A quick proof of the Seifert–Van Kampen theorem

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Abstract

This note contains a very short and elegant proof of the Seifert–Van Kampen theorem that is due to Grothendieck.

The Seifert–Van Kampen theorem [S, VK] says how to decompose the fundamental group of a space in terms of the fundamental groups of the constituents of an open cover of the space. The usual proof of it (as given for instance in Hatcher's book [H]) is tedious: one decomposes a loop in the space in terms of loops in the various open sets and then performs a rather involved combinatorial manipulation. In this note, we give a remarkably efficient alternate proof of it that we learned from Fulton's book [F]. This proof has the following properties:

- it is short and memorable, and
- it directly verifies the universal property in the Seifert–Van Kampen theorem rather than relying on generators and relations, and
- it uses techniques (covering space theory and descent) that are useful in many other contexts.

Its one downside is that it only works for spaces that have universal covers; however, spaces without universal covers are degenerate enough that their fundamental groups are of limited utility, so this is not a serious restriction. Fulton attributes this proof to Grothendieck. While it does not seem to appear explicitly in Grothendieck's work, it owes a lot to some of the elementary notions in Grothendieck's theory of the etalé fundamental group [SGA].

Remark. As we said, the proof uses covering spaces. This makes it difficult to include in a course that covers the fundamental group before discussing covering spaces. However, it is not hard to design a course in which covering spaces are discussed at the same time as the fundamental group; see Fulton's book [F] for one way to do this.

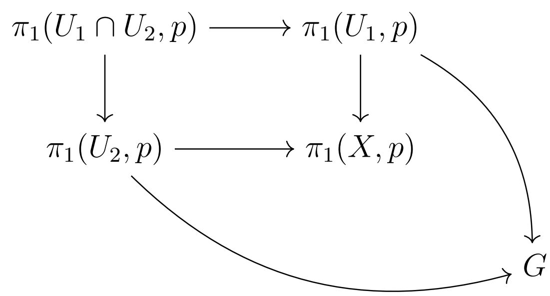
The statement of the Seifert–Van Kampen theorem is as follows. Say that a space is *reasonable* if it has a universal cover, that is, if it is semilocally simply connected.

Seifert–Van Kampen Theorem. *Let X be a reasonable topological space and let $X = U_1 \cup U_2$ be an open cover of X. Assume that U_1 and U_2 and $U_1 \cap U_2$ are all non-empty, path-connected, and reasonable. Then for all $p \in U_1 \cap U_2$, the commutative diagram*

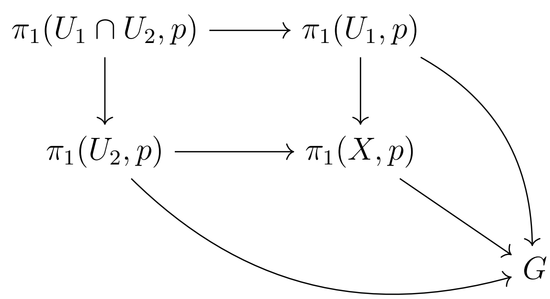
$$\begin{CD} \pi_1(U_1 \cap U_2, p) @>>> \pi_1(U_1, p) \\ @VVV @VVV \\ \pi_1(U_2, p) @>>> \pi_1(X, p) \end{CD}$$

is a pushout diagram.

Saying that the above diagram is a pushout diagram means that for all groups G and all commutative diagrams



we can find a unique homomorphism $\pi_1(X, p) \to G$ such that the diagram



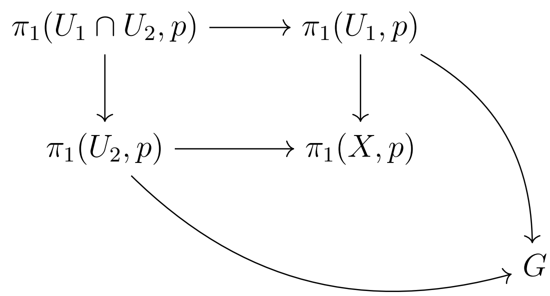
commutes. The key to this is to find a geometric avatar for a homomorphism coming out of the fundamental group of a space. This is provided by the following standard lemma, which summarizes a large amount of covering space theory.

Lemma 1. *Let Z be a reasonable nonempty path-connected space, let G be a group, and let $p \in Z$. Then there is a natural bijection*

$$\{\textit{homomorphisms } \pi_1(Z, p) \to G\} \leftrightarrow \{\textit{based regular } G\textit{-covers } (Y, q) \to (Z, p)\}.$$

Remark. The covers in the right hand side of Lemma 1 need not be connected; indeed, they will be connected exactly when the corresponding homomorphism is surjective. For instance, the trivial homomorphism corresponds to the product cover $(Z \times G, (p, 1)) \to (Z, p)$.

Proof of Seifert–Van Kampen theorem. Consider a group G and a commutative diagram



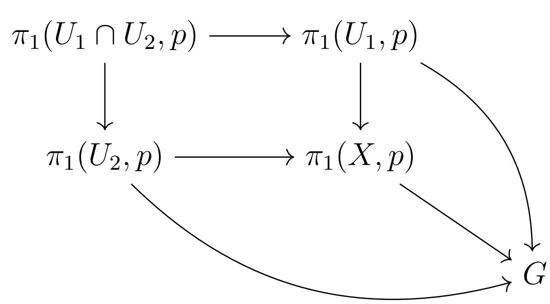
Using Lemma 1, we can associate to the homomorphisms $\pi_1(U_1, p) \to G$ and $\pi_1(U_2, p) \to G$ based regular G-coverings $f_1 : (\widetilde{U}_1, \widetilde{p}_1) \to (U_1, p)$ and $f_2 : (\widetilde{U}_2, \widetilde{p}_2) \to (U_2, p)$. For $i = 1, 2$, let $\widetilde{V}_i = f_i^{-1}(U_1 \cap U_2)$, so $(\widetilde{V}_i, \widetilde{p}_i) \to (U_1 \cap U_2, p)$ is a based regular G-covering representing the homomorphism

$$\pi_1(U_1 \cap U_2, p) \to \pi_1(U_i, p) \to G.$$

Since the homomorphisms

$$\pi_1(U_1 \cap U_2, p) \to \pi_1(U_1, p) \to G \qquad \text{and} \qquad \pi_1(U_1 \cap U_2, p) \to \pi_1(U_2, p) \to G$$

are equal, we see that $(\widetilde{V}_1, \widetilde{p}_1) \to (U_1 \cap U_2, p)$ and $(\widetilde{V}_2, \widetilde{p}_2) \to (U_1 \cap U_2, p)$ are isomorphic based regular G-coverings, and thus there exists a unique G-equivariant homeomorphism $\phi : (\widetilde{V}_1, \widetilde{p}_1) \to (\widetilde{V}_2, \widetilde{p}_2)$. Using ϕ, we can glue $f_1 : (\widetilde{U}_1, \widetilde{p}_1) \to (U_1, p)$ and $f_2 : (\widetilde{U}_2, \widetilde{p}_2) \to (U_2, p)$ together to obtain a based regular G-covering $(\widetilde{X}, \widetilde{p}) \to (X, p)$. Using Lemma 1 one final time, we see that this represents the desired homomorphism $\pi_1(X, p) \to G$ making the diagram



commute. The uniqueness of this homomorphism follows from the uniqueness in each step of the above proof. □

Remark. The gluing together of covers in the above proof is a (trivial) example of *descent*. See [Q] and the references therein for more sophisticated examples of this.

References

[F] W. Fulton, *Algebraic topology*, Graduate Texts in Mathematics, 153, Springer, New York, 1995.

[SGA] A. Grothendieck and M. Raynaud, *Revêtements étales et groupe fondamental (SGA 1)*, Documents Mathématiques (Paris), 3, Soc. Math. France, Paris, 2003.

[H] A. Hatcher, *Algebraic topology*, Cambridge Univ. Press, Cambridge, 2002.

[Q] Qfwfq (http://mathoverflow.net/users/4721/qfwfq), What is descent theory?, URL (version: 2010-04-21): http://mathoverflow.net/q/22032.

[S] H. Seifert, Konstruction drei dimensionaler geschlossener Raume, Berichte Sachs. Akad. Leipzig, Math.-Phys. Kl. 83 (1931), 26–66.

[VK] E. R. van Kampen, On the connection between the fundamental groups of some related spaces, Amer. J. Math. 55 (1933), 261–267.

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