$ on V. We cannot apply Theorem 4.4 to ν_1 since it does not have compact support. However, choose $0 < \epsilon < \epsilon' < r$ such that $\|p\| < \epsilon'$ and $\|q\| < \epsilon'$. Using Lemma 1.4, we can find a smooth function $h\colon V \to \mathbb{R}$ such that

$$h(x) = 1 \qquad (\|x\| \leq \epsilon')$$

and

$$h(x) = 0 \qquad (\|x\| > r - \epsilon).$$

Define $\nu \in \mathfrak{X}(V)$ via the formula

$$\nu(x) = h(x) \cdot \nu_1(x).$$

The support of ν is then compact, so we can apply Theorem 4.4 to obtain a flow $g_t\colon V \to V$. Unwinding the definitions, the map $[0,1] \to V$ taking $t \in [0,1]$ to $g_t(p)$ traces out the straight line segment connecting p to q; in particular, $g_1(p) = q$. Moreover, for $x \in V$ satisfying $\|x\| > r - \epsilon$ we have $\nu(x) = 0$, and thus $g_1(x) = x$, as desired.

STEP 2. *We construct f.*

Let $\gamma\colon [0,1] \to M^n$ be a continuous path with $\gamma(0) = p$ and $\gamma(1) = q$. We can then find some $N > 0$ such that for all $0 \leq k < N$, there exists a chart $\phi_k\colon U_k \to V_k$ with $\gamma(k/N), \gamma((k+1)/N) \in U_k$ and with V_k an open disc in $\mathbb{R}^n$. Let $g_k\colon V_k \to V_k$ be the diffeomorphism taking $\phi(\gamma(k/N))$ to $\phi(\gamma((k+1)/N))$ given by Step 1. Define $f_k\colon M^n \to M^n$ via the formula

$$g_k(p) = \begin{cases} \phi_k^{-1}(g_k(\phi_k(p))) & \text{if } p \in U_k, \\ p & \text{otherwise.} \end{cases}$$

It is clear that f_k is a diffeomorphism of M^n satisfying

$$f_k(\gamma(k/N)) = \gamma((k+1)/N).$$

Define

$$f = f_{N-1} \circ f_{N-2} \circ \cdots \circ f_1\colon M^n \to M^n.$$

Then f is a diffeomorphism satisfying $f(p) = q$, as desired. $\square$

CHAPTER 5

Differential 1-forms

In this chapter, we introduce the theory of differential 1-forms and path integrals.

5.1. Cotangent vectors

Recall that if V is a finite-dimensional $\mathbb{R}$-vector space, then the *dual* of V, denoted V^*, is the set of all $\mathbb{R}$-linear maps $V \to \mathbb{R}$. The dual V^* is an $\mathbb{R}$-vector space of the same dimension as V. If $\{\vec{e}_1, \ldots, \vec{e}_n\}$ is a basis for V, then we can define a dual basis $\{\vec{v}_1^*, \ldots, \vec{v}_n^*\}$ for V^* via the formula

$$\vec{v}_k^*(\sum_{i=1}^n c_i \vec{e}_i) = c_k \qquad (c_1, \ldots, c_n \in \mathbb{R}, 1 \leq k \leq n).$$

This leads to an isomorphism between V and V^* taking $\vec{e}_i$ to $\vec{e}_i^*$. However, we warn the reader that this isomorphism depends on the choice of basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$ and that there is no canonical (i.e. basis independent) choice of isomorphism between V^* and V.

The cotangent space is now defined as follows.

Definition. Let M^n be a smooth manifold and let $p \in M^n$. The *cotangent space* of M^n at p, denoted $T_p^* M^n$, is the dual $(T_p M^n)^*$ of the tangent space at p. □

If $\phi\colon U \to V$ is a chart for M^n with $p \in U$, then $T_p M^n$ is identified with $T_{\phi(p)} V$ and $T_p^* M^n$ is identified with $(T_{\phi(p)} V)^*$. Let $\{\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p\}$ be the standard basis for $T_p M^n$ with respect to $\phi\colon U \to V$. The dual basis for $T_p^* M^n$ is denoted $\{(\mathrm{dx}_1)_p, \ldots, (\mathrm{dx}_n)_p\}$, so by definition we have

$$(\mathrm{dx}_i)_p(\left(\frac{\partial}{\partial x_j}\right)_p) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i, j \leq n$. We will call $\{(\mathrm{dx}_1)_p, \ldots, (\mathrm{dx}_n)_p\}$ the *standard basis for the cotangent space* of M^n at p with respect to $\phi\colon U \to V$.

Now consider two charts $\phi_1\colon U_1 \to V_1$ and $\phi_2\colon U_2 \to V_2$ with $p \in U_1 \cap U_2$. Set $q_1 = \phi_1(p)$ and $q_2 = \phi_2(p)$ and let $\tau_{12}\colon \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ be the transition map. By definition, the composition

$$T_{q_1} V_1 = T_p M^n = T_{q_2} V_2$$

of our identifications is given by the derivative map

$$D_{q_1} \tau_{12}\colon T_{q_1} V_1 \to T_{q_2} V_2. \tag{3}$$

Now consider an element $\rho \in T_p^* M^n$. We can identify ρ with elements $\rho_1 \in (T_{q_1} V_1)^*$ and $\rho_2 \in (T_{q_2} V_2)^*$. If $\vec{v} \in T_p M^n$ is identified with $\vec{v}_1 \in T_{q_1} V_1$ and $\vec{v}_2 \in T_{q_2} V_2$, then we must have

$$\rho(\vec{v}) = \rho_1(\vec{v}_1) = \rho_2(\vec{v}_2).$$

By (3), we have $\vec{v}_1 = (D_{q_1} \tau_{12})^{-1}(\vec{v}_2)$, and thus

$$\rho_1((D_{q_1} \tau_{12})^{-1}(\vec{v}_2)) = \rho_2(\vec{v}_2).$$

In other words, ρ_1 and ρ_2 are related by the equation

$$\rho_2 = \rho_1 \circ (D_{q_1} \tau_{12})^{-1}. \tag{4}$$

We close this section by giving an important example of a cotangent vector.

Example. Let M^n be a smooth manifold, let $f \colon M^n \to \mathbb{R}$ be a smooth map, and let $p \in M^n$. We define $(\mathrm{df})_p \in T_p^* M^n$ to be the linear map $T_p M^n \to \mathbb{R}$ that takes $\vec{v} \in T_p M^n$ to the directional derivative of f in the direction of $\vec{v}$. In other words, $(\mathrm{df})_p(\vec{v})$ is the image of $\vec{v}$ under the derivative map

$$D_p f \colon T_p M^n \to T_{f(p)} \mathbb{R} = \mathbb{R}.$$

If $\phi \colon U \to V$ is a chart with $p \in U$ and $\{(\mathrm{dx}_1)_p, \ldots, (\mathrm{dx}_n)_p\}$ is the associated standard basis for the cotangent space of M^n at p, then the usual formulas from multivariable calculus show that

$$(\mathrm{df})_p = \sum_{i=1}^{n} \frac{\partial f \circ \phi^{-1}}{\partial x_i}(\phi(p)) \cdot (\mathrm{dx}_i)_p;$$

here observe that $f \circ \phi^{-1} \colon V \to \mathbb{R}$ is a smooth function on the open set V of $\mathbb{R}^n$. □

Remark. This is consistent with our previous notation: if $\phi \colon U \to V$ is a chart on a smooth manifold M^n, then for $1 \le i \le n$ we have the coordinate functions $x_i \colon U \to \mathbb{R}$. For $p \in U$, the element $(\mathrm{dx}_i)_p$ defined in the previous example agrees with the corresponding element of the standard basis for the cotangent space of M^n at p. □

5.2. The cotangent bundle

Let M^n be a smooth manifold. Our goal now is to globalize the construction in the previous section to define the *cotangent bundle* of a smooth manifold M^n. The construction will be very similar to that of the tangent bundle. As a set, we define

$$T^* M^n = \{(p, \rho) \mid p \in M^n \text{ and } \rho \in T_p^* M^n\}.$$

For all $p \in M^n$, we will identify $T_p^* M^n$ with the subset $\{(p, \rho) \mid \rho \in T_p^* M^n\}$ of $T^* M^n$. Under this identification, we have

$$T^* M^n = \bigsqcup_{p \in M^n} T_p^* M^n.$$

We now define a topology on $T^* M^n$ as follows. Let $\phi \colon U \to V$ be a chart on M^n. Define $T^* U$ to be the subset

$$\{(p, \rho) \mid p \in U \text{ and } \rho \in T_p^* M^n\}$$

of $T^* M^n$. For $p \in U$, our definition of $T_p^* M^n$ identifies it with $T_{\phi(p)}^* V = (\mathbb{R}^n)^*$. Define a map $T^* \phi \colon T^* U \to V \times (\mathbb{R}^n)^*$ via the formula

$$T^* \phi(p, \rho) = (\phi(p), \rho).$$

We want to construct a topology on T^*M^n such that if T^*U is given the subspace topology, then $T^*\phi$ is a homeomorphism. Define

$$\mathcal{U} = \{(T^*\phi)^{-1}(W) \mid \phi\colon U \to V \text{ a chart on } M^n \text{ and } W \subset V \times (\mathbb{R}^n)^* \text{ is open}\}.$$

It is easy to see that $\mathcal{U}$ is a basis for a topology, and that under this topology the induced topology on the subsets T^*U is such that $T^*\phi$ is a homeomorphism. We endow T^*M^n with this topology.

Now, the set $V \times (\mathbb{R}^n)^*$ is an open subset of $\mathbb{R}^n \times (\mathbb{R}^n)^* \cong \mathbb{R}^{2n}$. The maps $T^*\phi$ are thus charts, so T^*M^n is a $2n$-dimensional manifold. We now prove that it is in fact a smooth manifold.

Lemma 5.1. *Let M^n be a smooth manifold. Then the set*

$$\mathcal{A} = \{T^*\phi\colon T^*U \to V \times (\mathbb{R}^n)^* \mid \phi\colon U \to V \text{ a chart on } M^n\}$$

is a smooth atlas on T^*M^n.

Proof. Consider two charts $\phi_1\colon U_1 \to V_1$ and $\phi_2\colon U_2 \to V_2$ on M^n. Let $\tau_{12}\colon \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ be the transition map from ϕ_1 to ϕ_2. By definition, τ_{12} is smooth. Using equation (4) above, we see that the transition map on T^*M^n from $T^*\phi_1\colon T^*U_1 \to V_1 \times (\mathbb{R}^n)^*$ to $T^*\phi_2\colon T^*U_2 \to V_2 \times (\mathbb{R}^n)^*$ is the map

$$T^*\tau_{12}\colon \phi_1(U_1 \cap U_2) \times (\mathbb{R}^n)^* \longrightarrow \phi_2(U_1 \cap U_2) \times (\mathbb{R}^n)^*$$

defined via the formula

$$T^*\tau_{12}(q, \rho) = (\tau_{12}(q), \rho \circ (D_q\tau_{12})^{-1}) \in \phi_2(U_1 \cap U_2) \times (\mathbb{R}^n)^*.$$

This is clearly a smooth map, as desired. □

5.3. Differential 1-forms

Fix a smooth manifold M^n. We then make the following definition.

Definition. Let M^n be a smooth manifold. A *differential* 1*-form* on M^n is a smooth map $\omega\colon M^n \to T^*M^n$ such that $\omega(p) \in T_p^*M^n$ for all $p \in M^n$. The set of all differential 1-forms on M^n is denoted $\Omega^1(M^n)$. □

Example. Regarding S^1 as a smooth submanifold of $\mathbb{R}^2$, recall that for $(x, y) \in S^1$ the vector space $T_{(x,y)}S^1$ can be regarded as the subspace of $\mathbb{R}^2$ spanned by $(-y, x)$. We can then define an element $\theta \in \Omega^1(S^1)$ by letting $\theta(x, y) \in T^*_{(x,y)}S^1$ be the linear map that takes $t \cdot (-y, x)$ to t. □

Observe that the $\mathbb{R}$-vector space structure on the cotangent spaces of M^n makes $\Omega^1(M^n)$ into an $\mathbb{R}$-vector space. In fact, it has even more structure: if $f\colon M^n \to \mathbb{R}$ is a smooth function and $\omega \in \Omega^1(M^n)$, then we can define an element of $\Omega^1(M^n)$ that takes $p \in M^n$ to $f(p) \cdot \omega(p) \in T_p^*(M^n)$; we will denote this by $f\omega$. This makes $\Omega^1(M^n)$ into a module over the ring of smooth functions on M^n.

We now explain how to think about differential 1-forms locally. Fix a chart $\phi\colon U \to V$ on M^n. Letting $x_1, \ldots, x_n$ be the coordinate functions of V, for all $p \in U$ we have the standard basis $\{(\mathrm{dx}_1)_p, \ldots, (\mathrm{dx}_n)_p\}$ for $T_p^*M^n$. For all $1 \leq i \leq n$, the map $p \mapsto (\mathrm{dx}_i)_p$ is a differential 1-form on the open set U of M^n. If $\omega \in \Omega^1(M^n)$, then there exist functions $f_1, \ldots, f_n\colon U \to \mathbb{R}$ such that the restriction to U of ω equals $f_1\mathrm{dx}_1 + \cdots + f_n\mathrm{dx}_n$. The fact that ω is a smooth map from M^n to T^*M^n is equivalent to the fact that these functions f_i are smooth. The expression

$f_1 \mathrm{dx}_1 + \cdots + f_n \mathrm{dx}_n$ will be called the *expression for* ω *in the local coordinates* $\phi\colon U \to V$.

Example. Let us return to Example 5.3. Let $\theta \in \Omega^1(S^1)$ be the differential 1-form from that example. Set $U = S^1 \setminus \{(0,1)\}$ and let $\phi\colon U \to \mathbb{R}$ be given by stereographic projection, so $\phi(x,y) = x/(1-y)$. To keep our notation straight, we will let z be the coordinate function on $\mathbb{R}$. We will determine the function $f\colon \mathbb{R} \to \mathbb{R}$ such that the restriction of θ to U equals $f\mathrm{dz}$. Observe that

$$\phi^{-1}(z) = (\frac{2z}{z^2+1}, \frac{z^2-1}{z^2+1}),$$

and thus

$$D_z\phi^{-1}\colon T_z\mathbb{R} \to T_{\phi^{-1}(z)}S^1$$

is represented by the matrix

$$\begin{pmatrix} \frac{2(1-z^2)}{(1+z^2)^2} \\ \frac{4z}{(1+z^2)^2} \end{pmatrix}.$$

For $(x,y) \in S^1$, like in Example 5.3 we will regard $T_{(x,y)}S^1$ as the subspace of $\mathbb{R}^2$ spanned by $(-y,x)$. Using this, it follows that

$$\left(\frac{\partial}{\partial z}\right)_{(\frac{2z}{z^2+1}, \frac{z^2-1}{z^2+1})} = \frac{2}{z^2+1} \cdot (\frac{1-z^2}{z^2+1}, \frac{2z}{z^2+1}).$$

We deduce that

$$\theta(\left(\frac{\partial}{\partial z}\right)_{(\frac{2z}{z^2+1}, \frac{z^2-1}{z^2+1})}) = \frac{2}{z^2+1},$$

and hence that the restriction of θ to U is

$$\frac{2}{z^2+1}\mathrm{dz}.$$ □

Another important example is as follows.

Example. Let $f\colon M^n \to \mathbb{R}$ be a smooth map. Recall from Example 5.1 that for $p \in M^n$ we defined $\mathrm{df}_p \in T_p^*M^n$ to be the linear map $T_pM^n \to \mathbb{R}$ taking $\vec{v} \in T_pM^n$ to the directional derivative of f in the direction of $\vec{v}$. We then have $\mathrm{df} \in \Omega^1(M^n)$ defined via $\mathrm{df}(p) = \mathrm{df}_p$. If $\phi\colon U \to V$ are local coordinates and $x_1, \ldots, x_n$ are the coordinate functions of V, then the expression for df in terms of these local coordinates is

$$\frac{\partial f \circ \phi^{-1}}{\partial x_1}\mathrm{dx}_1 + \cdots + \frac{\partial f \circ \phi^{-1}}{\partial x_n}\mathrm{dx}_n.$$ □

Remark. In the next section, we will prove that the element $\theta \in \Omega^1(S^1)$ discussed in Examples 5.3 and 5.3 cannot be written as df for any smooth $f\colon S^1 \to \mathbb{R}$. This is a fundamental fact; underlying it is the fact that the 'first de Rham cohomology group" of S^1 is nontrivial. □

One important property of the previous example is as follows.

Lemma 5.2. *Let M^n be a smooth manifold and let $f\colon M^n \to \mathbb{R}$ be a smooth function. Then $df = 0$ if and only if f is constant.*

PROOF. Let $\phi\colon U \to V$ be a chart and let $x_1, \ldots, x_n$ be the coordinate functions on V. Then the restriction of df to U is

$$\frac{\partial f \circ \phi^{-1}}{\partial x_1}\mathrm{dx}_1 + \cdots + \frac{\partial f \circ \phi^{-1}}{\partial x_n}\mathrm{dx}_n.$$

This equals 0 if and only if all the $\frac{\partial f \circ \phi^{-1}}{\partial x_i}$ vanish, i.e. if and only if $f \circ \phi^{-1}$ is constant. This holds for all charts if and only if f is constant. $\square$

5.4. Pulling back 1-forms

Let $f\colon M_1^{n_1} \to M_2^{n_2}$ be a smooth map between smooth manifolds. For $p \in M_1^{n_1}$, we have a derivative map

$$D_p f\colon T_p M_1^{n_1} \to T_{f(p)} M_2^{n_2}.$$

However, there is no reasonable way to take a vector field on $M_1^{n_1}$ and push it forward along f to obtain a vector field on $M_2^{n_2}$. In contrast, for differential 1-forms we have the following construction.

CONSTRUCTION. Let $f\colon M_1^{n_1} \to M_2^{n_2}$ be a smooth map between smooth manifolds and let $\omega \in \Omega^1(M_2^{n_2})$. Define $f^*(\omega) \in \Omega^1(M_1^{n_1})$ as follows. For $p \in M_1$, we must construct an element $f^*(\omega)(p) \in T_p^* M_1^{n_1}$, i.e. a linear map $T_p M_1^{n_1} \to \mathbb{R}$. The 1-form ω gives us a linear map $\omega(f(p))\colon T_{f(p)} M_2^{n_2} \to \mathbb{R}$, and $f^*(\omega)(p)$ is obtained by composing this with $D_p f\colon T_p M_1^{n_1} \to T_{f(p)} M_2^{n_2}$, i.e. by setting

$$f^*(\omega)(p) = (\omega\left(f\left(p\right)\right)) \circ \left(D_p f\right).$$

It is easy to see that $f^*(\omega) \in \Omega^1(M_1^{n_1})$. $\square$

REMARK. We emphasize that this construction *pulls differential* 1*-forms back*; there is no reasonable way to push them forward. $\square$

Here is an important example.

EXAMPLE. Let $\theta \in \Omega^1(S^1)$ be the form discussed in Examples 5.3 and 5.3. Define $\pi\colon \mathbb{R} \to S^1$ via the formula $\pi(t) = (\cos(t), \sin(t))$. Then $\pi^*(\omega) = \mathrm{dt}$. $\square$

One important property of the above pull-back construction is as follows.

LEMMA 5.3. *Let $f\colon M_1^{n_1} \to M_2^{n_2}$ be a smooth function between smooth manifolds and let $g\colon M_2^{n_2} \to \mathbb{R}$ be a smooth map. Then $f^*(dg) = d(g \circ f)$.*

PROOF. By definition, for $q \in M_2^{n_2}$ the linear map $\mathrm{dg}(q)\colon T_q M_2^{n_2} \to \mathbb{R}$ takes $\vec{w} \in T_q M_2^{n_2}$ to its image under the map

$$D_q g\colon T_q M_2^{n_2} \to T_{g(q)}\mathbb{R} = \mathbb{R}.$$

For $p \in M_1^{n_1}$, it follows that the linear map $f^*(\mathrm{dg})(p)\colon T_p \to \mathbb{R}$ takes $\vec{v} \in T_p M_1^{n_1}$ to its image under the composition

$$T_p M_1^{n_1} \xrightarrow{D_p f} T_{f(p)} M_2^{n_2} \xrightarrow{D_{f(p)} g} T_{g(f(p))}\mathbb{R} = \mathbb{R}.$$

By the chain rule (Theorem 2.5), this composition equals $D_p(g \circ f)$. The lemma follows. $\square$

To illustrate some of the tools we have developed, we now prove the following.

LEMMA 5.4. *Let $\theta \in \Omega^1(S^1)$ be the form discussed in Examples 5.3 and 5.3 and 5.4. Then there does not exist a smooth function $f\colon S^1 \to \mathbb{R}$ such that $\theta = df$.*

PROOF. Assume that $f\colon S^1 \to \mathbb{R}$ satisfies $\mathrm{df} = \theta$. As in Example 5.4, let $\pi\colon \mathbb{R} \to S^1$ be the map defined via the formula $\pi(t) = (\cos(t), \sin(t))$, so $\pi^*(\omega) = \mathrm{dt}$. Set $g = f \circ \pi$, so by Lemma 5.3 we have $\mathrm{dg} = \pi^*(\omega) = \mathrm{dt}$. This implies that $\mathrm{d}(g - t) = 0$, so by Lemma 5.2 we have $g(t) = t - c$ for some constant $c \in \mathbb{R}$. However, since $g = f \circ \pi$ it must be the case that g is periodic, i.e. that $g(t+1) = g(t)$ for all $t \in \mathbb{R}$. This is a contradiction. $\square$

5.5. Path integrals

The main reason we introduced 1-forms was to define path integrals. Let M^n be a smooth manifold and let $\omega \in \Omega^1(M^n)$. Consider a smooth path in M^n, i.e. a smooth function $\gamma\colon [a, b] \to M^n$ for some $a < b$. Here $[a, b]$ is not a manifold, but rather a manifold with boundary; we will ignore this technicality for the moment. Let t be the coordinate function on $[a, b]$. We can then write $\gamma^*(\omega) = f(t)\mathrm{dt}$ for some smooth function $f\colon [a, b] \to \mathbb{R}$. Define

$$\int_\gamma \omega = \int_a^b f(t)\mathrm{dt}.$$

The following is an important example.

EXAMPLE. Let $\theta \in \Omega^1(S^1)$ be the form discussed in Examples 5.3 and 5.3 and 5.4. Define $\gamma\colon [0, 2\pi] \to S^1$ via the formula $\gamma(t) = (\cos(t), \sin(t))$. Then $\gamma^*(\theta) = \mathrm{dt}$, so

$$\int_\gamma \theta = \int_0^{2\pi} \mathrm{dt} = 2\pi.$$

$\square$

One important property of line integrals is the following lemma, which says that they do not depend on the parameterization of the curve.

LEMMA 5.5. *Let M^n be a smooth manifold, let $\gamma\colon [a, b] \to M^n$ be a smooth path, and let $\omega \in \Omega^1(M^n)$. Finally, let $h\colon [a, b] \to [a, b]$ be a smooth map such that $h(a) = a$ and $h(b) = b$. Define $\gamma_2 = \gamma \circ h$. Then $\int_\gamma \omega = \int_{\gamma_2} \omega$.*

PROOF. Homework. $\square$

REMARK. Lemma 5.5 is why we defined line integrals of differential 1-forms rather than of functions. If $f\colon M^n \to \mathbb{R}$ is a smooth function and $\gamma\colon [a, b] \to M^n$ is a smooth path, then one might be tempted to define $\int_\gamma f = \int_a^b (f \circ \gamma)\mathrm{dt}$. If one made this definition, then Lemma 5.5 would not hold. $\square$

Another is the following version of the fundamental theorem of calculus.

LEMMA 5.6. *Let M^n be a smooth manifold, let $\gamma\colon [a, b] \to M^n$ be a smooth path, and let $f\colon M^n \to \mathbb{R}$ be a smooth function. Then $\int_\gamma df = f(b) - f(a)$.*

PROOF. Homework. $\square$

To finish up this circle of ideas, we will prove the following theorem which completely describes the set of 1-forms on S^1.

THEOREM 5.7. *Let $\theta \in \Omega^1(S^1)$ be the form discussed in Examples 5.3 and 5.3 and 5.4 and let $\omega \in \Omega^1(S^1)$ be arbitrary. Then there exists some $c \in \mathbb{R}$ and some smooth function $f\colon S^1 \to \mathbb{R}$ such that $\omega = c\theta + df$.*

PROOF. Define $\pi\colon \mathbb{R} \to S^1$ via the formula $\pi(t) = (\cos(t), \sin(t))$ and let $\gamma\colon [0, 2\pi] \to S^1$ be the restriction of π to $[0, 2\pi]$. Set $c = \int_\gamma \omega$ and $\omega_1 = \omega - c\theta$. We thus have $\int_\gamma \omega_1 = 0$. Our goal is to find some smooth function $f\colon S^1 \to \mathbb{R}$ such that $\omega_1 = \mathrm{df}$.

Write $\pi^*(\omega_1) = g(t)\mathrm{dt}$. The function $g(t)$ is 2π-periodic in the sense that

$$g(t + 2\pi) = g(t) \qquad (t \in \mathbb{R}).$$

Define a function $F\colon \mathbb{R} \to \mathbb{R}$ via the formula

$$F(t) = \int_0^t g(t)\mathrm{dt}.$$

By the fundamental theorem of calculus, we know that $\mathrm{dF} = g(t)\mathrm{dt} = \pi^*(\omega_1)$. Moreover, since

$$\int_0^{2\pi} g(t)\mathrm{dt} = \int_\gamma \omega_1 = 0$$

and sicne $g(t)$ is 2π-periodic we see that $F(t)$ is 2π-periodic. This implies that F descends to S^1 in the sense that there exists a smooth function $f\colon S^1 \to \mathbb{R}$ such that $F = f \circ \pi$. We clearly have $\mathrm{df} = \omega_1$, as desired. $\square$

CHAPTER 6

Differential k-forms

In this chapter, we introduce the theory of differential k-forms for $k \geq 2$.

6.1. Multilinear forms

We begin by discussing some aspects of linear algebra. Let V be an $\mathbb{R}$-vector space.

Definition. A *k-multilinear form* on V is a function

$$\omega \colon \bigoplus_{i=1}^{k} V \to \mathbb{R}$$

such that for all $1 \leq i \leq \ell$ and all $\vec{v}_1, \ldots, \vec{v}_{\ell-1}, \vec{v}_{\ell+1}, \ldots, \vec{v}_k \in V$, the map $V \to \mathbb{R}$ defined via the rule

$$\vec{w} \mapsto \omega(\vec{v}_1, \ldots, \vec{v}_{\ell-1}, \vec{w}, \vec{v}_{\ell+1}, \ldots, \vec{v}_k)$$

is linear. $\square$

A 1-multilinear form is thus simply an element of the dual V^*, and a 2-multilinear form should satisfy

$$\omega(c\vec{w} + d\vec{w}', \vec{v}_2) = c\omega(\vec{w}, \vec{v}_2) + d\omega(\vec{w}', \vec{v}_2)$$

and

$$\omega(\vec{v}_1, c\vec{w} + d\vec{w}') = c\omega(\vec{v}_1, \vec{w}) + d\omega(\vec{v}_1, \vec{w}')$$

for all

$$\vec{v}_1, \vec{v}_2, \vec{w}, \vec{w}' \in V \quad \text{and} \quad c, d \in \mathbb{R}.$$

The vector space of all k-multilinear forms on V is denoted $\mathcal{T}^k(V)$.

Multilinear forms can be multiplied in the following way.

Construction. Let ω_1 be a k_1-multilinear form and ω_2 be a k_2-multilinear form. Define a function

$$\omega_1\omega_2 \colon \bigoplus_{i=1}^{k_1+k_2} V \to \mathbb{R}$$

via the formula

$$(\omega_1\omega_2)(\vec{v}_1, \ldots, \vec{v}_{k_1+k_2}) = \omega_1(\vec{v}_1, \ldots, \vec{v}_{k_1})\omega_2(\vec{v}_{k_1+1}, \ldots, \vec{v}_{k_1+k_2})$$

Then it is clear that $\omega_1\omega_2$ is a $(k_1 + k_2)$-multilinear form on V. $\square$

The following lemma provides a basis for $\mathcal{T}^k(V)$.

LEMMA 6.1. *Let V be an $\mathbb{R}$-vector space, let $\{\vec{e}_1, \ldots, \vec{e}_n\}$ be a basis for V, and let $\{\vec{e}_1^*, \ldots, \vec{e}_n^*\}$ be the dual basis for V^*. Then*

$$\{\vec{e}_{i_1}^* \cdots \vec{e}_{i_k}^* \mid 1 \leq i_1, i_2, \ldots, i_k \leq n\}$$

is a basis for $\mathcal{T}^k(V)$. In particular, $\mathcal{T}^k(V)$ is n^k-dimensional.

PROOF. We begin by introducing some notation. Let

$$\mathcal{I} = \{(i_1, \ldots, i_k \mid 1 \leq i_1, i_2, \ldots, i_k \leq n\}.$$

For an element $I = (i_1, \ldots, i_k)$ of $\mathcal{I}$, define

$$\vec{e}_I^* = \vec{e}_{i_1}^* \cdots \vec{e}_{i_k}^*.$$

For $\omega \in \mathcal{T}^k(V)$ and an element $I = (i_1, \ldots, i_k)$ of $\mathcal{I}$, define

$$\omega(\vec{e}_I) = \omega(\vec{e}_{i_1}, \ldots, \vec{e}_{i_k}).$$

Finally, let $\mathcal{B}$ be the purported basis for $\mathcal{T}^k(V)$.

We first prove that $\mathcal{B}$ is linearly independent. Assume that

$$\sum_{I \in \mathcal{I}} d_I \vec{e}_I^* = 0 \qquad (5)$$

for some $d_I \in \mathbb{R}$. For $I, I' \in \mathcal{I}$ we have

$$\vec{e}_I^*(\vec{e}_{I'}) = \begin{cases} 1 & \text{if } I = I', \\ 0 & \text{if } I \neq I'. \end{cases}$$

For all $I \in \mathcal{I}$, we can thus plug $\vec{e}_I$ into (5) and see that $d_I = 0$, as desired.

We next prove that $\mathcal{B}$ spans $\mathcal{T}^k(V)$. Consider $\omega \in \mathcal{T}^k(V)$. For $I \in \mathcal{I}$, define $c_I = \omega(\vec{e}_I) \in \mathbb{R}$. Set

$$\omega' = \sum_{I \in \mathcal{I}} c_I \vec{e}_I^*.$$

We then have $\omega'(\vec{e}_I) = \omega(\vec{e}_I)$ for all $I \in \mathcal{I}$. Using the multilinearity of ω, we see that for all $\vec{v} \in V$ the value of $\omega(\vec{v})$ is equal to an appropriate linear combination of the $\omega(\vec{e}_I)$ as I ranges over $\mathcal{I}$. A similar fact holds for ω'. We conclude that $\omega = \omega'$, as desired. $\square$

We now make the following important definition.

DEFINITION. An element $\omega \in \mathcal{T}^k(V)$ is *alternating* if flipping two inputs of ω changes its sign. More precisely, let S_k be the symmetric group on k letters. We then require that for $\vec{v}_1, \ldots, \vec{v}_k \in V$ and $\sigma \in S_k$ we have

$$\omega(\vec{v}_{\sigma(1)}, \ldots, \vec{v}_{\sigma(k)}) = (-1)^{|\sigma|}\omega(\vec{v}_1, \ldots, \vec{v}_k).$$

Here $(-1)^{|\sigma|}$ is the signature of the permutation σ. The set of all alternating k-multilinear forms on V is denoted $\mathcal{A}^k(V)$. $\square$

EXAMPLE. We can construct an alternating form $\det \in \mathcal{A}^n(\mathbb{R}^n)$ by letting $\det(\vec{v}_1, \ldots, \vec{v}_n)$ equal the determinant of the $n \times n$ matrix whose columns are $\vec{v}_1, \ldots, \vec{v}_n$ $\square$

REMARK. We will later see that $\mathcal{A}^n(\mathbb{R}^n) = \{s \det \mid s \in \mathbb{R}\}$; see Corollary 6.6. $\square$

To understand $\mathcal{A}^k(\mathbb{R}^n)$, we make the following construction.

Construction. Let V be a vector space and let $k \geq 1$. Define $\mathrm{alt}\colon \mathcal{T}^k(V) \to \mathcal{A}^k(V)$ via the formula

$$\mathrm{alt}(\omega)(\vec{v}_1, \ldots, \vec{v}_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{|\sigma|} \omega(\vec{v}_{\sigma(1)}, \ldots, \vec{v}_{\sigma(k)}) \qquad (\vec{v}_1, \ldots, \vec{v}_k \in V). \quad \square$$

The linear map alt has the following property.

Lemma 6.2. *For $\omega \in \mathcal{T}^k(V)$, we have $alt(\omega) = \omega$.*

Proof. An immediate consequence of the fact that S_k has $k!$ elements. $\square$

This allows us to make the following definition.

Definition. Consider $\omega_1 \in \mathcal{A}^{k_1}(V)$ and $\omega_2 \in \mathcal{A}^{k_2}(V)$. Define $\omega_1 \wedge \omega_2 = \mathrm{alt}(\omega_1 \omega_2) \in \mathcal{A}^{k_1+k_2}(V)$. $\square$

The wedge product is graded commutative in the sense that if $\omega_1 \in \mathcal{A}^{k_1}(V)$ and $\omega_2 \in \mathcal{A}^{k_2}(V)$, then

$$\omega_1 \wedge \omega_2 = (-1)^{k_1 k_2} \omega_2 \wedge \omega_1.$$

Our next goal is to prove that the wedge product is associative. This requires a lemma.

Lemma 6.3. *Consider $\omega_1 \in \mathcal{A}^{k_1}(V)$ and $\omega_2 \in \mathcal{A}^{k_2}(V)$. Then*

$$alt(\omega_1 \omega_2) = alt(alt(\omega_1)\omega_2) = alt(\omega_1\, alt(\omega_2)).$$

Proof. We will prove that

$$\mathrm{alt}(\omega_1\omega_2) = \mathrm{alt}(\mathrm{alt}(\omega_1)\omega_2);$$

the proof that

$$\mathrm{alt}(\omega_1\omega_2) = \mathrm{alt}(\omega_1 \mathrm{alt}(\omega_2))$$

is similar. Expanding out $\mathrm{alt}(\mathrm{alt}(\omega_1)\omega_2)$, we see that for

$$\vec{v}_1, \ldots, \vec{v}_{k_1+k_2} \in V$$

the number

$$\mathrm{alt}(\mathrm{alt}(\omega_1)\omega_2)(\vec{v}_1, \ldots, \vec{v}_{k_1+k_1})$$

equals

$$\frac{1}{(k_1+k_2)!} \sum_{\sigma \in S_{k_1+k_2}} \mathrm{alt}(\omega_1)(\vec{v}_{\sigma(1)}, \ldots, \vec{v}_{\sigma(k_1)}) \omega_2(\vec{v}_{\sigma(k_1+1)}, \ldots, \vec{v}_{\sigma(k_1+k_2)}). \tag{6}$$

By definition, we have

$$\mathrm{alt}(\omega_1)(\vec{v}_{\sigma(1)}, \ldots, \vec{v}_{\sigma(k_1)}) = \frac{1}{(k_1)!} \sum_{\delta \in S_{k_1}} \omega_1(\vec{v}_{\sigma(\delta(1))}, \ldots, \vec{v}_{\sigma(\delta(k_1))}).$$

Plugging this into (6) and regarding S_{k_1} as the subgroup of $S_{k_1+k_2}$ that acts on the first k_1 elements and fixes the rest, we get

$$\frac{1}{(k_1+k_2)!(k_1)!} \sum_{\substack{\sigma \in S_{k_1+k_2} \\ \delta \in S_{k_1}}} \omega_1(\vec{v}_{\sigma(\delta(1))}, \ldots, \vec{v}_{\sigma(\delta(k_1))}) \omega_2(\vec{v}_{\sigma(\delta(k_1+1))}, \ldots, \vec{v}_{\sigma(\delta(k_1+k_2))}).$$

Every element of $S_{k_1+k_2}$ can be written as $\sigma\delta$ with $\sigma \in S_{k_1+k_2}$ and $\delta \in S_{k_1}$ in precisely $(k_1)!$ ways. It follows that the above sum equals

$$\frac{(k_1)!}{(k_1+k_2)!(k_1)!} \sum_{\sigma \in S_{k_1+k_2}} \omega_1(\vec{v}_{\sigma(1)}, \ldots, \vec{v}_{\sigma(k_1)})\omega_2(\vec{v}_{\sigma(k_1+1)}, \ldots, \vec{v}_{\sigma(k_1+k_2)}).$$

Cancelling the $(k_1)!$'s, this is precisely

$$\mathrm{alt}(\omega_1\omega_2)(\vec{v}_1, \ldots, \vec{v}_{k_1+k_2}),$$

as desired. □

THEOREM 6.4. *For $\omega_1 \in \mathcal{A}^{k_1}(V)$ and $\omega_2 \in \mathcal{A}^{k_2}(V)$ and $\omega_3 \in \mathcal{A}^{k_3}(V)$, we have*

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3;$$

in fact, both sides equal

$$\mathit{alt}(\omega_1\omega_2\omega_3).$$

PROOF. Using Lemma 6.3, we see that

$$(\omega_1 \wedge \omega_2) \wedge \omega_3 = \mathrm{alt}(\mathrm{alt}(\omega_1\omega_2)\omega_3) = \mathrm{alt}(\omega_1\omega_2\omega_3).$$

Similarly,

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = \mathrm{alt}(\omega_1 \mathrm{alt}(\omega_2\omega_3)) = \mathrm{alt}(\omega_1\omega_2\omega_3).$$ □

Theorem 6.4 implies that the product operation $\wedge$ is associative. Moreover, applying it multiple times we see that if $\omega_i \in \mathcal{A}^{k_i}(V)$ for $1 \leq i \leq \ell$, then

$$\omega_1 \wedge \cdots \wedge \omega_\ell = \mathrm{alt}(\omega_1 \cdots \omega_\ell).$$

We close this section by proving the following lemma, which gives a basis for $\mathcal{A}^k(V)$. It should be compared to Lemma 6.1

LEMMA 6.5. *Let V be an $\mathbb{R}$-vector space, let $\{\vec{e}_1, \ldots, \vec{e}_n\}$ be a basis for V, and let $\{\vec{e}_1^*, \ldots, \vec{e}_n^*\}$ be the dual basis for V^*. Then*

$$\{\vec{e}_{i_1}^* \wedge \cdots \wedge \vec{e}_{i_k}^* \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

is a basis for $\mathcal{A}^k(V)$. In particular, $\mathcal{A}^k(V)$ is $\binom{n}{k}$-dimensional.

PROOF. This can be proved exactly like Lemma 6.1. The only difference is that an alternative k-multilinear form ω is determined by the set of values of

$$\omega(\vec{e}_{i_1}, \ldots, \vec{e}_{i_k})$$

with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ (while for a general k-multilinear form we would need to allow $1 \leq i_1, \ldots, i_k \leq n$). This follows from two facts.

- For $1 \leq i_1, \ldots, i_k \leq n$, if $i_j = i_{j'}$ for distinct $1 \leq j, j' \leq k$, then $\omega(\vec{e}_{i_1}, \ldots, \vec{e}_{i_k}) = 0$ since flipping $\vec{e}_{i_j}$ and $\vec{e}_{i_{j'}}$ multiplies it by -1 while not changing its value.
- For distinct $1 \leq i_1, \ldots, i_k \leq n$, there exists a unique $\sigma \in S_k$ such that $\sigma(i_1) < \sigma(i_2) < \cdots < \sigma(i_k)$, and

$$\omega(\vec{e}_{\sigma(i_1)}, \ldots, \vec{e}_{\sigma(i_k)}) = (-1)^{|\sigma|}\omega(\vec{e}_{i_1}, \ldots, \vec{e}_{i_k}).$$ □

We highlight the following corollary of Lemma 6.5.

COROLLARY 6.6. *Let V be an n-dimensional $\mathbb{R}$-vector space. Then $\mathcal{A}^n(V)$ is 1-dimensional. In particular, if $V = \mathbb{R}^n$ then every element of $\mathcal{A}^n(V)$ is a multiple of the determinant (see Example 6.1).*

PROOF. This follows from Lemma 6.5 together with the fact that $\binom{n}{n} = 1$. $\square$

6.2. Basics of k-forms

Let M^n be a smooth manifold and let $k \geq 1$. For $p \in M^n$, define

$$\mathcal{A}_p^k(M^n) = \mathcal{A}^k(T_p M^n).$$

We thus have

$$\mathcal{A}_p^1(M^n) = T_p^* M^n.$$

Lemma 6.5 implies that $\mathcal{A}_p^k(M^n)$ is an $\binom{n}{k}$-dimensional $\mathbb{R}$-vector space. Define

$$\begin{aligned}\mathcal{A}^k(M^n) &= \{(p, \omega_p) \mid p \in M^n \text{ and } \omega_p \in \mathcal{A}_p^k(M^n)\} \\ &= \sqcup_{p \in M^n} \mathcal{A}_p^k(M^n).\end{aligned}$$

Just like we did for the tangent bundle in §2.4 and for the cotangent bundle in §5.2, we can define a topology on $\mathcal{A}^k(M^n)$ which makes it into a smooth $n + \binom{n}{k}$-dimensional manifold. A *differential p-form* on M^n is a smooth map $\omega\colon M^n \to \mathcal{A}^k(M^n)$ such that $\omega(p) \in \mathcal{A}_p^k(M^n)$ for all $p \in M^n$. The $\mathbb{R}$-vector space of all differential k-forms on M^n is denoted $\Omega^k(M^n)$. Just like for differential 1-forms, $\Omega^k(M^n)$ is a module over the ring $C^\infty(M^n)$ of smooth functions from M^n to $\mathbb{R}$.

For $\omega_1 \in \Omega^{k_1}(M^n)$ and $\omega_2 \in \Omega^{k_2}(M^n)$, we can define an element $\omega_1 \wedge \omega_2 \in \Omega^{k_1+k_2}(M^n)$ via the formula

$$(\omega_1 \wedge \omega_2)(p) = \omega_1(p) \wedge \omega_2(p) \qquad (p \in M^n).$$

This operation is graded commutative in the sense that

$$\omega_2 \wedge \omega_1 = (-1)^{k_1 k_2} \omega_1 \wedge \omega_2$$

and is associative in the sense that if $\omega_3 \in \Omega^{k_3}(M^n)$ then

$$(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3).$$

Because of this latter property, we will omit frequently omit parentheses in our formulas.

Using this wedge product, we can construct many k-forms from our already existing store of 1-forms. Another interesting class of differential forms are volume forms, which are defined as follows.

DEFINITION. A *volume form* on a smooth n-dimensional manifold M^n is a differential n-form $\omega \in \Omega^n(M^n)$ such that $\omega(p) \neq 0$ for all $p \in M^n$. $\square$

Not all smooth manifolds support volume forms (we will later see that a necessary and sufficient condition is that the manifold be *orientable*). Here are some that do.

EXAMPLE. If U is an open subset of $\mathbb{R}^n$, then U can be given the volume form $\mathrm{dx}_1 \wedge \mathrm{dx}_2 \wedge \cdots \wedge \mathrm{dx}_n$. $\square$

EXAMPLE. Consider a smoothly embedded submanifold M^n of $\mathbb{R}^m$. One's first impulse might be to try to restrict the above volume form on $\mathbb{R}^m$ to M^n; however, this does not work unless $n = m$ since it would result in a differential form of the wrong dimension. Some additional structure is needed. We will deal with a particularly easy case, namely where $m = n + 1$ and where there exists a *unit normal vector field* on M^n, i.e. a smooth map $\mathfrak{n}\colon M^n \to \mathbb{R}^{n+1}$ with the following two properties.

- $\| \mathfrak{n}(p)\| = 1$ for all $p \in \mathbb{R}^{n+1}$, and
- The vector $\mathfrak{n}(p) \in \mathbb{R}^{n+1}$ is orthogonal to the tangent space $T_p(M^n) \subset \mathbb{R}^{n+1}$ for all $p \in M^n$.

For example, the sphere $S^n \subset \mathbb{R}^{n+1}$ supports the unit normal vector field that takes $p \in S^n$ to itself (considered as a point of $\mathbb{R}^{n+1}$). Similarly, it is easy to construct unit normal vector fields on the standard ways of embedding genus g surfaces into $\mathbb{R}^3$; however, they cannot be constructed on the Möbius band. Given a unit normal vector field $\mathfrak{n}$ on $M^n \subset \mathbb{R}^{n+1}$, we can define a volume form $\omega \in \Omega^n(M^n)$ by setting

$$\omega(p)(\vec{v}_1, \ldots, \vec{v}_n) = \omega_{\mathbb{R}^{n+1}}(p)(\mathfrak{n}(p), \vec{v}_1, \ldots, \vec{v}_n)$$

for $p \in M^n$ and $\vec{v}_1, \ldots, \vec{v}_n \in T_pM^n$. Here $\omega_{\mathbb{R}^{n+1}}$ is the above volume form on $\mathbb{R}^{n+1}$. □

6.3. The local picture, I

Let M^n be a smooth manifold, let $\omega \in \Omega^k(M^n)$, and let $\phi_1 \colon U_1 \to V_1$ and $\phi_2 \colon U_2 \to V_2$ be two charts. Consider $p \in U_1 \cap U_2$, and set $q_1 = \phi_1(p)$ and $q_2 = \phi_2(p)$. We thus have identifications

$$T_pM^n = T_{q_1}V_1 \quad \text{and} \quad T_pM^n = T_{q_2}V_2.$$

The alternating multilinear form $\omega(p)$ on T_pM^n can thus be identified with alternating multilinear forms $\omega_1(q_1)$ on $T_{q_1}V_1 = \mathbb{R}^n$ and $\omega_2(q_2)$ on $T_{q_2}V_2 = \mathbb{R}^n$. We can related $\omega_1(q_1)$ and $\omega_2(q_2)$ just like we did for cotangent vectors in §5.1. Namely, if $\tau_{12} \colon \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is the transition map, then $T_{q_1}V_1$ and $T_{q_2}V_2$ are identified via the derivative map

$$D_{q_1}\tau_{12} \colon T_{q_1}V_1 \to T_{q_2}V_2.$$

For $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$, we then have

$$\omega_2(q_2)(\vec{v}_1, \ldots, \vec{v}_k) = \omega_1(q_1)(D_{q_1}^{-1}(\vec{v}_1), \ldots, D_{q_1}^{-1}(\vec{v}_k)).$$

6.4. The local picture, II

Again let M^n be a smooth manifold and let $\omega \in \Omega^k(M^n)$. Fix a chart $\phi \colon U \to V$ on M^n and consider some $\omega \in \Omega^k(M^n)$. Letting $x_1, \ldots, x_n$ be the coordinate functions of V, we have the differential 1-forms $\{\mathrm{dx}_1, \ldots, \mathrm{dx}_n\}$ on U. For a sequence $(i_1, \ldots, i_k)$ of numbers satisfying $1 \leq i_1, \ldots, i_k \leq n$, define

$$\mathrm{dx}_I = \mathrm{dx}_{i_1} \wedge \cdots \wedge \mathrm{dx}_{i_k}.$$

Setting

$$\mathcal{I} = \{(i_1, \ldots, i_k) \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\},$$

we can apply Lemma 6.5 to deduce that the restriction of ω to U can be uniquely written as

$$\sum_{I \in \mathcal{I}} f_I \mathrm{dx}_I \tag{7}$$

for functions $f_I \colon M^n \to \mathbb{R}$. The fact that ω is a smooth map from M^n to $\mathcal{A}^k(M^n)$ is equivalent to the fact that these functions f_I are smooth. The expression (7) will be called the *expression for ω in the local coordinates $\phi \colon U \to V$*. The expressions for ω in different local coordinates are related as in §6.3.

6.5. Pulling forms back

Now let $f\colon M_1^{n_1} \to M_2^{n_2}$ be a smooth map. Recall that earlier we showed how to construct a pull-back map

$$f^*\colon \Omega^1(M_2^{n_2}) \to \Omega^1(M_1^{n_1})$$

by composing with the derivative map

$$D_p\colon T_pM_1^{n_1} \to T_{f(p)}M_2^{n_2}$$

for all $p \in M_1^{n_1}$. The same idea works for k-forms. Namely, we can define a pull-back map

$$f^*\colon \Omega^k(M_2^{n_2}) \to \Omega^k(M_1^{n_1})$$

as follows. Consider $\omega \in \Omega^k(M_2^{n_2})$. The desired element $f^*(\omega) \in \Omega^k(M_1^{n_1})$ consists of an alternating k-multilinear form $f^*(\omega)(p)$ on $T_pM_1^{n_1}$ for all $p \in M_1^{n_1}$. The form ω gives us an alternating k-multilinear form $\omega(f(p))$ on $T_{f(p)}M_2^{n_2}$, and we define $f^*(\omega)(p)$ via the formula

$$f^*(\omega)(p)(\vec{v}_1, \ldots, \vec{v}_k) = \omega(f(p))((D_pf)(\vec{v}_1), \ldots, (D_pf)(\vec{v}_k))$$

for $\vec{v}_1, \ldots, \vec{v}_k \in T_pM_1^{n_1}$.

This pull-back respects the wedge product of forms in the following sense.

LEMMA 6.7. *Let $f\colon M_1^{n_1} \to M_2^{n_2}$, let $\omega_1 \in \Omega^{k_1}(M_2^{n_2})$, and let $\omega_2 \in \Omega^{k_2}(M_2^{n_2})$. Then*

$$f^*(\omega_1 \wedge \omega_2) = f^*(\omega_1) \wedge f^*(\omega_2).$$

PROOF. Immediate from the definitions. □

6.6. The d-operator: big picture

Let M^n be a smooth manifold and let $f\colon M^n \to \mathbb{R}$ be a smooth map. Recall that we have a natural element $\mathrm{df} \in \Omega^1(M^n)$ defined as follows. For $p \in M^n$, the element $\mathrm{df}(p) \in T_p^*(M^n)$ should be a linear map $T_pM^n \to \mathbb{R}$. This linear map takes $\vec{v} \in T_pM^n$ to the directional derivative of f in the direction of $\vec{v}$, i.e. to the image of $\vec{v}$ under the linear map

$$D_pf\colon T_pM^n \to T_{f(p)}\mathbb{R} = \mathbb{R}.$$

We will regard d as a linear map

$$d\colon C^\infty(M^n) \to \Omega^1(M^n).$$

In the next three sections, we will show how this can be generalized to a linear map

$$d\colon \Omega^k(M^n) \to \Omega^{k+1}(M^n)$$

for all $k \geq 1$. This linear map will satisfy the following three properties. To simplify their statements, we will write $\Omega^0(M^n)$ for $C^\infty(M^n)$, and also for $f \in \Omega^0(M^n)$ and $\omega \in \Omega^k(M^n)$ we will write $f \wedge \omega$ for the product $f\omega \in \Omega^k(M^n)$.

- For $\omega_1 \in \Omega^{k_1}(M^n)$ and $\omega_2 \in \Omega^{k_2}(^n)$, we have
$$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge \mathrm{d}\omega_2.$$
- For $\omega \in \Omega^k(M^n)$, we have $\mathrm{d}(\mathrm{d}\omega) = 0$.
- For $\omega \in \Omega^k(M^n)$ and $f\colon M_2^{n_2} \to M^n$ a smooth map between smooth manifolds, we have
$$f^*(\mathrm{d}\omega) = \mathrm{d}(f^*(\omega)).$$

The most elegant way to construct this would be to give a global definition that does not depend on a choice of a local coordinate system (like our definition of df above). However, the formulas for doing this are a little complicated and difficult to parse on a first reading, so we will instead do this in a more low-brow way. Namely, in §6.7 we will construct the d operator for M^n an open subset of $\mathbb{R}^n$ and prove that this has the above two properties. In §6.8 we will extract from the previous local definition a key property. Finally, in §6.9 we will show that this definition "glues together" to give an appropriate operator on an arbitrary smooth manifold.

6.7. The d-operator: local definition

Consider an open set $V \subset \mathbb{R}^n$. Letting $x_1, \ldots, x_n$ be the coordinate functions of $\mathbb{R}^n$ and letting

$$\mathcal{I}_k = \{(i_1, \ldots, i_k) \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\},$$

a k-form $\omega \in \Omega^k(V)$ with $k \geq 1$ can be uniquely written as

$$\omega = \sum_{I \in \mathcal{I}_k} f_I \mathrm{dx}_I$$

for some smooth functions $f_I \colon V \to \mathbb{R}$. We then define

$$\mathrm{d}\omega = \sum_{I \in \mathcal{I}_k} \mathrm{df} \wedge \mathrm{dx}_I \in \Omega^{k+1}(V).$$

It is clear that this gives an $\mathbb{R}$-linear map $d \colon \Omega^k(V) \to \Omega^{k+1}(V)$ for all $k \geq 1$. The following three lemmas show that it has the three properties discussed in §6.6.

Lemma 6.8. *Let V be an open subset of $\mathbb{R}^n$. For some $k_1, k_2 \geq 0$ let $\omega_1 \in \Omega^{k_1}(V)$ and $\omega^2 \in \Omega^{k_2}(V)$. Then $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge d\omega_2$.*

Proof. We divide this into three cases.

Case 1. *For $f, g \in \Omega^0(V)$, we have $d(fg) = g(df) + f(dg)$.*

Using the product rule, we have

$$\begin{aligned}
\mathrm{d}(fg) &= \sum_{i=1}^{n} \frac{\partial fg}{\partial x_i} \mathrm{dx}_i \\
&= \sum_{i=1}^{n} \left(g \frac{\partial f}{\partial x_i} + f \frac{\partial g}{\partial x_i} \right) \mathrm{dx}_i \\
&= g(\mathrm{df}) + f(\mathrm{dg}),
\end{aligned}$$

as desired.

Case 2. *For $f \in \Omega^0(V)$ and $\omega \in \Omega^k(V)$ with $k \geq 1$, we have $d(f\omega) = (df) \wedge \omega + f \wedge d\omega$.*

Write

$$\omega = \sum_{I \in \mathcal{I}_k} g_I \mathrm{dx}_I.$$

Using the first case, we then have

$$\begin{aligned} \mathrm{d}f\omega &= \sum_{I\in\mathcal{I}_k} \mathrm{d}(fg_I)\wedge \mathrm{dx}_I \\ &= \sum_{I\in\mathcal{I}_k} (g_I(\mathrm{df}) + f(\mathrm{dg}_I))\wedge \mathrm{dx}_I \\ &= (\mathrm{df})\wedge\omega + f\wedge \mathrm{d}\omega, \end{aligned}$$

as desired.

Case 3. *For $\omega_1 \in \Omega^{k_1}(V)$ and $\omega_2 \in \Omega^{k_2}(V)$ with $k_1, k_2 \geq 1$, we have $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge d\omega_2$.*

Write

$$\omega_1 = \sum_{I\in\mathcal{I}_{k_1}} f_I \mathrm{dx}_I$$

and

$$\omega_2 = \sum_{J\in\mathcal{I}_{k_2}} g_J \mathrm{dx}_J.$$

We now calculate that

$$\mathrm{d}\omega_1 = \sum_{I\in\mathcal{I}_{k_1}} \mathrm{df}_I \wedge \mathrm{dx}_I$$

and

$$\mathrm{d}\omega_2 = \sum_{J\in\mathcal{I}_{k_2}} \mathrm{dg}_J \wedge \mathrm{dx}_J,$$

and hence $\mathrm{d}\omega_1 \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge \mathrm{d}\omega_2$ equals

$$\begin{aligned} &(\sum_{I\in\mathcal{I}_{k_1}} \mathrm{df}_I\wedge \mathrm{dx}_I)\wedge(\sum_{J\in\mathcal{I}_{k_2}} g_J\mathrm{dx}_J) + (-1)^{k_1}(\sum_{I\in\mathcal{I}_{k_1}} f_I\mathrm{dx}_I)\wedge(\sum_{J\in\mathcal{I}_{k_2}} \mathrm{dg}_J\wedge \mathrm{dx}_J) \\ &= \sum_{\substack{I\in\mathcal{I}_{k_1}\\ J\in\mathcal{I}_{k_2}}} (g_J\mathrm{df}_I\wedge\mathrm{dx}_I\wedge\mathrm{dx}_J + (-1)^{k_1} f_I\wedge \mathrm{dx}_I\wedge\mathrm{dg}_J\wedge\mathrm{dx}_J). \end{aligned}$$

For all $I \in \mathcal{I}_{k_1}$ and $J \in \mathcal{I}_{k_2}$, we have $\mathrm{dx}_I \wedge \mathrm{dg}_J = (-1)^{k_1}\mathrm{dg}_J \wedge \mathrm{dx}_I$. It follows that the above expression equals

$$\sum_{\substack{I\in\mathcal{I}_{k_1}\\ J\in\mathcal{I}_{k_2}}} (g_J\mathrm{df}_I + f_I\mathrm{dg}_J)\wedge\mathrm{dx}_I\wedge\mathrm{dx}_J.$$

Using the previous cases, we see that this equals

$$\sum_{\substack{I\in\mathcal{I}_{k_1}\\ J\in\mathcal{I}_{k_2}}} \mathrm{d}(f_I g_J)\wedge\mathrm{dx}_I\wedge\mathrm{dx}_J.$$

For $I \in \mathcal{I}_{k_1}$ and $J \in \mathcal{I}_{k_2}$, we have $\mathrm{dx}_I \wedge \mathrm{dx}_J = 0$ if I and J share any entries. Otherwise, $\mathrm{dx}_I \wedge \mathrm{dx}_J = \pm\mathrm{dx}_{I'}$ for some unique $I' \in \mathcal{I}_{k_1+k_2}$ and some choice of sign. It follows that $\mathrm{d}(f_I g_J) \wedge \mathrm{dx}_I \wedge \mathrm{dx}_J = \mathrm{d}((f_I\mathrm{dx}_I) \wedge (g_J \wedge \mathrm{dx}_J))$, and thus the above expression equals

$$\mathrm{d}\left(\left(\sum_{I\in\mathcal{I}_{k_1}} f_I\mathrm{dx}_I\right)\wedge\left(\sum_{J\in\mathcal{I}_{k_2}} g_J\mathrm{dx}_J\right)\right) = \mathrm{d}(\omega_1\wedge\omega_2),$$

as desired. □

LEMMA 6.9. *Let V be an open subset of $\mathbb{R}^n$ and let $\omega \in \Omega^k(V)$ for some $k \geq 0$. Then $d(d\omega) = 0$.*

PROOF. There are two cases.

CASE 1. $k = 0$.

In this case, ω is a smooth function $f\colon V \to \mathbb{R}^n$. We then calculate that

$$\begin{aligned}
\mathrm{d}(\mathrm{df}) &= \mathrm{d}\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{dx}_i\right) \\
&= \sum_{i=1}^{n} \mathrm{d}\left(\frac{\partial f}{\partial x_i}\right) \wedge \mathrm{dx}_i \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i} \mathrm{dx}_j \wedge \mathrm{dx}_i.
\end{aligned}$$

For $i = j$, we have $\mathrm{dx}_j \wedge \mathrm{dx}_i = 0$. Applying the equality of mixed partial derivatives, we thus see that the above expression equals

$$\sum_{\substack{1 \leq i,j \leq n \\ i \neq j}} \frac{\partial^2 f}{\partial x_j \partial x_i} \mathrm{dx}_j \wedge \mathrm{dx}_i = \sum_{1 \leq i < j \leq n} \frac{\partial^2 f}{\partial x_j \partial x_i} (\mathrm{dx}_j \wedge \mathrm{dx}_i - \mathrm{dx}_i \wedge \mathrm{dx}_j) = 0,$$

as desired.

CASE 2. $k \geq 1$.

Write

$$\omega = \sum_{I \in \mathcal{I}_k} f_I \mathrm{dx}_I.$$

Using Lemma 6.8, we have

$$\begin{aligned}
\mathrm{d}(\mathrm{d}\omega) &= \mathrm{d}\left(\sum_{I \in \mathcal{I}_k} \mathrm{df}_I \wedge \mathrm{dx}_I\right) \\
&= \sum_{I \in \mathcal{I}_k} \mathrm{d}(\mathrm{df}_I) \wedge \mathrm{dx}_I - \mathrm{df}_I \wedge \mathrm{d}(\mathrm{dx}_I).
\end{aligned}$$

By the first case, all of these terms equal 0, as desired. □

LEMMA 6.10. *Let $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ be open subsets and let $f\colon V \to W$ be a smooth map. Consider $\omega \in \Omega^k(V)$ for some $k \geq 0$. Then $f^*(d\omega) = d(f^*(\omega))$.*

PROOF. An immediate consequence of the definitions together with the naturality of the wedge product (Lemma 6.7) and the naturality of the d-operator on smooth functions (Lemma 5.3). □

6.8. The d-operator: a key lemma

In this section, we apply the local results of the previous section to prove the following lemma.

LEMMA 6.11. *Let M^n be a smooth manifold. For some $k \geq 1$, let $\omega \in \Omega^k(M^n)$. Assume that we can write*

$$\omega = \sum_{i=1}^{m} f_i \, dg_{i,1} \wedge dg_{i,2} \wedge \cdots \wedge dg_{i,k} \tag{8}$$

for some smooth functions $f\colon M^n \to \mathbb{R}$ and $g_{i,j}\colon M^n \to \mathbb{R}$. Then

$$\sum_{i=1}^{m} df_i \wedge dg_{i,1} \wedge dg_{i,2} \wedge \cdots \wedge dg_{i,k} \in \Omega^{k+1}(M^n).$$

does not depend on the decomposition (8).

PROOF. Let

$$\omega = \sum_{i=1}^{m'} f'_i \mathrm{dg}'_{i,} \wedge \mathrm{dg}'_{i,2} \wedge \cdots \wedge \mathrm{dg}'_{i,k}$$

be another such decomposition. Our goal is to prove that

$$\sum_{i=1}^{m} \mathrm{df}_i \wedge \mathrm{dg}_{i,1} \wedge \mathrm{dg}_{i,2} \wedge \cdots \wedge \mathrm{dg}_{i,k} = \sum_{i=1}^{m'} \mathrm{df}'_i \wedge \mathrm{dg}'_{i,1} \wedge \mathrm{dg}'_{i,2} \wedge \cdots \wedge \mathrm{dg}'_{i,k}. \tag{9}$$

Assume first that M^n is an open subset V of $\mathbb{R}^n$. In §6.7, we constructed a form $\mathrm{d}\omega \in \Omega^{k+1}(V)$ from $\omega \in \Omega^k(V)$. We claim that

$$\mathrm{d}\omega = \sum_{i=1}^{m} \mathrm{df}_i \wedge \mathrm{dg}_{i,1} \wedge \mathrm{dg}_{i,2} \wedge \cdots \wedge \mathrm{dg}_{i,k} \in \Omega^{k+1}(M^n).$$

The proof of this is by induction on k. The base case $k = 0$ is trivial, so assume that $k \geq 1$ and that the desired result is true for all smaller values of k. Using Lemmas 6.8 and 6.9 together with our inductive hypothesis, we have

$$\begin{aligned}
\mathrm{d}\omega &= \sum_{i=1}^{m} \mathrm{d}(f_i \mathrm{dg}_{i,1} \wedge \cdots \wedge \mathrm{dg}_{i,k-1}) + (-1)^{k-1}(f_i \mathrm{dg}_{i,1} \wedge \cdots \wedge \mathrm{dg}_{i,k-1}) \wedge \mathrm{d}(\mathrm{dg}_{i,k}) \\
&= \sum_{i=1}^{m} \mathrm{df}_i \wedge \mathrm{dg}_{i,1} \wedge \mathrm{dg}_{i,2} \wedge \cdots \wedge \mathrm{dg}_{i,k} + 0,
\end{aligned}$$

as desired. A similar argument shows that

$$\mathrm{d}\omega = \sum_{i=1}^{m'} \mathrm{df}'_i \wedge \mathrm{dg}'_{i,1} \wedge \mathrm{dg}'_{i,2} \wedge \cdots \wedge \mathrm{dg}'_{i,k},$$

so we conclude that the desired equation (9) holds.

Now assume that M^n is a general manifold. Letting $\phi\colon U \to V$ be a chart, the previous paragraph proves that the restrictions to U of the left and right hand sides of (9) are the same. Since M^n can be covered by such U, we deduce that in (9) holds, as desired. $\square$

6.9. The d-operator: global definition

Fix some smooth manifold M^n and some $k \geq 1$. Our goal is to construct a linear map

$$d \colon \Omega^k(M^n) \to \Omega^{k+1}(M^n).$$

Consider some $\omega \in \Omega^k(M^n)$. We can find an open cover $\{U_\alpha\}_{\alpha \in A}$ of M^n with the following property. For each $\alpha \in A$, the restriction of ω to U_α can be written as

$$\omega|_{U_\alpha} = \sum_{i=1}^{m} f_i \mathrm{d}\mathrm{g}_{i,1} \wedge \mathrm{d}\mathrm{g}_{i,2} \wedge \cdots \wedge \mathrm{d}\mathrm{g}_{i,k} \tag{10}$$

for some smooth functions $f_i \colon M^n \to \mathbb{R}$ and $g_{i,j} \colon M^n \to \mathbb{R}$. For example, we can take the U_α to be the domains of charts $\phi_\alpha \colon U_\alpha \to V_\alpha$ and the $g_{i,j}$'s to be appropriate coordinate functions on these charts. Define $\mathrm{d}\omega_\alpha \in \Omega^{k+1}(U_\alpha)$ via the formula

$$\mathrm{d}\omega_\alpha = \sum_{i=1}^{m} \mathrm{d}\mathrm{f}_i \wedge \mathrm{d}\mathrm{g}_{i,1} \wedge \mathrm{d}\mathrm{g}_{i,2} \wedge \cdots \wedge \mathrm{d}\mathrm{g}_{i,k}.$$

Lemma 6.11 implies that this only depends on ω and U_α, not on the decomposition (10). Lemma 6.11 also shows that for all $\alpha, \alpha' \in A$, the restrictions of $\mathrm{d}\omega_\alpha$ and $\mathrm{d}\omega_{\alpha'}$ to $U_\alpha \cap U_{\alpha'}$ are equal; indeed, these restrictions can be computed by restricting the decompositions (10) associated to U_α and $U_{\alpha'}$ to $U_\alpha \cap U_{\alpha'}$. From this, we see that the $\mathrm{d}\omega_\alpha$ glue together to give a well-defined element $\mathrm{d}\omega \in \Omega^{k+1}(M^n)$ such that the restriction of $\mathrm{d}\omega$ to U_α equals $\mathrm{d}\omega_\alpha$ for all $\alpha \in A$.

One might worry that $\mathrm{d}\omega$ depends on the choice of open cover $\{U_\alpha\}_{\alpha \in A}$. However, if $\{U'_\beta\}_{\beta \in B}$ is another choice of cover, then so is the set

$$\{U_\alpha\}_{\alpha \in A} \cup \{U'_\beta\}_{\beta \in B}.$$

From this, we see that $\mathrm{d}\omega$ does not depend on our choice of cover.

This completes the construction of d. The following theorem summarizes its properties.

THEOREM 6.12. *Let M^n be a smooth manifold. Then the following hold.*

- *For $\omega_1 \in \Omega^{k_1}(M^n)$ and $\omega_2 \in \Omega^{k_2}(^n)$, we have*
$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge d\omega_2.$$
- *For $\omega \in \Omega^k(M^n)$, we have $d(d\omega) = 0$.*
- *For $\omega \in \Omega^k(M^n)$ and $f \colon M_2^{n_2} \to M^n$ a smooth map between smooth manifolds, we have*
$$f^*(d\omega) = d(f^*(\omega)).$$

PROOF. Immediate from Lemmas 6.8 and 6.9 and 6.10, which prove the corresponding local results. □

For later use, we make the following definitions.

DEFINITION. Let M^n be a smooth manifold and let $\omega \in \Omega^k(M^n)$. We say that ω is *closed* if $\mathrm{d}\omega = 0$ and is *exact* if there exists some $\omega' \in \Omega^{k-1}(M^n)$ such that $\mathrm{d}\omega' = \omega$. □

LEMMA 6.13. *If $\omega \in \Omega^k(M^n)$ is an exact form, then ω is closed.*

PROOF. Write $\omega = \mathrm{d}\omega'$ for some $\omega' \in \Omega^{k-1}(M^n)$. Then $\mathrm{d}\omega = \mathrm{d}(\mathrm{d}\omega') = 0$ by the third property in Theorem 6.12. □

EXAMPLE. Since $\Omega^k(M^n) = 0$ for $k > n$, all n-forms on M^n are closed. In particular, all volume forms on M^n are closed. We will later see that on *compact* manifolds volume forms are not exact. $\square$

CHAPTER 7

Orientations

This brief chapter is devoted to orientations on manifolds.

7.1. Vector spaces

We begin by discussing orientations on vector spaces, which are defined as follows.

Definition. Let V be an n-dimensional real vector space with $n \geq 1$. An *orientation* on V is an equivalence class of ordered basis $(\vec{v}_1, \ldots, \vec{v}_n)$ for V under the following equivalence relation:

- If $b = (\vec{v}_1, \ldots, \vec{v}_n)$ and $b' = (\vec{v}_1', \ldots, \vec{v}_n')$ are ordered bases for V, then $b \sim b'$ if $\det(f) > 0$, where $f \colon V \to V$ is the linear map satisfying $f(\vec{v}_i) = \vec{v}_i'$ for $1 \leq i \leq n$.

If V is equipped with a fixed orientation, then we will call V an *oriented vector space* and any ordered basis representing that orientation an *oriented basis* for V. $\square$

The first basic property of orientations is as follows.

Lemma 7.1. *Let V be an n-dimensional real vector space with $n \geq 1$. Then V has exactly two orientations.*

Proof. Let $b = (\vec{v}_1, \ldots, \vec{v}_n)$ and $b' = (\vec{v}_1', \ldots, \vec{v}_n')$ be two ordered bases for V. Define $b'' = (-\vec{v}_1', \vec{v}_2', \ldots, \vec{v}_n')$. Since multiplying a column of a matrix by -1 has the effect of multiplying its determinant by -1, it follows that b represents the same orientation as either b' or b''. $\square$

This lemma implies that the following definition makes sense.

Definition. Let V be an n-dimensional real vector space and let b be an orientation of V. Then $-b$ will denote the other orientation. $\square$

As notation, we will write $\mathcal{O}(V)$ for the set of orientations on V. If $f \colon V \to V'$ is a vector space isomorphism, then f induces a bijection $f_* \colon \mathcal{O}(V) \to \mathcal{O}(V')$. If $V = V'$, then $f \in \text{Aut}(V)$ and we can write this bijection using the formula

$$f_*(b) = \begin{cases} b & \text{if } \det(f) > 0, \\ -b & \text{if } \det(f) < 0. \end{cases}$$

Finally, the *standard orientation* on $\mathbb{R}^n$ is the orientation corresponding to the standard basis of $\mathbb{R}^n$.

7.2. The orientation bundle

Let M^n be a smooth manifolds. Informally, an orientation on a smooth manifold M^n is a choice of orientation on each tangent space T_pM^n that varies smoothly. To make this precise, we define an orientation bundle on M^n. The definition is similar to that of the tangent and cotangent bundles.

Given a finite-dimensional real vector space V, let $\mathcal{O}(V)$ be the set consisting of the two orientations on V. Define $\mathcal{O}(M^n)$ to be the set

$$\mathcal{O}(M^n) = \{(p, b) \mid p \in M^n \text{ and } b \in \mathcal{O}(T_pM^n)\}.$$

We define a topology on $\mathcal{O}(M^n)$ as follows. Let $\phi\colon U \to V$ be a chart on M^n. Define $\mathcal{O}(U)$ to be the subset

$$\{(p, b) \mid p \in U \text{ and } b \in \mathcal{O}(T_pM^n)\}$$

of $\mathcal{O}(M^n)$. For $p \in U$, our definition of $\mathcal{O}(T_pM^n)$ identifies it with $\mathcal{O}(T_{\phi(p)}V) = \mathcal{O}(\mathbb{R}^n)$. Define a map $\mathcal{O}(\phi)\colon \mathcal{O}(U) \to V \times \mathcal{O}(\mathbb{R}^n)$ via the formula

$$\mathcal{O}(\phi)(p, b) = (\phi(p), b).$$

Giving $\mathcal{O}(\mathbb{R}^n)$ the discrete topology, we want to construct a topology on $\mathcal{O}(M^n)$ such that if $\mathcal{O}(U)$ is given the subspace topology, then $\mathcal{O}(\phi)$ is a homeomorphism. Define

$$\mathcal{U} = \{\mathcal{O}(\phi)^{-1}(W) \mid \phi\colon U \to V \text{ a chart on } M^n \text{ and } W \subset V \times \mathcal{O}(\mathbb{R}^n) \text{ is open}\}.$$

It is easy to see that $\mathcal{U}$ is a basis for a topology, and that under this topology the induced topology on the subsets $\mathcal{O}(U)$ is such that $\mathcal{O}(\phi)$ is a homeomorphism. We endow $\mathcal{O}(M^n)$ with this topology.

Now, the set $V \times \mathcal{O}(\mathbb{R}^n)$ is homeomorphic to two disjoint copies of V. This is not itself an open subset of $\mathbb{R}^n$, but since $\mathbb{R}^n$ contains two disjoint open copies of itself the set $V \times \mathcal{O}(\mathbb{R}^n)$ is diffeomorphic to an open subset of $\mathbb{R}^n$. Using this diffeomorphism, we can view the maps $\mathcal{O}(\phi)$ as providing charts on $\mathcal{O}(M^n)$, so $\mathcal{O}(M^n)$ is an n-dimensional manifold. We now prove that it is in fact a smooth manifold.

LEMMA 7.2. *Let M^n be a smooth manifold. Then the set*

$$\mathcal{A} = \{\mathcal{O}(\phi)\colon \mathcal{O}(U) \to V \times \mathcal{O}(\mathbb{R}^n) \mid \phi\colon U \to V \text{ a chart on } M^n\}$$

is a smooth atlas on $\mathcal{O}(M^n)$.

PROOF. Consider two charts $\phi_1\colon U_1 \to V_1$ and $\phi_2\colon U_2 \to V_2$ on M^n. Let $\tau_{12}\colon \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ be the transition map from ϕ_1 to ϕ_2. By definition, τ_{12} is smooth. The transition map on $\mathcal{O}(M^n)$ from $\mathcal{O}(\phi_1)\colon \mathcal{O}(U_1) \to V_1 \times \mathcal{O}(\mathbb{R}^n)$ to $\mathcal{O}(\phi_2)\colon \mathcal{O}(U_2) \to V_2 \times \mathcal{O}(\mathbb{R}^n)$ is the map

$$\mathcal{O}(\tau_{12})\colon \phi_1(U_1 \cap U_2) \times \mathcal{O}(\mathbb{R}^n) \longrightarrow \phi_2(U_1 \cap U_2) \times \mathcal{O}(\mathbb{R}^n)$$

defined via the formula

$$\mathcal{O}(\tau_{12})(q, b) = (\tau_{12}(q), (D_q\tau_{12})_*(b)) \in \phi_2(U_1 \cap U_2) \times \mathcal{O}(\mathbb{R}^n).$$

Here $(D_q\tau_{12})_*\colon \mathcal{O}(\mathbb{R}^n) \to \mathcal{O}(\mathbb{R}^n)$ is the map on orientations induced by the derivative map $D_q\tau_{12}$, which is an isomorphism of vector spaces. This transition map is clearly smooth, as desired. $\square$

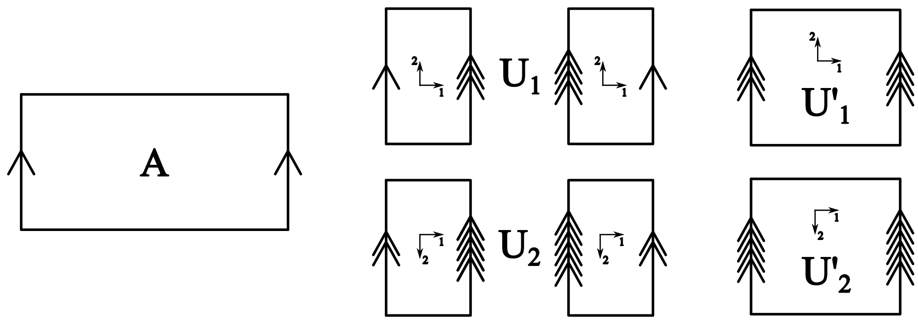

FIGURE 7.1. On the left is the open annulus A. The top and bottom lines are not included. On the right are $\mathcal{O}(U) = U_1 \sqcup U_2$ and $\mathcal{O}(U') = U_1' \sqcup U_2'$. These glue together to give two annuli.

7.3. Orientations on manifolds

If M^n is a smooth manifold, then an *orientation* on M^n is a smooth map $\beta \colon M^n \to \mathcal{O}(M^n)$ such that $\beta(p) \in \mathcal{O}(T_p(M^n))$ for all $p \in M^n$. A smooth manifold equipped with an orientation is an *oriented manifold*. A smooth manifold for which there exists an orientation is an *orientable manifold*; if no orientation exists, then the manifold is *nonorientable*.

EXAMPLE. If U is an open subset of $\mathbb{R}^n$, then U is orientable. Indeed, in this case we have $\mathcal{O}(U) = U \times \mathcal{O}(\mathbb{R}^n)$. Letting $b \in \mathcal{O}(\mathbb{R}^n)$ be the standard orientation, we can define an orientation $\beta \colon U \to \mathcal{O}(\mathbb{R}^n)$ via the formula $\beta(p) = (p, b)$. This will be called the *standard orientation* on an open subset of $\mathbb{R}^n$. □

EXAMPLE. Consider a 2-dimensional open annulus A. Of course, A can be realized as an open subset of $\mathbb{R}^2$, so we know that $\mathcal{O}(A)$ consists of two disjoint copies of A. However, to help understand the next example we will work this out in a different way. As in Figure 7.1 we will think of A as the quotient of $[0,1] \times (0,1)$ by the equivalence relation $\sim$ that identifies $(0, y)$ with $(1, y)$ for all $y \in (0,1)$. Define

$$U = \{(x,y) \mid 0 < y < 1,\ 1/4 < x < 3/4\}/\sim \subset A$$

and

$$U' = \{(x,y) \mid 0 < y < 1 \text{ and either } 0 \leq x < 1/3 \text{ or } 2/3 < x \leq 1\}/\sim \subset A.$$

Both U and U' are diffeomorphic to open rectangles in $\mathbb{R}^2$. We thus have $\mathcal{O}(U) = U_1 \sqcup U_2$ and $\mathcal{O}(U') = U_1' \sqcup U_2'$, where U_1 and U_2 are the components corresponding to the two possible orientations on $\mathbb{R}^2$, and similarly for U_1' and U_2'. These are both depicted in Figure 7.1. In $\mathcal{O}(A)$, open neighborhoods of the boundary segments of $U_1 \sqcup U_2$ are identified with open neighborhoods of the boundary segments of $U_1' \sqcup U_2'$ like in Figure 7.1. Examining that figure, we see that in fact $\mathcal{O}(A)$ is diffeomorphic to two disjoint copies of A (as we already knew!). In particular, A is orientable via either of the two evident maps $A \to \mathcal{O}(A)$. □

EXAMPLE. Consider a 2-dimensional open Möbius band M. As in Figure 7.2 we will think of M as the quotient of $[0,1] \times (0,1)$ by the equivalence relation $\sim$ that identifies $(0, y)$ with $(1, 1-y)$ for all $y \in (0,1)$. Define

$$U = \{(x,y) \mid 0 < y < 1,\ 1/4 < x < 3/4\}/\sim \subset M$$

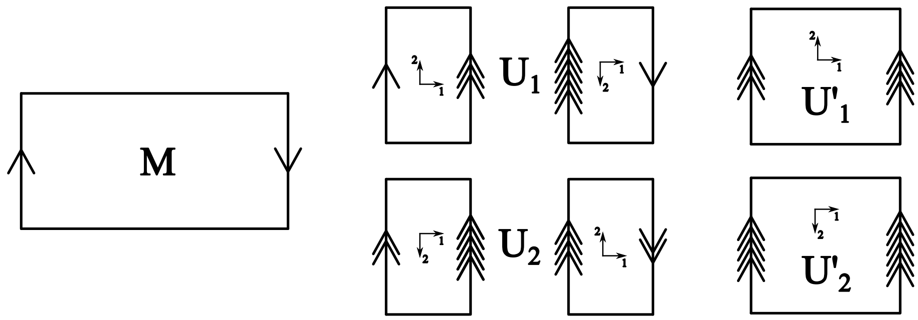

FIGURE 7.2. On the left is the open Möbius band M. The top and bottom lines are not included. On the right are $\mathcal{O}(U) = U_1 \sqcup U_2$ and $\mathcal{O}(U') = U_1' \sqcup U_2'$. These glue together to give a single annulus.

and

$$U' = \{(x, y) \mid 0 < y < 1 \text{ and either } 0 \leq x < 1/3 \text{ or } 2/3 < x \leq 1\}/\sim \subset M.$$

Both U and U' are diffeoomorphic to open rectangles in $\mathbb{R}^2$. We thus have $\mathcal{O}(U) = U_1 \sqcup U_2$ and $\mathcal{O}(U') = U_1' \sqcup U_2'$, where U_1 and U_2 are the components corresponding to the two possible orientations on $\mathbb{R}^2$, and similarly for U_1' and U_2'. These are both depicted in Figure 7.2; the one difference from the annulus in Figure 7.1 is that the orientations are flipped when crossing between the two pieces of the U_i. In $\mathcal{O}(M)$, open neighborhoods of the boundary segments of $U_1 \sqcup U_2$ are identified with open neighborhoods of the boundary segments of $U_1' \sqcup U_2'$ like in Figure 7.1. Examining that figure, we see that in fact $\mathcal{O}(M)$ is diffeomorphic to a single annulus. It is intuitively clear that there is no orientation $M \to \mathcal{O}(M)$; this will be justified in Lemma 7.3 below. □

The following lemma clarifies what it means for a manifold to be orientable.

LEMMA 7.3. *Let M^n be a connected smooth manifold. Then exactly one of the following holds.*

- *$\mathcal{O}(M^n)$ is diffeomorphic to two disjoint copies of M^n and M^n is orientable, or*
- *$\mathcal{O}(M^n)$ is connected and M^n is not orientable.*

PROOF. There is a natural projection $\pi\colon \mathcal{O}(M^n) \to M^n$. For $p \in M^n$, the fiber $\pi^{-1}(p)$ equals $= \{(p, b), (p, -b)\}$, where $\{b, -b\} = \mathcal{O}(T_p(M^n))$. From the definition of the topology on $\mathcal{O}(M^n)$, it is clear that π is a 2-fold covering map. The manifold $\mathcal{O}(M^n)$ is connected if and only if π is a nontrivial cover. We thus must prove that M^n is orientable if and only if π is a trivial cover.

Assume first that $\pi\colon \mathcal{O}(M^n) \to M^n$ is a trivial covering map, so $\mathcal{O}(M^n) = M^n \sqcup M^n$. In this case, M^n is orientable, and indeed there are two orientations $\beta\colon M^n \to \mathcal{O}(M^n)$ taking M^n diffeomorphically onto these two copies of M^n.

Now assume that M^n is orientable and let $\beta\colon M^n \to \mathcal{O}(M^n)$ be an orientation. Since $\pi \circ \beta = \mathrm{id}$, we see that β is a diffeomorphism onto its image. Define $\beta'\colon M^n \to \mathcal{O}(M^n)$ via $\beta'(p) = (p, -b)$, where $\beta(p) = (p, b)$. Then β' is another orientation, and by the same reasoning β' is a diffeomorphism onto its image. Since $\mathcal{O}(M^n) = \mathrm{Im}(\beta) \sqcup \mathrm{Im}(\beta')$, we see that $\mathcal{O}(M^n)$ is diffeomorphic to two disjoint copies of M^n and that $\pi\colon \mathcal{O}(M^n) \to M^n$ is a trivial cover, as desired. □

7.4. Oriented atlases

We now make the following definition.

DEFINITION. If M^n is a smooth manifold, then an *oriented atlas* for M^n is a smooth atlas $\{\phi_\alpha\colon U_\alpha \to V_\alpha\}_{\alpha\in I}$ for M^n with the following property. Consider $\alpha, \beta \in I$. Let $\tau_{\alpha\beta}\colon \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ be the transition function. For all $q \in \phi_\alpha(U_\alpha \cap U_\beta)$, identifying $T_q V_\alpha$ and $T_{\tau_{\alpha\beta}(q)} V_\beta$ with $\mathbb{R}^n$ we require that the determinant of the linear map

$$D_q \tau_{\alpha\beta}\colon T_q V_\alpha \to T_{\tau_{\alpha\beta}(q)} V_\beta$$

is positive. □

As is usual, we will say that two oriented atlases are equivalent if their union is an oriented atlas. The following characterization of orientations on manifolds can be taken as the definition of an orientation.

THEOREM 7.4. *Let M^n be a smooth manifold. Then equivalence classes of oriented atlases are in bijection with orientations on M^n. In particular, M^n is orientable if and only if it has an oriented atlas.*

PROOF. PROVE IT!!! □

7.5. Orientations and volume forms

Our goal now is to investigate the relationship between orientations and volume forms. We will need the following important lemma, which will also find use elsewhere.

LEMMA 7.5. *Let V be an n-dimensional real vector space, let $\omega \in \mathcal{A}^n(V)$, and let $f\colon V \to V$ be a linear map. Then*

$$\omega(f(\vec{v}_1), \ldots, f(\vec{v}_n)) = \det(f) \cdot \omega(\vec{v}_1, \ldots, \vec{v}_n)$$

for all $\vec{v}_1, \ldots, \vec{v}_n \in V$.

PROOF. For a linear map $f\colon V \to V$, define $\nu_f \in \mathcal{A}^n(V)$ via the formula

$$\nu_f(\vec{v}_1, \ldots, \vec{v}_n) = \omega(f(\vec{v}_1), \ldots, f(\vec{v}_n))$$

and define $\nu'_f \in \mathcal{A}^n(V)$ via the formula

$$\nu'_f(\vec{v}_1, \ldots, \vec{v}_n) = \det(f) \cdot \omega(\vec{v}_1, \ldots, \vec{v}_n).$$

Our goal is to prove that $\nu_f = \nu'_f$ for all f. Since diagonalizable linear maps $f\colon V \to V$ are dense in the space of all linear maps, it is enough to prove that $\nu_f = \nu'_f$ for a diagonalizable f. Moreover, since $\mathcal{A}^n(V)$ is 1-dimensional it is enough to find a single basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for V such that $\nu_f(\vec{v}_1, \ldots, \vec{v}_n) = \nu'_f(\vec{v}_1, \ldots, \vec{v}_n)$. Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis of eigenvectors for f with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, so $f(\vec{v}_i) = \lambda_i \vec{v}_i$ for all $1 \leq i \leq n$. We then calculate:

$$\begin{aligned}
\nu_f(\vec{v}_1, \ldots, \vec{v}_n) &= \omega(f(\vec{v}_1), \ldots, f(\vec{v}_n)) \\
&= \omega(\lambda_1 \vec{v}_1, \ldots, \lambda_n \vec{v}_n) \\
&= \lambda_1 \cdots \lambda_n \omega(\vec{v}_1, \ldots, \vec{v}_n) \\
&= \det(f) \omega(\vec{v}_1, \ldots, \vec{v}_n) \\
&= \nu'_f(\vec{v}_1, \ldots, \vec{v}_n),
\end{aligned}$$

as desired. □

We now prove the following theorem. Recall that a volume form on a smooth n-manifold M^n is an n-form $\omega \in \Omega^n(M^n)$ such that $\omega(p) \neq 0$ for all $p \in M^n$.

THEOREM 7.6. *Let M^n be a smooth manifold. Then M^n has a volume form if and only if M^n is orientable.*

PROOF. Assume first that M^n has a volume form $\omega \in \Omega^n(M^n)$. For $p \in M^n$, let $b_p = (\vec{v}_1, \ldots, \vec{v}_n)$ be any basis for T_pM^n such that

$$\omega(p)(\vec{v}_1, \ldots, \vec{v}_n) > 0.$$

We claim that if $b'_p = (\vec{w}_1, \ldots, \vec{w}_n)$ is another such basis, then $b_p \sim b'_p$. Indeed, let $f\colon T_pM^n \to T_pM^n$ be the linear map satisfying $f(\vec{v}_i) = \vec{w}_i$ for $1 \leq i \leq n$. By Lemma 7.5, we have

$$\omega(p)(\vec{w}_1, \ldots, \vec{w}_n) = \det(f)\omega(p)(\vec{v}_1, \ldots, \vec{v}_n).$$

Since

$$\omega(p)(\vec{w}_1, \ldots, \vec{w}_n) > 0 \qquad \text{and} \qquad \omega(p)(\vec{v}_1, \ldots, \vec{v}_n),$$

it follows that $\det(f) > 0$ and thus that $b_p \sim b'_p$, as claimed. This implies that we can define an orientation $\beta\colon M^n \to \mathcal{O}(M^n)$ via the formula $\beta(p) = (p, b_p)$. That β is smooth is immediate from the definitions.

Assume now that M^n is orientable. Let $\{\phi_\alpha\colon U_\alpha \to V_\alpha\}_{\alpha \in I}$ be an oriented atlas for M^n. Let $\{f_\alpha\colon M^n \to \mathbb{R}\}_{\alpha \in I}$ be a smooth partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha \in I}$ of M^n (see Theorem 1.2). Letting $x_1, \ldots, x_n$ be the coordinate functions on $\mathbb{R}^n$, define

$$\overline{\omega}_\alpha = \phi_\alpha^*(\mathrm{dx}_1 \wedge \cdots \wedge \mathrm{dx}_n) \in \Omega^n(U_\alpha).$$

Next, define $\omega_\alpha \in \Omega^n(M^n)$ via the formula

$$\omega_\alpha(p) = \begin{cases} f_\alpha(p) \cdot \overline{\omega}_\alpha(p) & \text{if } p \in U_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Since the support of f_α lies in U_α, the form ω_α is smooth. Finally, define

$$\omega = \sum_{\alpha \in I} \omega_\alpha \in \Omega^n(M^n).$$

While this is a priori an infinite sum, for any point p it is a finite sum since the f_α form a partition of unity. We claim that ω is a volume form on M^n. To check this, consider $p \in M^n$. We must show that $\omega(p) \neq 0$. Let $(\vec{v}_1, \ldots, \vec{v}_n)$ be an oriented basis for T_pM^n. Fixing some $\alpha \in I$ such that $p \in U_\alpha$, it is enough to prove that

$$\overline{\omega}_\alpha(p)(\vec{v}_1, \ldots, \vec{v}_n) > 0;$$

indeed, this will imply that there is no cancellation in the sum defining $\omega(p)$. The basis

$$\left((\phi_\alpha^{-1})_* \left(\frac{\partial}{\partial x_1} \right), \ldots, (\phi_\alpha^{-1})_* \left(\frac{\partial}{\partial x_n} \right) \right)$$

is also an oriented basis for T_pM^n. Letting $f\colon T_pM^n \to T_pM^n$ be the isomorphism defined via the formula

$$f\left((\phi_\alpha^{-1})_* \left(\frac{\partial}{\partial x_i} \right) \right) = \vec{v}_i \qquad (1 \leq i \leq n),$$

we thus have $\det(f) > 0$. Applying Lemma 7.5, we then have

$$\overline{\omega}_\alpha(p)(\vec{v}_1, \ldots, \vec{v}_n) = \det(f) \cdot (\mathrm{dx}_1 \wedge \cdots \mathrm{dx}_n)(\phi_\alpha(p))(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) = \det(f) > 0,$$

as desired. $\square$