Math 60330: Basic Geometry and Topology
Midterm Exam

This exam is due on Friday, October 14th in class. You have unlimited time. You are welcome to consult the course notes and other textbooks. However, you may not talk to anyone other than Prof. Andrew Putman about this exam before it is due; do so will be a violation of the honor code.

1. Let $T^2 = S^1 \times S^1$ be the 2-dimensional torus. Show that any map $f\colon S^2 \to T^2$ is homotopic to a constant map.

2. (a) Let $x_1, \ldots, x_k \in S^2$ be distinct points and let $p \in S^2 \setminus \{x_1, \ldots, x_k\}$ be a basepoint. Calculate a presentation for the group $\pi_1(S^2 \setminus \{x_1, \ldots, x_k\}, p)$.

   (b) Let $L_1, \ldots, L_k$ be distinct non-intersecting lines in $\mathbb{R}^3$ and let $q \in \mathbb{R}^3 \setminus (L_1 \cup \cdots \cup L_k)$ be a basepoint. Calculate a presentation for the group $\pi_1(\mathbb{R}^3 \setminus (L_1 \cup \cdots \cup L_k), q)$.

3. Let X be a space. Define the *suspension of* X, denoted ΣX, to be the quotient of $X \times [0,1]$ that identifies $X \times \{0\}$ to a single point and $X \times \{1\}$ to a single point (these are two different points!). Fix a basepoint $p \in \Sigma X$. Prove that if X is path-connected, then $\pi_1(\Sigma X, p) = 1$. Give an example to show that path-connectedness is necessary.

4. Let X be a path-connected Hausdorff space and let $f\colon \widetilde{X} \to X$ be a covering space with $\widetilde{X}$ compact. Prove that $f\colon \widetilde{X} \to X$ is a finite-sheeted cover.

5. Using covering space theory, construct a free basis for the commutator subgroup $[F_2, F_2]$ of the free group F_2 on two generators a and b.