Basic Geometry and Topology

Stephan Stolz

September 7, 2015

Contents

1 Pointset Topology

1.1 Metric spaces

We recall that a map $f\colon \mathbb{R}^m \to \mathbb{R}^n$ between Euclidean spaces is *continuous* if and only if

$$\forall x \in X \quad \forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall y \in X \quad d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon, \tag{1.1}$$

where

$$d(x,y) = ||x-y|| = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2} \in \mathbb{R}_{\geq 0}$$

is the *Euclidean distance* between two points x, y in $\mathbb{R}^n$.

Example 1.2. (Examples of continuous maps.)

1. The addition map $a\colon \mathbb{R}^2 \to \mathbb{R}$, $x = (x_1, x_2) \mapsto x_1 + x_2$;
2. The multiplication map $m\colon \mathbb{R}^2 \to \mathbb{R}$, $x = (x_1, x_2) \mapsto x_1 x_2$;

The proofs that these maps are continuous are simple estimates that you probably remember from calculus. Since the continuity of *all* the maps we'll look at in these notes is proved by expressing them in terms of the maps a and m, we include the proofs of continuity of a and m for completeness.

Proof. To prove that the addition map a is continuous, suppose $x = (x_1, x_2) \in \mathbb{R}^2$ and $\epsilon > 0$ are given. We claim that for $\delta := \epsilon/2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ with $d(x, y) < \delta$ we have $d(a(x), a(y)) < \epsilon$ and hence a is a continuous function. To prove the claim, we note that

$$d(x, y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$$

and hence $|x_1 - y_1| \leq d(x, y)$, $|x_1 - y_1| \leq d(x, y)$. It follows that

$$d(a(x), a(y)) = |a(x) - a(y)| = |x_1 + x_2 - y_1 - y_2| \leq |x_1 - y_1| + |x_2 - y_2| \leq 2d(x, y) < 2\delta = \epsilon.$$

To prove that the multiplication map m is continuous, we claim that for

$$\delta := \min\{1, \epsilon/(|x_1| + |x_2| + 1)\}$$

and $y = (y_1, y_2) \in \mathbb{R}^2$ with $d(x, y) < \delta$ we have $d(m(x), m(y)) < \epsilon$ and hence m is a continuous function. The claim follows from the following estimates:

$$\begin{aligned}
d(m(y), m(x)) &= |y_1y_2 - x_1x_2| = |y_1y_2 - x_1y_2 + x_1y_2 - x_1x_2| \\
&\leq |y_1y_2 - x_1y_2| + |x_1y_2 - x_1x_2| = |y_1 - x_1||y_2| + |x_1||y_2 - x_2| \\
&\leq d(x, y)(|y_2| + |x_1|) \leq d(x, y)(|x_2| + |y_2 - x_2| + |x_1|) \\
&\leq d(x, y)(|x_1| + |x_2| + 1) < \delta(|x_1| + |x_2| + 1) \leq \epsilon
\end{aligned}$$

□

Lemma 1.3. *The function $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ has the following properties:*

1. *$d(x, y) = 0$ if and only if $x = y$;*
2. *$d(x, y) = d(y, x)$ (symmetry);*
3. *$d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)*

Definition 1.4. A *metric space* is a set X equipped with a map

$$d\colon X \times X \to \mathbb{R}_{\geq 0}$$

with properties (1)-(3) above. A map $f\colon X \to Y$ between metric spaces X, Y is

continuous if condition (1.1) is satisfied.

an isometry if $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$;

Two metric spaces X, Y are *homeomorphic* (resp. *isometric*) if there are continuous maps (resp. isometries) $f\colon X \to Y$ and $g\colon Y \to X$ which are inverses of each other.

Example 1.5. An important class of examples of metric spaces are subsets of $\mathbb{R}^n$. Here are particular examples we will be talking about during the semester:

1. The *n-disk* $D^n := \{x \in \mathbb{R}^n \mid |x| \le 1\} \subset \mathbb{R}^n$, and $D^n_r := \{x \in \mathbb{R}^n \mid |x| \le r\}$, the *$n$-disk of radius* $r > 0$.

   The dilation map
   $$D^n \longrightarrow D^n_r \qquad x \mapsto rx$$
   is a homeomorphism between D^n and D^n_r with inverse given by multiplication by $1/r$. However, these two metric spaces are *not* isometric for $r \neq 1$. To see this, define the *diameter* $\operatorname{diam}(X)$ of a metric space X by
   $$\operatorname{diam}(X) := \sup\{d(x,y) \mid x, y \in X\} \in \mathbb{R}_{\ge 0} \cup \{\infty\}.$$
   For example, $\operatorname{diam}(D^n_r) = 2r$. It is easy to see that if two metric spaces X, Y are isometric, then their diameters agree. In particular, the disks D^n_r and $D^n_{r'}$ are not isometric unless $r = r'$.

2. The *n-sphere* $S^n := \{x \in \mathbb{R}^{n+1} \mid |x| = 1\} \subset \mathbb{R}^{n+1}$.

3. The *torus* $T = \{v \in \mathbb{R}^3 \mid d(v, C) = r\}$ for $0 < r < 1$. Here
   $$C = \{(x, y, 0) \mid x^2 + y^2 = 1\} \subset \mathbb{R}^3$$
   is the unit circle in the xy-plane, and $d(v, C) = \inf_{w \in C} d(v, w)$ is the distance between v and C.

4. The *general linear group*
   $$\begin{aligned} GL_n(\mathbb{R}) &= \{\text{vector space isomorphisms } f\colon \mathbb{R}^n \to \mathbb{R}^n\} \\ &\longleftrightarrow \{(v_1, \dots, v_n) \mid v_i \in \mathbb{R}^n,\ \det(v_1, \dots, v_n) \neq 0\} \\ &= \{\text{invertible } n \times n\text{-matrices}\} \subset \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n} = \mathbb{R}^{n^2} \end{aligned}$$
   Here we think of $(v_1, \dots, v_n)$ as an $n \times n$-matrix with column vectors v_i, and the bijection is the usual one in linear algebra that sends a linear map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ to the matrix $(f(e_1), \dots, f(e_n))$ whose column vectors are the images of the standard basis elements $e_i \in \mathbb{R}^n$.

5. The *special linear group*
   $$SL_n(\mathbb{R}) = \{(v_1, \dots, v_n) \mid v_i \in \mathbb{R}^n,\ \det(v_1, \dots, v_n) = 1\} \subset \mathbb{R}^{n^2}$$

6. The *orthogonal group*
   $$\begin{aligned} O(n) &= \{\text{linear isometries } f\colon \mathbb{R}^n \to \mathbb{R}^n\} \\ &= \{(v_1, \dots, v_n) \mid v_i \in \mathbb{R}^n,\ v_i\text{'s are orthonormal}\} \subset \mathbb{R}^{n^2} \end{aligned}$$
   We recall that a collection of vectors $v_i \in \mathbb{R}^n$ is *orthonormal* if $|v_i| = 1$ for all i, and v_i is perpendicular to v_j for $i \neq j$.

7. The *special orthogonal group*

$$SO(n) = \{(v_1, \dots, v_n) \in O(n) \mid \det(v_1, \dots, v_n) = 1\} \subset \mathbb{R}^{n^2}$$

8. The *Stiefel manifold*

$$\begin{aligned} V_k(\mathbb{R}^n) &= \{\text{linear isometries } f \colon \mathbb{R}^k \to \mathbb{R}^n\} \\ &= \{(v_1, \dots, v_k) \mid v_i \in \mathbb{R}^n,\ v_i\text{'s are orthonormal}\} \subset \mathbb{R}^{kn} \end{aligned}$$

Example 1.6. The following maps between metric spaces are continuous. While it is possible to prove their continuity using the definition of continuity, it will be much simpler to prove their continuity by 'building' these maps using compositions and products from the continuous maps a and m of Example 1.2. We will do this below in Lemma 1.22.

1. Every polynomial function $f \colon \mathbb{R}^n \to \mathbb{R}$ is continuous. We recall that a polynomial function is of the form $f(x_1, \dots, x_n) = \sum_{i_1, \dots, i_n} a_{i_1, \dots, i_n} x_1^{i_1} \cdot \dots \cdot x_n^{i_n}$ for $a_{i_1, \dots, i_n} \in \mathbb{R}$.

2. Let $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$ be the set of $n \times n$ matrices. Then the map

$$M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R}) \qquad (A, B) \mapsto AB$$

   given by matrix multiplication is continuous. Here we use the fact that a map to the product $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2} = \mathbb{R} \times \dots \times \mathbb{R}$ is continuous if and only if each component map is continuous (see Lemma 1.21), and each matrix entry of AB is a polynomial and hence a continuous function of the matrix entries of A and B. Restricting to the invertible matrices $GL_n(\mathbb{R}) \subset M_{n\times n}(\mathbb{R})$, we see that the multiplication map

$$GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$$

   is continuous. The same holds for the subgroups $SO(n) \subset O(n) \subset GL_n(\mathbb{R})$.

3. The map $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $A \mapsto A^{-1}$ is continuous (this is a homework problem). The same statement follows for the subgroups of $GL_n(\mathbb{R})$.

The Euclidean metric on $\mathbb{R}^n$ given by $d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$ for $x, y \in \mathbb{R}^n$ is not the only reasonable metric on $\mathbb{R}^n$. Another metric on $\mathbb{R}^n$ is given by

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|. \tag{1.7}$$

The question arises whether it can happen that a map $f \colon \mathbb{R}^n \to \mathbb{R}^n$ is continuous with respect to one of these metrics, but not with respect to the other. To see that this doen't happen, it is useful to characterize continuity of a map $f \colon X \to Y$ between metric spaces X, Y in a way that involves the metrics on X and Y less directly than Definition 1.4 does. This alternative characterization will be based on the following notion of "open subsets" of a metric space.

Definition 1.8. Let X be a metric space. A subset $U \subset X$ is *open* if for every point $x \in U$ there is some $\epsilon > 0$ such that $B_\epsilon(x) \subset U$. Here $B_\epsilon(x) = \{y \in X \mid d(y,x) < \epsilon\}$ is the *ball of radius ϵ around x.*

To illustrate this, lets look at examples of subsets of $\mathbb{R}^n$ equipped with the Euclidean metric. The subset $D^n_r = \{v \in \mathbb{R}^n \mid ||v|| \leq r\} \subset \mathbb{R}^n$ is not open, since for for a point $v \in D^n_r$ with $||v|| = r$ any open ball $B_\epsilon(v)$ with center v will contain points not in D^n_r. By contrast, the subset $B_r(0) \subset \mathbb{R}^n$ is open, since for any $x \in B_r(0)$ the ball $B_\delta(x)$ of radius $\delta = r - ||x||$ is contained in $B_r(0)$, since for $y \in B_\delta(x)$ by the triangle inequality we have

$$d(y,0) \leq d(y,x) + d(x,0) < \delta + ||x|| = (r - ||x||) + ||x|| = r.$$

Lemma 1.9. *A map $f: X \to Y$ between metric spaces is continuous if and only if $f^{-1}(V)$ is an open subset of X for every open subset $V \subset Y$.*

Corollary 1.10. *If $f: X \to Y$ and $g: Y \to Z$ are continuous maps, then so it their composition $g \circ f: X \to Z$.*

Exercise 1.11. (a) Prove Lemma 1.9

(b) Assume that d, d' are two metrics on a set X which are equivalent in the sense that there are constants $C, C' > 0$ such that $d(x,y) \leq C d_1(x,y)$ and $d_1(x,y) \leq C' d(x,y)$ for all $x, y \in X$. Show that a subset $U \subset X$ is open with respect to d if and only if it is open with respect to d'.

(c) Show that the Euclidean metric d and the metric (1.7) on $\mathbb{R}^n$ are equivalent. This shows in particular that a map $f: \mathbb{R}^n \to \mathbb{R}^n$ is continuous w.r.t. d if and only if it is continuous w.r.t. d_1.

1.2 Topological spaces

Lemma 1.9 and Exercise (b) above shows that it is better to *define* continuity of maps between metric spaces in terms of the *open subsets* of these metric space instead of the original ϵ-δ-definition. In fact, we can go one step further, forget about the metric on a set X altogether, and just consider a collection $\mathcal{T}$ of subsets of X that we declare to be "open". The next result summarizes the basic properties of open subsets of a metric space X, which then motivates the restrictions that we wish to put on such collections $\mathcal{T}$.

Lemma 1.12. *Open subsets of a metric space X have the following properties.*

(i) X and $\emptyset$ are open.

(ii) Any union of open sets is open.

(iii) The intersection of any finite number of open sets is open.

Definition 1.13. A *topological space* is a set X together with a collection $\mathcal{T}$ of subsets of X, called *open sets* which are required to satisfy conditions (i), (ii) and (iii) of the lemma above. The collection $\mathcal{T}$ is called a *topology* on X. The sets in $\mathcal{T}$ are called the *open sets*, and their complements in X are called *closed sets*. A subset of X may be neither closed nor open, either closed or open, or both.

A map $f: X \to Y$ between topological spaces X, Y is *continuous* if the inverse image $f^{-1}(V)$ of every open subset $V \subset Y$ is an open subset of X.

It is easy to see that the composition of continuous maps is again continuous.

Examples of topological spaces.

1. Let X be a metric space, and $\mathcal{T}$ the collection of those subsets of X that are unions of balls $B_\epsilon(x)$ in X (i.e., the subsets which are open in the sense of Definition 1.8). Then $\mathcal{T}$ is a topology on X, the *metric topology*.
2. Let X be a set. Then $\mathcal{T} = \{\text{all subsets of } X\}$ is a topology, the *discrete topology*. We note that *any* map $f\colon X \to Y$ to a topological space Y is continuous. We will see later that the only continuous maps $\mathbb{R}^n \to X$ are the constant maps.
3. Let X be a set. Then $\mathcal{T} = \{\emptyset, X\}$ is a topology, the *indiscrete topology*.

Sometimes it is convenient to define a topology $\mathcal{U}$ on a set X by first describing a smaller collection $\mathcal{B}$ of subsets of X, and then defining $\mathcal{U}$ to be those subsets of X that can be written as *unions* of subsets belonging to $\mathcal{B}$. We've done this already when defining the metric topology: Let X be a metric space and let $\mathcal{B}$ be the collection of subsets of X of the form $B_\epsilon(x) := \{y \in X \mid d(y,x) < \epsilon\}$ (the balls in X). Then the metric topology $\mathcal{U}$ on X consists of those subsets U which are unions of subsets belonging to $\mathcal{B}$.

Lemma 1.14. *Let $\mathcal{B}$ be a collection of subsets of a set X satisfying the following conditions*

1. *Every point $x \in X$ belongs to some subset $B \in \mathcal{B}$.*
2. *If $B_1, B_2 \in \mathcal{B}$, then for every $x \in B_1 \cap B_2$ there is some $B \in \mathcal{B}$ with $x \in B$ and $B \subset B_1 \cap B_2$.*

Then $\mathcal{T} := \{\text{unions of subsets belonging to } \mathcal{B}\}$ is a topology on X.

Definition 1.15. If the above conditions are satisfied, we call the collection $\mathcal{B}$ is called a *basis for the topology* $\mathcal{T}$ or we say that $\mathcal{B}$ *generates the topology* $\mathcal{T}$.

It is easy to check that the collection of balls in a metric space satisfies the above conditions and hence the collection of open subsets is a topology as claimed by Lemma 1.12.

1.3 Constructions with topological spaces

1.3.1 Subspace topology

Definition 1.16. Let X be a topological space, and $A \subset X$ a subset. Then

$$\mathcal{T} = \{A \cap U \mid U \underset{open}{\subset} X\}$$

is a topology on A called the *subspace topology*.

Lemma 1.17. *Let X be a metric space and $A \subset X$. Then the metric topology on A agrees with the subspace topology on A (as a subset of X equipped with the metric topology).*

Lemma 1.18. *Let X, Y be topological spaces and let A be a subset of X equipped with the subspace topology. Then the inclusion map $i\colon A \to X$ is continuous and a map $f\colon Y \to A$ is continuous if and only if the composition $i \circ f\colon Y \to X$ is continuous.*

1.3.2 Product topology

Definition 1.19. The *product topology* on the Cartesian product $X \times Y = \{(x,y) \mid x \in X,\, y \in Y\}$ of topological spaces X, Y is the topology with basis

$$\mathcal{B} = \{U \times V \mid U \underset{open}{\subset} X, V \underset{open}{\subset} Y\}$$

The collection $\mathcal{B}$ obviously satisfies property (1) of a basis; property (2) holds since $(U \times V) \cap (U' \times V') = (U \cap U') \times (V \cap V')$. We note that the collection $\mathcal{B}$ is *not* a topology since the union of $U \times V$ and $U' \times V'$ is typically not a Cartesian product (e.g., draw a picture for the case where $X = Y = \mathbb{R}$ and U, U', V, V' are open intervals).

Lemma 1.20. *The product topology on $\mathbb{R}^m \times \mathbb{R}^n$ (with each factor equipped with the metric topology) agrees with the metric topology on $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$.*

Proof: homework.

Lemma 1.21. *Let X, Y_1, Y_2 be topological spaces. Then the projection maps $p_i \colon Y_1 \times Y_2 \to Y_i$ is continuous and a map $f \colon X \to Y_1 \times Y_2$ is continuous if and only if the* component maps

$$X \xrightarrow{f} Y_1 \times Y_2 \xrightarrow{p_i} Y_i$$

are continuous for $i = 1, 2$.

Proof: homework

Lemma 1.22.

1. *Let X be a topological space and let $f, g \colon X \to \mathbb{R}$ be continuous maps. Then $f + g$ and $f \cdot g$ continuous maps from X to $\mathbb{R}$. If $g(x) \neq 0$ for all $x \in X$, then also f/g is continuous.*
2. *Any polynomial function $f \colon \mathbb{R}^n \to \mathbb{R}$ is continuous.*
3. *The multiplication map $\mu \colon GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is continuous.*

Proof. To prove part (1) we note that the map $f + g \colon X \to \mathbb{R}$ can be factored in the form

$$X \xrightarrow{f \times g} \mathbb{R} \times \mathbb{R} \xrightarrow{a} \mathbb{R}$$

The map $f \times g$ is continuous by Lemma 1.21 since its component maps f, g are continuous; the map a is continuous by Example 1.2, and hence the composition $f + g$ is continuous. The argument for $f \cdot g$ is the same, with a replaced by m. To prove that f/g is continuous, we factor it in the form

$$X \xrightarrow{f \times g} \mathbb{R} \times \mathbb{R}^\times \xrightarrow{p_1 \times (I \circ p_2)} \mathbb{R} \times \mathbb{R}^\times \xrightarrow{m} \mathbb{R},$$

where $\mathbb{R}^\times = \{t \in \mathbb{R} \mid t \neq 0\}$, p_1 (resp. p_2) is the projection to the first (resp. second) factor of $\mathbb{R} \times \mathbb{R}^\times$, and $I \colon \mathbb{R}^\times \to \mathbb{R}^\times$ is the inversion map $t \mapsto t^{-1}$. By Lemma 1.21 the p_i's are continuous, in calculus we learned that I is continuous, and hence again by Lemma 1.21 the map $p_1 \times (I \circ p_2)$ is continuous.

To prove part (2), we note that the constant map $\mathbb{R}^n \to \mathbb{R}$, $x = (x_1, \ldots, x_n) \mapsto a$ is obviously continuous, and that the projection map $p_i \colon \mathbb{R}^n \to \mathbb{R}$, $x = (x_1, \ldots, x_n) \mapsto x_i$ is continuous by Lemma 1.21. Hence by part (1) of this lemma, the monomial function $x \mapsto a x_1^{i_1} \cdots x_n^{i_n}$ is continuous. Any polynomial function is a sum of monomial functions and hence continuous.

For the proof of (3), let $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$ be the set of $n \times n$ matrices and let

$$\mu \colon M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R}) \qquad (A, B) \mapsto AB$$

be the map given by matrix multiplication. By Lemma 1.21 the map μ is continuous if and only if the composition

$$M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \xrightarrow{\mu} M_{n\times n}(\mathbb{R}) \xrightarrow{p_{ij}} \mathbb{R}$$

is continuous for all $1 \leq i, j \leq n$, where p_{ij} is the projection map that sends a matrix A to its entry $A_{ij} \in \mathbb{R}$. Since the $p_{ij}(\mu(A, B)) = (A \cdot B)_{ij}$ is a *polynomial* in the entries of the matrices A and B, this is a continuous map by part (2) and hence μ is continuous.

Restricting μ to invertible matrices, we obtain the multiplication map

$$\mu_{|} \colon GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$$

that we want to show is continuous. We will argue that in general if $f \colon X \to Y$ is a continuous map with $f(A) \subset B$ for subsets $A \subset X$, $B \subset Y$, then the *restriction* $f_{|A} \colon A \to B$ is continuous. To prove this, consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f_{|A}} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{f} & Y \end{array}$$

where i, j are the obvious inclusion maps. These inclusion maps are continuous w.r.t. the subspace topology on A, B by Lemma 1.18. The continuity of f and i implies the continuity of $f \circ i = j \circ f_{|A}$ which again by Lemma 1.18 implies the continuity of $f_{|A}$. □

1.3.3 Quotient topology.

Definition 1.23. Let X be a topological space and let $\sim$ be an equivalence relation on X. We denote by $X/\sim$ be the set of equivalence classes and by

$$p \colon X \to X/\sim \qquad x \mapsto [x]$$

be the projection map that sends a point $x \in X$ to its equivalence class $[x]$. The *quotient topology* on $X/\sim$ is given by the collection of subsets

$$\mathcal{U} = \{U \subset X/\sim \mid p^{-1}(U) \text{ is an open subset of } X\}.$$

The set $X/\sim$ equipped with the quotient topology is called the *quotient space*.

The quotient topology is often used to construct a topology on a set Y which is not a subset of some Euclidean space $\mathbb{R}^n$, or for which it is not clear how to construct a metric. If there is a surjective map

$$p\colon X \longrightarrow Y$$

from a topological space X, then Y can be identified with the quotient space $X/\sim$, where the equivalence relation is given by $x \sim x'$ if and only if $p(x) = p(x')$. In particular, $Y = X/\sim$ can be equipped with the quotient topology. Here are important examples.

Example 1.24. 1. The *real projective space of dimension n* is the set

$$\mathbb{RP}^n := \{\text{1-dimensional subspaces of } \mathbb{R}^{n+1}\}.$$

The map

$$S^n \longrightarrow \mathbb{RP}^n \qquad \mathbb{R}^{n+1} \ni v \mapsto \text{subspace generated by } v$$

is surjective, leading to the identification

$$\mathbb{RP}^n = S^n/(v \sim \pm v),$$

and the quotient topology on $\mathbb{RP}^n$.

2. Similarly, working with complex vector spaces, we obtain a quotient topology on the the *complex projective space*

$$\mathbb{CP}^n := \{\text{1-dimensional subspaces of } \mathbb{C}^{n+1}\} = S^{2n+1}/(v \sim zv), \qquad z \in S^1$$

3. Generalizing, we can consider the *Grassmann manifold*

$$G_k(\mathbb{R}^{n+k}) := \{k\text{-dimensional subspaces of } \mathbb{R}^{n+k}\}.$$

There is a surjective map

$$V_k(\mathbb{R}^{n+k}) = \{(v_1, \dots, v_k) \mid v_i \in \mathbb{R}^{n+k},\ v_i\text{'s are orthonormal}\} \quad \twoheadrightarrow \quad G_k(\mathbb{R}^{n+k})$$

given by sending $(v_1, \dots, v_k) \in V_k(\mathbb{R}^{n+k})$ to the k-dimensional subspace of $\mathbb{R}^{n+k}$ spanned by the v_i's. Hence the subspace topology on the Stiefel manifold $V_k(\mathbb{R}^{n+k}) \subset \mathbb{R}^{(n+k)k}$ gives a quotient topology on the Grassmann manifold $G_k(\mathbb{R}^{n+k}) = V_k(\mathbb{R}^{n+k})/\sim$. The same construction works for the complex Grassmann manifold $G_k(\mathbb{C}^{n+k})$.

As the examples below will show, sometimes a quotient space $X/\sim$ is homeomorphic to a topological space Z constructed in a different way. To establish the homeomorphism between $X/\sim$ and Z, we need to construct continuous maps

$$f\colon X/\sim \longrightarrow Z \qquad g\colon Z \to X/\sim$$

that are inverse to each other. The next lemma shows that it is easy to check continuity of the map f, the map *out of the quotient space.*

Lemma 1.25. *The projection map $p\colon X \to X/\sim$ is continuous and a map $f\colon X/\sim \to Z$ to a topological space Z is continuous if and only if the composition $f \circ p\colon X \to Z$ is continuous.*

As we will see in the next section, there are many situations where the continuity of the inverse map for a continuous bijection f is automatic. So in the examples below, and for the exercises in this section, we will defer checking the continuity of f^{-1} to that section.

Notation. Let A be a subset of a topological space X. Define a equivalence relation $\sim$ on X by $x \sim y$ if $x = y$ or $x, y \in A$. We use the notation X/A for the quotient space $X/\sim$.

Example 1.26. (1) We claim that the quotient space $[-1, +1]/\{\pm 1\}$ is homeomorphic to S^1 via the map $f\colon [-1, +1]/\{\pm 1\} \to S^1$ given by $[t] \mapsto e^{\pi i t}$. Geometrically speaking, the map f wraps the interval $[-1, +1]$ once around the circle. Here is a picture.

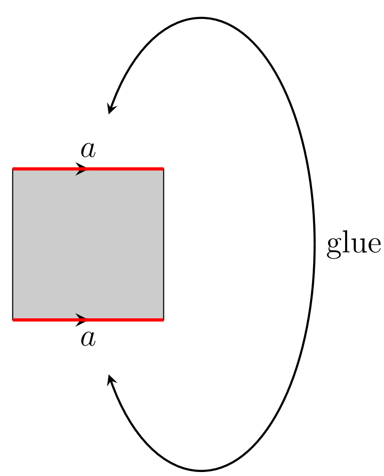
It is easy to check that the map f is a bijection. To see that f is continuous, consider the composition

$$[-1, +1] \xrightarrow{p} [-1, +1]/\{\pm 1\} \xrightarrow{f} S^1 \xrightarrow{i} \mathbb{C} = \mathbb{R}^2,$$

where p is the projection map and i the inclusion map. This composition sends $t \in [-1, +1]$ to $e^{\pi i t} = (\sin \pi t, \cos \pi t) \in \mathbb{R}^2$. By Lemma 1.21 it is a continuous function, since its component functions $\sin \pi t$ and $\cos \pi t$ are continuous functions. By Lemma 1.25 the continuity of $i \circ f \circ p$ implies the continuity of $i \circ f$, which by Lemma 1.18 implies the continuity of f. As mentioned above, we'll postpone the proof of the continuity of the inverse map f^{-1} to the next section.

(2) More generally, D^n/S^{n-1} is homeomorphic to S^n. (proof: homework)

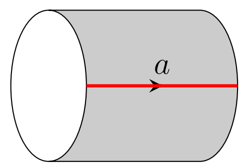
(3) Consider the quotient space of the square $[-1, +1] \times [-1, +1]$ given by identifying $(s, -1)$ with $(s, 1)$ for all $s \in [-1, 1]$. It can be visualized as a square whose top edge is to be glued with its bottom edge. In the picture below we indicate that identification by labeling those two edges by the same letter.



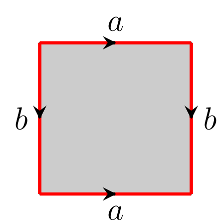
The quotient $([-1,+1]\times[-1,+1])/(s,-1)\sim(s,+1)$ is homeomorphic to the cylinder

$$C=\{(x,y,z)\in\mathbb{R}^3 \mid x\in[-1,+1], y^2+z^2=1\}.$$

The proof is essentially the same as in (1). A homeomorphism from the quotient space to C is given by $f([s,t])=(s,\sin\pi t,\cos\pi t)$. The picture below shows the cylinder C with the image of the edge a indicated.



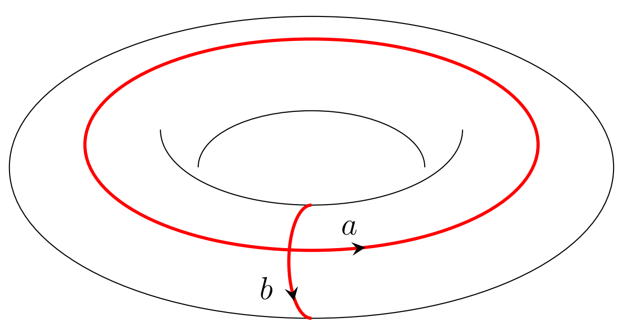
(4) Consider again the square, but this time using an equivalence relations that identifies more points than the one in the previous example. As before we identify $(s,-1)$ and $(s,1)$ for $s\in[-1,1]$, and in addition we identify $(-1,t)$ with $(1,t)$ for $t\in[-1,1]$. Here is the picture, where again corresponding points of edges labeled by the same letter are to be identified.



We claim that the quotient space is homeomorphic to the torus

$$T:=\{x\in\mathbb{R}^3 \mid d(x,K)=d\},$$

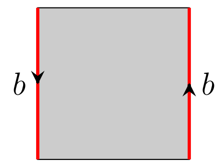
where $K=\{(x_1,x_2,0)\mid x_1^2+x_2^2=1\}$ is the unit circle in the xy-plane and $0<d<1$ is a real number (see) via a homeomorphism that maps the edges of the square to the loops in T indicated in the following picture below.



Exercise: prove this by writing down an explicity map from the quotient space to T, and arguing that this map is a continuous bijection (as always in this section, we defer the proof of the continuity of the inverse to the next section).

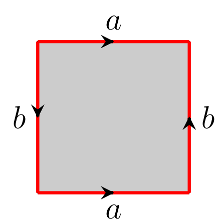
(5) We claim that the quotient space $D^n/\sim$ with equivalence relation generated by $v\sim -v$ for $v\in S^{n-1}\subset D^n$ is homeomorphic to the real projective space $\mathbb{RP}^n$. Proof: exercise. In particular, $\mathbb{RP}^1=S^1/v\sim -v$ is homeomorphic to $D^1/\sim=[-1,1]/-1\sim 1$, which by example (1) is homeomorphic to S^1.

(6) The quotient space $[-1,1] \times [-1,1]/\sim$ with the equivalence relation generated by $(-1,t) \sim (1,-t)$ is represented graphically by the following picture.



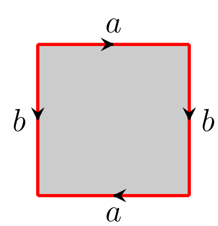
This topological space is called the *Möbius band*. It is homeomorphic to a subspace of $\mathbb{R}^3$ shown by the following picture

(7) The quotient space of the square by edge identifications given by the picture



is the *Klein bottle*. It is harder to visualize, since it is not homeomorphic to a subspace of $\mathbb{R}^3$ (which can be proved by the methods of algebraic topology).

(8) The quotient space of the square given by the picture



is homeomorphic to the real projective plane $\mathbb{RP}^2$. Exercise: prove this (hint: use the statement of example (5)). Like the Klein bottle, it is challenging to visualize the real projective plane, since it is not homeomorphic to a subspace of $\mathbb{R}^3$.

1.4 Properties of topological spaces

In the previous subsection we described a number of examples of topological spaces X, Y that we claimed to be homeomorphic. We typically constructed a bijection $f: X \to Y$ and argued that f is continuous. However, we did not finish the proof that f is a homeomorphism, since we defered the argument that the inverse map $f^{-1}: Y \to X$ is continuous. We note that not every continuous bijection is a homeomorphism. For example if X is a set, X_δ (resp. X_{ind}) is the topological space given by equipping the set X with the discrete (resp. indiscrete) topology, then the identity map is a continuous bijection from X_δ to X_{ind}. However its inverse, the identity map $X_{\mathrm{ind}} \to X_\delta$ is not continuous if X contains at least two points.

Fortunately, there are situations where the continuity of the inverse map is automatic as the following proposition shows.

Proposition 1.27. *Let $f\colon X \to Y$ be a continuous bijection. Then f is a homeomorphism provided X is compact and Y is Hausdorff.*

The goal of this section is to define these notions, prove the proposition above, and to give a tools to recognize that a topological space is compact and/or Hausdorff.

1.4.1 Hausdorff spaces

Definition 1.28. Let X be a topological space, $x_i \in X$, $i = 1, 2, \dots$ a sequence in X and $x \in X$. Then *x is the limit of the x_i's* if for any open subset $U \subset X$ containing x there is some N such that $x_i \in U$ for all $i \geq N$.

Caveat: If X is a topological space with the indiscrete topology, *every point* is the limit of every sequence. The limit is *unique* if the topological space has the following property:

Definition 1.29. A topological space X is *Hausdorff* if for every $x, y \in X$, $x \neq y$, there are disjoint open subsets $U, V \subset X$ with $x \in U$, $y \in V$.

Note: if X is a metric space, then the metric topology on X is Hausdorff (since for $x \neq y$ and $\epsilon = d(x, y)/2$, the balls $B_\epsilon(x)$, $B_\epsilon(y)$ are disjoint open subsets). In particular, any subset of $\mathbb{R}^n$, equipped with the subspace topology, is Hausdorff.

Warning: The notion of *Cauchy sequences* can be defined in metric spaces, but not in general for topological spaces (even when they are Hausdorff).

Lemma 1.30. *Let X be a topological space and A a closed subspace of X. If $x_n \in A$ is a sequence with limit x, then $x \in A$.*

Proof. Assume $x \notin A$. Then x is a point in the open subset $X \setminus A$ and hence by the definition of limit, all but finitely many elements x_n must belong to $X \setminus A$, contradicting our assumptions. □

1.4.2 Compact spaces

Definition 1.31. An *open cover* of a topological space X is a collection of open subsets of X whose union is X. If for every open cover of X there is a finite subcollection which also covers X, then X is called *compact*.

Some books (like Munkres' *Topology*) refer to open covers as *open coverings*, while newer books (and wikipedia) seem to prefer to above terminology, probably for the same reasons as me: to avoid confusions with *covering spaces*, a notion we'll introduce soon.

Now we'll prove some useful properties of compact spaces and maps between them, which will lead to the important Corollaries **??** and 1.34.

Lemma 1.32. *If $f\colon X \to Y$ is a continuous map and X is compact, then the image $f(X)$ is compact.*

In particular, if X is compact, then any quotient space $X/\sim$ is compact, since the projection map $X \to X/\sim$ is continuous with image $X/\sim$.

Proof. To show that $f(X)$ is compact assume that $\{U_a\}$, $a \in A$ is an open cover of the subspace $f(X)$. Then each U_a is of the form $U_a = V_a \cap f(X)$ for some open subset $V_a \in Y$. Then $\{f^{-1}(V_a)\}$, $a \in A$ is an open cover of X. Since X is compact, there is a finite subset A' of A such that $\{f^{-1}(V_a)\}$, $a \in A'$ is a cover of X. This implies that $\{U_a\}$, $a \in A'$ is a finite cover of $f(X)$, and hence $f(X)$ is compact. □

Lemma 1.33. *1. If K is a closed subspace of a compact space X, then K is compact.*

2. If K is compact subspace of a Hausdorff space X, then K is closed.

Proof. To prove (1), assume that $\{U_a\}$, $a \in A$ is an open covering of K. Since the U_a's are open w.r.t. the subspace topology of K, there are open subsets V_a of X such that $U_a = V_a \cap K$. Then the V_a's together with the open subset $X \setminus K$ form an open covering of X. The compactness of X implies that there is a finite subset $A' \subset A$ such that the subsets V_a for $a \in A'$, together with $X \setminus K$ still cover X. It follows that U_a, $a \in A'$ is a finite cover of K, showing that K is compact.

The proof of part (2) is a homework problem. □

Corollary 1.34. *If $f \colon X \to Y$ is a continuous bijection with X compact and Y Hausdorff, then f is a homeomorphism.*

Proof. We need to show that the map $g \colon Y \to X$ inverse to f is continuous, i.e., that $g^{-1}(U) = f(U)$ is an open subset of Y for any open subset U of X. Equivalently (by passing to complements), it suffices to show that $g^{-1}(C) = f(C)$ is a closed subset of Y for any closed subset C of C.

Now the assumption that X is compact implies that the closed subset $C \subset X$ is compact by part (1) of Lemma 1.33 and hence $f(C) \subset Y$ is compact by Lemma 1.32. The assumption that Y is Hausdorff then implies by part (2) of Lemma 1.33 that $f(C)$ is closed. □

Lemma 1.35. *Let K be a compact subset of $\mathbb{R}^n$. Then K is* bounded*, meaning that there is some $r > 0$ such that K is contained in the open ball $B_r(0) := \{x \in \mathbb{R}^n \mid d(x, 0) < r\}$.*

Proof. The collection $B_r(0) \cap K$, $r \in (0, \infty)$, is an open cover of K. By compactness, K is covered by a *finite* number of these balls; if R is the maximum of the radii of these finitely many balls, this implies $K \subset B_R(0)$ as desired. □

Corollary 1.36. *If $f \colon X \to \mathbb{R}$ is a continuous function on a compact space X, then f has a maximum and a minimum.*

Proof. $K = f(X)$ is a compact subset of $\mathbb{R}$. Hence K is bounded, and thus K has an infimum $a := \inf K \in \mathbb{R}$ and a supremum $b := \sup K \in \mathbb{R}$. The infimum (resp. supremum) of K is the limit of a sequence of elements in K; since K is closed (by Lemma 1.33 (2)), the limit points a and b belong to K by Lemma 1.30. In other words, there are elements $x_{min}, x_{max} \in X$ with $f(x_{min}) = a \leq f(x)$ for all $x \in X$ and $f(x_{max}) = b \geq f(x)$ for all $x \in X$. □

In order to use Corollaries 1.34 and 1.36, we need to be able to show that topological spaces we are interested in, are in fact compact. Note that this is *quite difficult* just working from the definition of compactness: you need to ensure that *every* open cover has a finite subcover. That sounds like a lot of work...

Fortunately, there is a very simple classical characterization of compact subspaces of Euclidean spaces:

Theorem 1.37. (Heine-Borel Theorem) *A subspace $X \subset \mathbb{R}^n$ is compact if and only if X is closed and bounded.*

We note that we've already proved that if $K \subset \mathbb{R}^n$ is compact, then K is a closed subset of $\mathbb{R}^n$ (Lemma 1.33(2)), and K is bounded (Lemma 1.35).

There two important ingredients to the proof of the converse, namely the following two results:

Lemma 1.38. *A closed interval $[a, b]$ is compact.*

This lemma has a short proof that can be found in any pointset topology book, e.g., [**?**].

Theorem 1.39. *If $X_1, \dots, X_n$ are compact topological spaces, then their product $X_1 \times \cdots \times X_n$ is compact.*

For a proof see e.g. [**?**, Ch. 3, Thm. 5.7]. The statement is true more generally for a product of *infinitely many* compact space (as discussed in [**?**, p. 113], the correct definition of the product topology for infinite products requires some care), and this result is called *Tychonoff's Theorem*, see [**?**, Ch. 5, Thm. 1.1].

Proof of the Heine-Borel Theorem. Let $K \subset \mathbb{R}^n$ be closed and bounded, say $K \subset B_r(0)$. We note that $B_r(0)$ is contained in the n-fold product

$$P := [-r, r] \times \cdots \times [-r, r] \subset \mathbb{R}^n$$

which is compact by Theorem 1.39. So K is a closed subset of P and hence compact by Lemma 1.33(1). □

1.4.3 Connected spaces

Definition 1.40. A topological space X is *connected* if it can't be written as decomposed in the form $X = U \cup V$, where U, V are two non-empty disjoint open subsets of X.

For example, if a, b, c, d are real numbers with $a < b < c < d$, consider the subspace $X = (a, b) \amalg (c, d) \subset \mathbb{R}$. The topological space X is not connected, since $U = (a, b)$, $V = (c, d)$ are open disjoint subsets of X whose union is X. This remains true if we replace the open intervals by closed intervals. The space $X' = [a, b] \amalg [c, d]$ is not connected, since it is the disjoint union of the subsets $U' = [a, b]$, $V' = [c, d]$. We want to emphasize that while U' and V' are not open as subsets of $\mathbb{R}$, they are *open subsets of* X', since they can be written as

$$U' = (-\infty, c) \cap X' \qquad V' = (b, \infty) \cap X',$$

showing that they are open subsets for the subspace topology of $X' \subset \mathbb{R}$.

Lemma 1.41. *Any interval I in $\mathbb{R}$ (open, closed, half-open, bounded or not) is connected.*

Proof. Using proof by contradiction, let us assume that I has a decomposition $I = U \cup V$ as the union of two non-empty disjoint open subsets. Pick points $u \in U$ and $v \in V$, and let us assume $u < v$ without loss of generality. Then

$$[u, v] = U' \cup V' \qquad \text{with} \qquad U' := U \cap [u, v] \quad V' := U \cap [u, v]$$

is a decomposition of $[u, v]$ as the disjoint union of non-empty disjoint open subsets U', V' of $[u, v]$. We claim that the supremum $c := \sup U'$ belongs to both, U' and V', thus leading to the desired contradiction. Here is the argument.

- Assuming that c doesn't belong to U', for any $\epsilon > 0$, there must be some element of U' belonging to the interval $(c - \epsilon, c)$, allowing us to construct a sequence of elements $u_i \in U'$ converging to c. This implies $c \in U'$ by Lemma 1.30, since U' is a *closed* subspace of $[u, v]$ (its complement V' is open).
- By construction, every $x \in [u, v]$ with $x > c = \sup U'$ belongs to V'. So we can construct a sequence $v_i \in V'$ converging to c. Since V' is a closed subset of $[u, v]$, we conclude $c \in V'$.

□

Theorem 1.42. (Intermediate Value Theorem) *Let X be a connected topological space, and $f : X \to \mathbb{R}$ a continuous map. If elements $a, b \in \mathbb{R}$ belong to the image of f, then also any real number c between a and b belongs to the image of f.*

Proof. Assume that c is not in the image of f. Then $X = f^{-1}(-\infty, c) \cup f^{-1}(c, \infty)$ is a decomposion of X as a union of non-empty disjoint open subsets. □

There is another notion, closely related to the notion of connected topological space, which might be easier to think of geometrically.

Definition 1.43. A topological space X is *path connected* if for any points $x, y \in X$ there is a path connecting them. In other words, there is a continuous map $\gamma : [a, b] \to X$ from some interval to X with $\gamma(a) = x$, $\gamma(b) = y$.

Lemma 1.44. *Any path connected topological space is connected.*

Proof. Using proof by contradiction, let us assume that the topological space X is path connected, but not connected. So there is a decomposition $X = U \cup V$ of X as the union of non-empty open subsets $U, V \subset X$. The assumption that X is path connected allows us to find a path $\gamma : [a, b] \to X$ with $\gamma(a) \in U$ and $\gamma(b) \in V$. Then we obtain the decomposition

$$[a, b] = f^{-1}(U) \cup f^{-1}(V)$$

of the interval $[a, b]$ as the disjoint union of open subsets. These are non-empty since $a \in f^{-1}(U)$ and $b \in f^{-1}(V)$. This implies that $[a, b]$ is not connected, the desired contradiction. □

For typical topological spaces we will consider, the properties "connected" and "path connected" are equivalent. But here is an example known as the *topologist's sine curve* which is connected, but not path connected, see [**?**, Example 7, p. 156]. It is the following subspace of $\mathbb{R}^2$:

$$X = \{(x, \sin\frac{1}{x}) \in \mathbb{R}^2 \mid 0 < x < 1\} \cup \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}.$$