CHAPTER 4

Integration on Manifolds

§1 Introduction

Until now we have studied manifolds from the point of view of differential calculus. The last chapter introduces to our study the methods of integral calculus. As the new tools are developed in the next sections, the reader may be somewhat puzzled about their relevancy to the earlier material. Hopefully, the puzzlement will be resolved by the chapter's end, but a preliminary example may be helpful.

Let

$$p(z) = z^m + a_1 z^{m-1} + \cdots + a_m$$

be a complex polynomial on Ω, a smooth compact region in the plane whose boundary contains no zero of p. In Section 3 of the previous chapter we showed that the number of zeros of p inside Ω, counting multiplicities, equals the degree of the map

$$\frac{p}{|p|} : \partial\Omega \to S^1.$$

A famous theorem in complex variable theory, called the *argument principle*, asserts that this may also be calculated as an integral,

$$\oint_{\partial\Omega} d(\arg p).$$

You needn't understand the precise meaning of the expression in order to appreciate our point. What is important is that the number of zeros can be computed both by an intersection number and by an integral formula.

Theorems like the argument principal abound in differential topology. We shall show that their occurrence is not arbitrary or fortuitous but is a geometric consequence of a general theorem known as *Stokes theorem.* Low-dimensional versions of Stokes theorem are probably familiar to you from calculus:

1. Second Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

2. Green's Theorem in the plane:

$$\oint_{\partial\Omega} f_1\,dx + f_2\,dy = \iint_{\Omega} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy.$$

3. Divergence Theorem in 3-space:

$$\iiint_{\Omega} \operatorname{div} \vec{F}\,dv = \iint_{\partial\Omega} \vec{F}\cdot\vec{n}\,dA, \qquad \vec{F} = (f_1, f_2, f_3).$$

4. Classical Stokes Theorem in 3-space:

$$\begin{aligned}\oint_{\partial S} f_1\,dx + f_2\,dy + f_3\,dz &= \iint_S \vec{n}\cdot\operatorname{curl}\vec{F}\,dA \\ &= \iint_S \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) dy \wedge dz \\ &\quad + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) dz \wedge dx \\ &\quad + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx \wedge dy.\end{aligned}$$

The argument principal, in particular, may be easily deduced from Green's theorem provided that you know a little about complex analytic functions.

The preceding formulas share a common spirit; they all explain how to calculate an integral over a bounded region by an integral over the boundary alone. It is reasonable to expect that such relations can be generalized to arbitrary manifolds of any dimension, but it is not at all clear how they should be formulated. Part of the problem is just making sense of infinitesimal expressions like

$$f_1\,dx \quad \text{or} \quad \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy.$$

Elementary calculus books tend not to worry too much about these integrands, but one must be more scrupulous in higher dimensions. So we shall have to endure some formalism as the next section develops the algebra needed to define and manipulate infinitesimals. Then in Section 3 we introduce the appropriate integrands, abstract objects called differential forms.

It seems hopeless to provide at the outset a satisfactory, intuitive motivation for differential forms. Only experience with them can convince the student that these formal entities, with their special algebraic properties, provide the most natural basis for integration theory. Expressed in the language of forms, the general Stokes theorem becomes simple and elegant. And, in truth, isn't simplicity the ultimate test of the basic concepts of any theory? Why, for instance, did physics formulate the laws of mechanics in terms of momentum and energy rather than with some other arbitrary functions of mass and velocity? The reason is that those quantities enormously simplify the basic equations, thereby making the subject easier to understand. As for intuition, how forbiddingly formal were such notions as electric fields or complex numbers until their obvious convenience forced us to employ them, developing experience with time?

§2 Exterior Algebra

In order to establish an algebraic basis for differential forms, we begin by defining certain generalizations of the dual space to a real vector space V. A p-tensor on V is a real-valued function T on the cartesian product

$$V^p = \underbrace{V \times \cdots \times V}_{p \text{ times}},$$

which is separately linear in each variable, or *multilinear.* That is, holding all but the jth variable constant, we have the usual linearity condition

$$T(v_1, \ldots, v_j + av'_j, \ldots, v_p) = T(v_1, \ldots, v_j, \ldots, v_p) + aT(v_1, \ldots, v'_j, \ldots, v_p).$$

In particular, 1-tensors are just linear functionals on V. A familiar 2-tensor is dot product on $\mathbf{R}^k$. Also on $\mathbf{R}^k$ you know a k-tensor—namely, the determinant. For any k vectors $v_1, \ldots, v_k \in \mathbf{R}^k$ can be arranged into a $k \times k$

matrix

$$\begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_k \end{pmatrix},$$

and the determinant of this matrix is multilinear with respect to the row vectors; denote it by det $(v_1, \ldots, v_k)$.

As sums and scalar multiples of multilinear functions are still multilinear, the collection of all p-tensors is a vector space $\mathfrak{J}^p(V^*)$. Note that $\mathfrak{J}^1(V^*) = V^*$. Tensors may also be multiplied in a simple way; if T is a p-tensor and S a q-tensor, we define a $p + q$ tensor $T \otimes S$ by the formula

$$T \otimes S(v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q}) = T(v_1, \ldots, v_p) \cdot S(v_{p+1}, \ldots, v_{p+q}).$$

$T \otimes S$ is called the *tensor product* of T with S. Note that the tensor product operation is not commutative,

$$T \otimes S \neq S \otimes T,$$

but it is easy to check that it is associative and that it distributes over addition.

Tensor product clarifies the manner in which $\mathfrak{J}^p(V^*)$ extends V^*.

Theorem. Let $\{\phi_1, \ldots, \phi_k\}$ be a basis for V^*. Then the p-tensors $\{\phi_{i_1} \otimes \cdots \otimes \phi_{i_p}: 1 \leq i_1, \ldots, i_p \leq k\}$ form a basis for $\mathfrak{J}^p(V^*)$. Consequently, dim $\mathfrak{J}^p(V^*) = k^p$.

Proof. During the proof (only) we shall use the following notation. If $I = (i_1, \ldots, i_p)$ is a sequence of integers each between 1 and k, let

$$\phi_I = \phi_{i_1} \otimes \ldots \otimes \phi_{i_p}.$$

Let $\{v_1, \ldots, v_k\}$ be the dual basis in V, and denote by v_I the sequence $(v_{i_1}, \ldots, v_{i_p})$. By definition, if I and J are two such index sequences, $\phi_I(v_J)$ is 1 if $I = J$ and 0 if $I \neq J$. It is clear from multilinearity that two p-tensors T and S are equal if and only if $T(v_J) = S(v_J)$ for every index sequence J. Thus if we are given T, the tensor

$$S = \sum_I T(v_I)\phi_I$$

must equal T; hence the $\{\phi_I\}$ span $\mathfrak{J}^p(V^*)$. The ϕ_I are also independent, for if

$$S = \sum_I a_I\phi_I = 0,$$

then

$$0 = S(v_J) = a_J,$$

for each J. Q.E.D.

A tensor T is *alternating* if the sign of T is reversed whenever two variables are transposed:

$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_p) = -T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_p).$$

All 1-tensors are automatically alternating. The determinant is also alternating, but the dot product is not. It is useful to rephrase this condition slightly. Let S_p denote the group of permutations of the numbers 1 to p. Recall that a permutation $\pi \in S_p$ is called *even* or *odd*, depending on whether it is expressible as a product of an even or odd number of simple transpositions. Let $(-1)^\pi$ be $+1$ or -1, depending on whether π is even or odd. For any p-tensor T and any $\pi \in S_p$, define another p-tensor T^π by

$$T^\pi(v_1, \ldots, v_p) = T(v_{\pi(1)}, \ldots, v_{\pi(p)}).$$

Then, clearly, the alternating p-tensors are those satisfying

$$T^\pi = (-1)^\pi T \qquad \text{for all } \pi \in S_p$$

Note that $(T^\pi)^\sigma = T^{\pi\circ\sigma}$ always holds.

There is a standard procedure for making alternating tensors out of arbitrary ones. If T is any p-tensor, define a new one Alt (T) by

$$\operatorname{Alt}(T) = \frac{1}{p!} \sum_{\pi \in S_p} (-1)^\pi T^\pi.$$

Note that Alt (T) is in fact alternating, for it is obvious that $(-1)^{\pi\circ\sigma} = (-1)^\pi(-1)^\sigma$. Thus

$$[\operatorname{Alt}(T)]^\sigma = \frac{1}{p!} \sum_{\pi \in S_p} (-1)^\pi (T^\pi)^\sigma = \frac{1}{p!}(-1)^\sigma \sum_{\pi \in S_p} (-1)^{\pi\circ\sigma} T^{\pi\circ\sigma}.$$

If we set $\tau = \pi \circ \sigma$, then, because S_p is a group, as π ranges through S_p so does τ. Thus

$$[\operatorname{Alt}(T)]^\sigma = (-1)^\sigma \frac{1}{p!} \sum_{\tau \in S_p} (-1)^\tau T^\tau = (-1)^\sigma \operatorname{Alt}(T),$$

as claimed.

Also note that if T is already alternating, then Alt $(T) = T$, for each summand $(-1)^\pi T^\pi$ equals T, and there are exactly $p!$ permutations in S_p.

Since sums and scalar multiples of alternating functions continue to alternate, the alternating p-tensors form a vector subspace $\Lambda^p(V^*)$ of $\mathfrak{I}^p(V^*)$. Unhappily, tensor products of alternating tensors do not alternate, but here the Alt operator can be useful. If $T \in \Lambda^p(V^*)$ and $S \in \Lambda^q(V^*)$, we define their *wedge product*

$$T \wedge S \in \Lambda^{p+q}(V^*)$$

to be Alt $(T \otimes S)$.† The wedge product clearly distributes over addition and scalar multiplication, because Alt is a linear operation; however, proving associativity will require some work. We need a calculation.

Lemma. If Alt $(T) = 0$, then $T \wedge S = 0 = S \wedge T$.

Proof. S_{p+q} carries a natural copy of S_p—namely, the subgroup G consisting of all permutations of $(1, \ldots, p+q)$ that fix $p+1, \ldots, p+q$. The correspondence between G and S_p assigns to each $\pi \in G$ the permutation π' induced by restricting π to $(1, \ldots, p)$. Note that $(T \otimes S)^\pi = T^{\pi'} \otimes S$, and $(-1)^\pi = (-1)^{\pi'}$. Thus

$$\sum_{\pi \in G} (-1)^\pi (T \otimes S)^\pi = \left[\sum_{\pi' \in S_p} (-1)^{\pi'} T^{\pi'}\right] \otimes S = \mathrm{Alt}\,(T) \otimes S = 0.$$

Now a subgroup G decomposes S_{p+q} into a disjoint union of right cosets $G \circ \sigma = \{\pi \circ \sigma : \pi \in G\}$. But for each such coset,

$$\sum_{\pi \in G} (-1)^{\pi \circ \sigma} (T \otimes S)^{\pi \circ \sigma} = (-1)^\sigma \left[\sum_{\pi \in G} (-1)^\pi (T \otimes S)^\pi\right]^\sigma = 0.$$

Since $T \wedge S = \mathrm{Alt}\,(T \otimes S)$ is the sum of these partial summations over the right cosets of G, then $T \wedge S = 0$. Similarly, $S \wedge T = 0$. Q.E.D.

Theorem. Wedge product is associative,

$$(T \wedge S) \wedge R = T \wedge (S \wedge R),$$

justifying the notation $T \wedge S \wedge R$.

Proof. We claim that $(T \wedge S) \wedge R$ equals Alt $(T \otimes S \otimes R)$. By definition,

$$(T \wedge S) \wedge R = \mathrm{Alt}\,((T \wedge S) \otimes R),$$

so the linearity of Alt implies

$$(T \wedge S) \wedge R - \mathrm{Alt}\,(T \otimes S \otimes R) = \mathrm{Alt}\,([T \wedge S - T \otimes S] \otimes R).$$

†There is always some question about normalizing the definition of $\wedge$. Spivak, for example, includes some factorial multipliers. We have chosen to keep the algebra simple, but we shall be haunted by the ghost of the vanished factorials when we discuss volumes.

Since $T \wedge S$ is alternating,

$$\begin{aligned}\mathrm{Alt}\,(T \wedge S - T \otimes S) &= \mathrm{Alt}\,(T \wedge S) - \mathrm{Alt}\,(T \otimes S) \\ &= T \wedge S - T \wedge S = 0.\end{aligned}$$

So the lemma implies

$$\mathrm{Alt}\,([T \wedge S - T \otimes S] \otimes R) = 0,$$

as needed. A similar argument shows

$$T \wedge (S \wedge R) = \mathrm{Alt}\,(T \otimes S \otimes R) \quad \text{Q.E.D.}$$

The formula $T \wedge S \wedge R = \mathrm{Alt}\,(T \otimes S \otimes R)$, derived in the proof above, obviously extends to relate the wedge and tensor products of any number of tensors. We can use it to derive a basis for $\Lambda^p(V^*)$. For if T is a p-tensor, then we may write

$$T = \Sigma\, t_{i_1, \ldots, i_p} \phi_{i_1} \otimes \cdots \otimes \phi_{i_p},$$

where $\{\phi_1, \ldots, \phi_k\}$ is a basis for V^*, and the sum ranges over all index sequences $(i_1, \ldots, i_p)$ for which each index is between 1 and k. If T alternates, then $T = \mathrm{Alt}\,(T)$, so

$$T = \Sigma\, t_{i_1, \ldots, i_p} \mathrm{Alt}\,(\phi_{i_1} \otimes \cdots \otimes \phi_{i_p}) = \Sigma\, t_{i_1, \ldots, i_p} \phi_{i_1} \wedge \cdots \wedge \phi_{i_p}.$$

Henceforth we shall denote the alternating tensors $\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}$ by ϕ_I, where $I = (i_1, \ldots, i_p)$. We have shown that the ϕ_I span $\Lambda^p(V^*)$; however, due to a fundamental property of the wedge product, they are not independent.

Suppose that ϕ and ψ are any linear functionals on V, $\phi, \psi \in \Lambda^1(V^*)$. In this case, the Alt operator takes a very simple form:

$$\phi \wedge \psi = \tfrac{1}{2}(\phi \otimes \psi - \psi \otimes \phi).$$

Observe that

$$\phi \wedge \psi = -\psi \wedge \phi \quad \text{and} \quad \phi \wedge \phi = 0,$$

showing that $\wedge$ is anticommutative on $\Lambda^1(V^*)$. As you will discover, the anticommutativity of wedge product on 1-forms is its fundamental property. In fact, the essential reason for developing the algebra of alternating tensors is to build anticommutativity into the foundation of integration theory.

Anticommutativity introduces some relations into the set of spanning tensors $\{\phi_I\}$. If two index sequences I and J differ only in their orderings, iterated application of anticommutativity shows that $\phi_I = \pm\phi_J$. And if any of the indices of I are equal, $\phi_I = 0$. Consequently, we can eliminate redundancy in the spanning set by allowing only those ϕ_I for which the index sequence is strictly increasing: $1 \leq i_1 < i_2 < \cdots < i_p \leq k$. The number of such sequences is the number of ways one can choose p items from the set $\{1, \ldots, k\}$, namely,

$$\binom{k}{p} = \frac{k!}{p!(k-p)!}.$$

It is easy to see that the remaining tensors are linearly independent. Let $\{v_1, \ldots, v_k\}$ be the basis for V dual to $\{\phi_1, \ldots, \phi_k\}$. For any increasing index sequence $I = (i_1, \ldots, i_p)$, let $v_I = (v_{i_1}, \ldots, v_{i_p})$. The definition of the Alt operator then shows that $\phi_I(v_I) = 1/p!$, but if J is a different increasing index sequence, then $\phi_I(v_J) = 0$. Thus if $\sum a_I \phi_I = 0$ is a relation among the new spanning set, then

$$0 = \Sigma\, a_I \phi_I(v_J) = \frac{1}{p!} a_J$$

shows that each $a_J = 0$. We have proved

Theorem. If $\{\phi_1, \ldots, \phi_k\}$ is a basis for V^*, then $\{\phi_I = \phi_{i_1} \wedge \cdots \wedge \phi_{i_p}$: $1 \leq i_1, \ldots, i_p \leq k\}$ is a basis for $\Lambda^p(V^*)$. Consequently,

$$\dim \Lambda^p(V^*) = \binom{k}{p} = \frac{k!}{p!(k-p)!}$$

Suppose that the index sequence I has length p, while J has length q. From the anticommutativity of $\wedge$ on $\Lambda^1(V^*)$, you can quickly show that

$$\phi_I \wedge \phi_J = (-1)^{pq} \phi_J \wedge \phi_I.$$

The foregoing basis theorem therefore implies

Corollary. The wedge product satisfies the following anticommutativity relation:

$$T \wedge S = (-1)^{pq} S \wedge T,$$

whenever

$$T \in \Lambda^p(V^*) \quad \text{and} \quad S \in \Lambda^q(V^*).$$

The basis theorem also implies that $\Lambda^k(V^*)$ is one dimensional, where $k = \dim V$. You have probably known this for years, although in this form the statement looks unfamiliar. We already know one nonzero alternating k-tensor on $\mathbf{R}^k$, the determinant tensor det. So $\dim \Lambda^k(\mathbf{R}^{k*}) = 1$ just means that every alternating k-multilinear function on $\mathbf{R}^k$ is a multiple of the determinant, a fact you know as "the uniqueness of the determinant function."

If the length of the index sequence I is greater than the dimension k of V, then I must repeat at least one integer; thus $\phi_I = 0$. We conclude that $\Lambda^p(V^*) = 0$ if $p > k$, so the sequence of vector spaces $\Lambda^1(V^*)$, $\Lambda^2(V^*)$, ... terminates at $\Lambda^k(V^*)$. It is useful to add one more entry to this list by defining $\Lambda^0(V^*) = \mathbf{R}$, which is interpreted as the constant functions on V. We extend $\wedge$ by simply letting the wedge product of any element in $\mathbf{R}$ with any tensor in $\Lambda^p(V^*)$ be the usual scalar multiplication. The wedge product then makes the direct sum

$$\Lambda(V^*) = \Lambda^0(V^*) \oplus \Lambda^1(V^*) \oplus \cdots \oplus \Lambda^k(V^*)$$

a noncommutative algebra, called the *exterior algebra* of V^*, whose identity element is $1 \in \Lambda^0(V^*)$.

One further construction is basic. Suppose that $A: V \longrightarrow W$ is a linear map. Then the transpose map $A^*: W^* \longrightarrow V^*$ extends in an obvious manner to the exterior algebras, $A^*: \Lambda^p(W^*) \longrightarrow \Lambda^p(V^*)$ for all $p \geq 0$. If $T \in \Lambda^p(W^*)$, just define $A^*T \in \Lambda^p(V^*)$ by

$$A^*T(v_1, \ldots, v_p) = T(Av_1, \ldots, Av_p)$$

for all vectors $v_1, \ldots, v_p \in V$. It is easy to check that A^* is linear and that

$$A^*(T \wedge S) = A^*T \wedge A^*S.$$

So A^* is an algebra homomorphism: $\Lambda(W^*) \longrightarrow \Lambda(V^*)$. Note that if $B: W \longrightarrow U$ is another linear map, then $(BA)^* = A^*B^*$.

In particular, suppose that $A: V \longrightarrow V$ is an isomorphism, and $\dim V = k$ as usual. Then $A^*: \Lambda^k(V^*) \longrightarrow \Lambda^k(V^*)$ is a linear map of a one-dimensional vector space, hence it must be multiplication by some constant $\lambda \in \mathbf{R}$, i.e., $A^*T = \lambda T$ for all $T \in \Lambda^k(V^*)$. We claim that λ is just the determinant of A. We know that $\det \in \Lambda^k(\mathbf{R}^{k*})$. So choose any isomorphism $B: V \longrightarrow \mathbf{R}^k$, and consider $T = B^*(\det) \in \Lambda^k(V^*)$. Then $A^*B^*(\det) = \lambda B^*(\det)$, implying

$$B^{*-1}A^*B^*(\det) = \lambda(B^*)^{-1}B^*(\det) = \lambda(BB^{-1})^*(\det) = \lambda(\det)$$

or

$$(BAB^{-1})^*(\det) = \lambda(\det).$$

Now evaluate both sides of this equation on the standard ordered basis $\{e_1, \ldots, e_k\}$ for $\mathbf{R}^k$. A quick check of the definition of the tensor det shows that, for any linear map C, $\det(Ce_1, \ldots, Ce_n) = \det(C)$. Thus

$$\lambda = \det(BAB^{-1}) = \det(A)$$

as claimed, proving

Determinant Theorem. If $A: V \longrightarrow V$ is a linear isomorphism, then $A^*T = (\det A)T$ for every $T \in \Lambda^k(V)$, where $k = \dim V$. In particular, if $\phi_1, \ldots, \phi_k \in \Lambda^1(V^*)$, then

$$A^*\phi_1 \wedge \cdots \wedge A^*\phi_k = (\det A)\, \phi_1 \wedge \cdots \wedge \phi_k.$$

EXERCISES

1. Suppose that $T \in \Lambda^p(V^*)$ and $v_1, \ldots, v_p \in V$ are linearly dependent. Prove that $T(v_1, \ldots, v_p) = 0$ for all $T \in \Lambda^p(V^*)$.

2. Dually, suppose that $\phi_1, \ldots, \phi_p \in V^*$ are linearly dependent, and prove that $\phi_1 \wedge \cdots \wedge \phi_p = 0$.

3. Suppose that $\phi_1, \ldots, \phi_k \in V^*$ and $v_1, \ldots, v_k \in V$, where $k = \dim V$. Prove that

$$\phi_1 \wedge \cdots \wedge \phi_k(v_1, \ldots, v_k) = \frac{1}{k!} \det [\phi_i(v_j)],$$

where $[\phi_i(v_j)]$ is a $k \times k$ real matrix. [HINT: If the ϕ_i are dependent, then the matrix has linearly dependent rows, so Exercise 2 suffices. If not, the formula is easily checked for the dual basis in V. Now verify that the matrix does specify an alternating k-tensor on V, and use $\dim \Lambda^k(V^*) = 1$.]

4. More generally, show that whenever $\phi_1, \ldots, \phi_p \in V^*$ and $v_1, \ldots, v_p \in V$, then

$$\phi_1 \wedge \cdots \wedge \phi_p(v_1, \ldots, v_p) = \frac{1}{p!} \det [\phi_i(v_j)].$$

[HINT: If the v_i are dependent, use Exercise 1. If not, apply Exercise 3 to the restrictions $\overline{\phi_i}$ of ϕ_i to the p-dimensional subspace spanned by $v_1, \ldots, v_p$.]

5. Specifically write out Alt $(\phi_1 \otimes \phi_2 \otimes \phi_3)$ for $\phi_1, \phi_2, \phi_3 \in V^*$.

*6. (a) Let T be a nonzero element of $\Lambda^k(V^*)$, where $\dim V = k$. Prove that two ordered bases $\{v_1, \ldots, v_k\}$ and $\{v'_1, \ldots, v'_k\}$ for V are equivalently oriented if and only if $T(v_1, \ldots, v_k)$ and $T(v'_1, \ldots, v'_k)$ have the same sign. [HINT: Determinant theorem.]
(b) Suppose that V is oriented. Show that the one-dimensional vector space $\Lambda^k(V^*)$ acquires a natural orientation, by defining the sign of a nonzero element $T \in \Lambda^k(V^*)$ to be the sign of $T(v_1, \ldots, v_k)$ for any positively oriented ordered basis $\{v_1, \ldots, v_k\}$ for V.

(c) Conversely, show that an orientation of $\Lambda^k(V^*)$ naturally defines an orientation on V by reversing the above.

7. For a $k \times k$ matrix A, let A^t denote the transpose matrix. Using the fact that det (A) is multilinear in both the rows and columns of A, prove that $\det(A^t) = \det(A)$. [HINT: Use $\dim \Lambda^k(\mathbf{R}^{k*}) = 1$]

8. Recall that a matrix A is *orthogonal* if $AA^t = I$. Conclude that if A is orthogonal, $\det(A) = \pm 1$.

9. Let V be a k-dimensional subspace of $\mathbf{R}^N$. Recall that a basis $v_1, \ldots, v_k$ of V is *orthonormal* if

$$v_i \cdot v_j = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Let $A\colon V \longrightarrow V$ be a linear map, and prove the following three conditions equivalent:
(a) $Av \cdot Aw = v \cdot w$ for all $v, w \in V$.
(b) A carries orthonormal bases to orthonormal bases.
(c) The matrix of A with respect to any orthonormal basis is orthogonal.
Such an A is called an *orthogonal transformation.* [Note, by (b), it must be an isomorphism.]

*10. (a) Let V be an oriented k-dimensional vector subspace of $\mathbf{R}^N$. Prove there is an alternating k-tensor $T \in \Lambda^k(V^*)$ such that $T(v_1, \ldots, v_k) = 1/k!$ for all positively oriented ordered orthonormal bases. Furthermore, show that T is unique; it is called the *volume element* of V. [HINT: Use the determinant theorem, Exercises 8 and 9, plus $\dim \Lambda^k(V^*) = 1$ for uniqueness.]
(b) In fact, suppose that $\phi_1, \ldots, \phi_k \in V^*$ is an ordered basis dual to some positively oriented ordered orthonormal basis for V. Show that the volume element for V is $\phi_1 \wedge \cdots \wedge \phi_k$. [HINT: Exercise 3.]

*11. Let T be the volume element of $\mathbf{R}^2$. Prove that for any vectors $v_1, v_2 \in \mathbf{R}^2$, $T(v_1, v_2)$ is $\pm$ one half the volume of the parallelogram spanned by v_1 and v_2. Furthermore, when v_1 and v_2 are independent, then the sign equals the sign of the ordered basis $\{v_1, v_2\}$ in the standard orientation of $\mathbf{R}^2$. Generalize to $\mathbf{R}^3$. Now how would you define the volume of a parallelepiped in $\mathbf{R}^k$?

12. (a) Let V be a subspace of $\mathbf{R}^N$. For each $v \in V$, define a linear functional $\phi_v \in V^*$ by $\phi_v(w) = v \cdot w$. Prove that the map $v \longrightarrow \phi_v$ is an isomorphism of V with V^*.

(b) Now suppose that V is oriented and $\dim V = 3$. Let T be the volume element on V. Given $u, v \in V$, define a linear functional on V by $w \longrightarrow 3!T(u, v, w)$. By part (a), there exists a vector, which we denote $u \times v$, such that $T(u, v, w) = (u \times v) \cdot w$ for all $w \in V$. Prove that this *cross product* satisfies $u \times v = -v \times u$. Furthermore, show that if $\{v_1, v_2, v_3\}$ is a positively oriented orthonormal basis for V, then $v_1 \times v_2 = v_3$, $v_2 \times v_3 = v_1$, and $v_3 \times v_1 = v_2$. (Also, $v \times v = 0$ always.)

§3 Differential Forms

In classical differential geometry, forms were symbolic quantities that looked like

$$\sum_i f_i \, dx_i$$

$$\sum_{i<j} f_{ij} \, dx_i \wedge dx_j$$

$$\sum_{i<j<k} f_{ijk} \, dx_i \wedge dx_j \wedge dx_k.$$

These expressions were integrated and differentiated, and because experience proved anticommutativity to be convenient, they were manipulated like alternating tensors. Modern differential forms locally reduce to the same symbolic quantities, but they possess the indispensable attribute of being globally defined on manifolds. Global definition of integrands makes possible global integration.

Definition. Let X be a smooth manifold with or without boundary. A *p-form* on X is a function ω that assigns to each point $x \in X$ an alternating p-tensor $\omega(x)$ on the tangent space of X at x; $\omega(x) \in \Lambda^p[T_x(X)^*]$.

Two p-forms ω_1 and ω_2 may be added point by point to create a new p-form $\omega_1 + \omega_2$:

$$(\omega_1 + \omega_2)(x) = \omega_1(x) + \omega_2(x).$$

Similarly, the wedge product of forms is defined point by point. If ω is a p-form and θ is a q-form, the $p + q$ form $\omega \wedge \theta$ is given by $(\omega \wedge \theta)(x) = \omega(x) \wedge \theta(x)$. Anticommutativity $\omega \wedge \theta = (-1)^{pq}\theta \wedge \omega$ follows from the analogous equation at each point.

0-forms are just arbitrary real-valued functions on X.

Many examples of 1-forms can be manufactured from smooth functions. If $\phi : X \longrightarrow \mathbf{R}$ is a smooth function, $d\phi_x : T_x(X) \longrightarrow \mathbf{R}$ is a linear map at each point x. Thus the assignment $x \longrightarrow d\phi_x$ defines a 1-form $d\phi$ on X, called the *differential* of ϕ. In particular, the coordinate functions $x_1, \ldots, x_k$ on $\mathbf{R}^k$ yield 1-forms $dx_1, \ldots, dx_k$ on $\mathbf{R}^k$. Now at each $z \in \mathbf{R}^k$, $T_z(\mathbf{R}^k) = \mathbf{R}^k$; check that the important differentials $dx_1, \ldots, dx_k$ have the specific action $dx_i(z)(a_1, \ldots, a_k) = a_i$. Thus at each $z \in \mathbf{R}^k$, the linear functionals $dx_1(z), \ldots, dx_k(z)$ are just the standard basis for $(\mathbf{R}^k)^*$.

In terms of $dx_1, \ldots, dx_k$ it is easy to write down all forms on an open subset U of Euclidean space. For each strictly increasing index sequence $I = (i_1, \ldots, i_p)$, let

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

a p-form on $\mathbf{R}^k$. Then from the basis theorem for the individual vector spaces

$$\Lambda^p[T_z(U)^*] = \Lambda^p(R^{k*}),$$

we obtain at once

Proposition. Every p-form on an open set $U \subset \mathbf{R}^k$ may be uniquely expressed as a sum $\sum_I f_I \, dx_I$, over increasing index sequences $I = (i_1 < \cdots < i_p)$, the f_I being functions on U.

The proposition says that forms in Euclidean space are really the symbolic quantities of classical mathematics—except that our symbols have precise meanings. Here is an easy exercise that will force you to check your understanding of the definitions: show that if ϕ is a smooth function on $\mathbf{R}^k$, then

$$d\phi = \sum \frac{\partial \phi}{\partial x_i} dx_i.$$

(The formula itself is obvious, but what do the two sides mean?)

One of the important features of forms is that they pull back naturally under smooth mappings. If $f\colon X \longrightarrow Y$ is a smooth map and ω is a p-form on Y, we define a p-form $f^*\omega$ on X as follows. If $f(x) = y$, then f induces a derivative map $df_x : T_x(X) \longrightarrow T_y(Y)$. Since $\omega(y)$ is an alternating p-tensor on $T_y(Y)$, we can pull it back to $T_x(X)$ using the transpose $(df_x)^*$, as described in the previous section. Define

$$f^*\omega(x) = (df_x)^*\omega[f(x)].$$

Then $f^*\omega(x)$ is an alternating p-tensor on $T_x(X)$, so $f^*\omega$ is a p-form on X, called the *pullback* of ω by f. When ω is a 0-form—that is, a function on Y—then $f^*\omega = \omega \circ f$, a function on X. Remember that f^* pulls *back* forms, it does not push them forward: when $f\colon X \longrightarrow Y$, f^* carries forms from Y to X.

Before we unravel the definition of f^*, prove the following formulas:

$$f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2$$
$$f^*(\omega \wedge \theta) = (f^*\omega) \wedge (f^*\theta)$$
$$(f \circ h)^*\omega = h^* f^*\omega.$$

Now let's see explicitly what f^* does on Euclidean space. Let $U \subset \mathbf{R}^k$ and $V \subset \mathbf{R}^l$ be open subsets, and let $f\colon V \longrightarrow U$ be smooth. Use $x_1, \ldots, x_k$ for the standard coordinate functions on $\mathbf{R}^k$ and $y_1, \ldots, y_l$ on $\mathbf{R}^l$. Write f concretely as $f = (f_1, \ldots, f_k)$, each f_i being a smooth function on V. The derivative df_y at a point $y \in V$ is represented by the matrix

$$\left(\frac{\partial f_i}{\partial y_j}(y)\right),$$

and its transpose map $(df_y)^*$ is represented by the transpose matrix. Consequently,

$$f^*\, dx_i = \sum_{j=1}^{l} \frac{\partial f_i}{\partial y_j} dy_j = df_i.$$

(You should convince yourself of this formula's validity before proceeding further. Can you explain the exact meaning of each term?)

Knowing the behavior of f^* on the 0-forms and on the basic 1-forms dx_i determines it completely. For an arbitrary form ω on U may be written uniquely as

$$\omega = \sum_I a_I\, dx_I.$$

Now application of the abstract properties of f^* listed above gives

$$f^*(\omega) = \sum_I (f^*a_I)\, df_I.$$

Here the function $f^*a_I = a_I \circ f$ is the pullback of the 0-form a_I from U to V, and we use f_I to denote $f_{i_1} \wedge \cdots \wedge f_{i_p}$.

One example is crucially important. Suppose that $f\colon V \longrightarrow U$ is a diffeomorphism of two open sets in $\mathbf{R}^k$ and $\omega = dx_1 \wedge \cdots \wedge dx_k$ (the so-called *volume form* on U). If $f(y) = x$, both $T_y(V)$ and $T_x(U)$ are equal to $\mathbf{R}^k$, although in the coordinate notation we have been using, the standard basis of linear functions on $\mathbf{R}^k$ is written as $dy_1(y), \ldots, dy_k(y)$ for $T_y(V)$ but as $dx_1(x), \ldots, dx_k(x)$ for $T_x(U)$. The determinant theorem of Section 2 now gives us the formula

$$\begin{aligned} f^*\omega(y) &= (df_y)^*\, dx_1(x) \wedge \cdots \wedge (df_y)^*\, dx_k(x) \\ &= \det(df_y)\, dy_1(y) \wedge \cdots \wedge dy_k(y). \end{aligned}$$

More succinctly,

$$f^*(dx_1 \wedge \cdots \wedge dx_k) = \det(df)\, dy_1 \wedge \cdots \wedge dy_k,$$

where we denote by det (df) the function $y \longrightarrow \det(df_y)$ on V.

A form ω on an open set $U \subset \mathbf{R}^k$ is said to be *smooth* if each coefficient function a_I in its expansion $\sum_I a_I\, dx_I$ is smooth. It is clear from the preceding general calculation that when $f: V \longrightarrow U$ is a smooth map of open subsets of two Euclidean spaces, then $f^*\omega$ is smooth if ω is.

More generally, we define smoothness for a form ω on X to mean that for every local parametrization $h: U \longrightarrow X$, $h^*\omega$ is a smooth form on the open set U in R^k. Of course, one need not really check every parametrization, only enough of them to cover X; if $h_\alpha: U_\alpha \longrightarrow X$ is any collection of local parametrizations covering X (i.e., $\bigcup_\alpha h_\alpha(U_\alpha) = X$), then the form ω is smooth provided that each pullback $h_\alpha^*\omega$ is smooth. For if $h: U \longrightarrow X$ is another local parametrization, then we can write the domain U as the union of open subdomains $h^{-1}[h_\alpha(U_\alpha)]$. And on each $h^{-1}[h_\alpha(U_\alpha)]$, the identity $h^*\omega = (h_\alpha^{-1} \circ h)^* h_\alpha^* \omega$ shows that $h^*\omega$ is smooth.

Since we are only interested in smooth forms, from now on the word "form" will implicitly mean "smooth form."

We close the section with one final exercise to test your grasp of these formalities. Work it out directly from the definitions, without recourse to any of the computations we made in Euclidean space.

Exercise. Let $f: X \longrightarrow Y$ be a smooth map of manifolds, and let ϕ be a smooth function on Y. Then

$$f^*(d\phi) = d(f^*\phi).$$

§4 Integration on Manifolds

Forms were created for integration, but what makes them the appropriate integrands? Perhaps their fundamental quality is that they automatically transform correctly when coordinates are changed. Recall the following important theorem from calculus (proofs of which can be found in many texts—for example, Spivak [2], p. 67).

Change of Variables in $\mathbf{R}^k$. Assume that $f: V \longrightarrow U$ is a diffeomorphism of open sets in $\mathbf{R}^k$ and that a is an integrable function on U. Then

$$\int_U a\, dx_1 \cdots dx_k = \int_V (a \circ f)\, |\det(df)|\, dy_1 \cdots dy_k.$$

Here we follow our earlier notation by using det (df) to denote the function

$y \longrightarrow \det(df_y)$ on V. It is also easy to see that the theorem holds in the half-space H^k as well.

When we change variables by such a mapping f, functions like a are transformed into their obvious pullbacks $a \circ f$. Yet this transformation is not natural from the point of view of integration. Since f distorts volume as it forces V onto U, the integral of $a \circ f$ is not the same as the integral of a. One must compensate by including the factor $|\det(df)|$, which measures the infinitesimal alteration of volume. Forms automatically counteract this volume change. Consider, for example, the original integrand to be not the function a, with $dx_1 \cdots dx_k$ serving as a formal symbol of integration, but the k-form $\omega = a\, dx_1 \wedge \cdots \wedge dx_k$. Then define

$$\int_U \omega = \int_U a\, dx_1 \cdots dx_k.$$

As calculated in the last section, ω pulls back to the form

$$f^*(\omega) = (a \circ f)\det(df)\, dy_1 \wedge \cdots \wedge dy_k$$

If f preserves orientation, then $\det(df) > 0$, so $f^*\omega$ is exactly the integrand on the right in the Change of Variables Theorem. Every k-form ω on U is $a\, dx_1 \wedge \cdots \wedge dx_k$ for some function a, so if we call ω "integrable" when a is, the theorem attains a very natural form.

Change of Variables in $\mathbf{R}^k$. Assume that $f: V \longrightarrow U$ is an orientation-preserving diffeomorphism of open sets in $\mathbf{R}^k$ or H^k, and let ω be an integrable k-form on U. Then

$$\int_U \omega = \int_V f^*\omega.$$

If f reverses orientation, then

$$\int_U \omega = -\int_V f^*\omega.$$

If you trace back through the previous sections, you will discover that the automatic appearance of the compensating factor $\det(df)$ is a mechanical consequence of the anticommutative behavior of 1-forms: $dx_i \wedge dx_j = -dx_j \wedge dx_i$. So you see, the entire algebraic apparatus of forms exists in order to provide integrands that transform properly for integration.

This transformation property is important because it allows us to integrate forms on manifolds, where we have no recourse to standard coordinates as in Euclidean space. Let ω be a smooth k-form on X, a k-dimensional manifold with boundary. The *support* of ω is defined as the closure of the set of points where $\omega(x) \neq 0$; we assume this closure to be compact, in which case ω is said to be *compactly supported*. At first, assume also that the support of ω is contained inside a single parametrizable open subset W of X. Then if $h: U \longrightarrow W$ is an orientation-preserving diffeomorphism of W with an open subset $U \subset H^k$, $h^*\omega$ is a compactly supported, smooth k-form on U. Therefore $h^*\omega$ is integrable, and we define $\int_X \omega = \int_U h^*\omega$. What if $g: V \longrightarrow W$ is another such parametrization of W? Then $f = h^{-1} \circ g$ is an orientation-preserving diffeomorphism $V \longrightarrow U$, so

$$\int_U h^*\omega = \int_V f^*h^*\omega = \int_V g^*\omega.$$

Thanks to the transformation properties of forms, $\int_X \omega$ has an intrinsic meaning, independent of the choice of parametrization. If our propaganda has not yet made you a true believer in forms, we invite you to try defining the integral of a function.

Now, in order to define the integral of an arbitrary, compactly supported, smooth k-form ω on X, we simply use a partition of unity to break up ω into pieces with parametrizable support. The collection of parametrizable open subsets of X forms an open cover; choose a subordinate partition of unity $\{\rho_i\}$. (Recall the definition, page 52.) The local finiteness property of $\{\rho_i\}$ implies that all but finitely many of them are identically zero on the compact support of ω. Thus only finitely many of the forms $\rho_i\omega$ are nonzero, and each one has compact support inside a parametrizable open set. Define

$$\int_X \omega = \sum_i \int \rho_i\omega.$$

Showing that $\int_X \omega$ does not depend on the particular partition of unity is easy. First, however, observe that if the support of ω is actually inside some parametrizable open set, then the two definitions of $\int_X \omega$ just given agree. Since $\sum_i \rho_i(x) = 1$ at every $x \in X$,

$$\sum_i \rho_i\omega = \omega.$$

Then the linearity of pullback and of integration on Euclidean space imply

$$\int_X \omega = \sum_i \int_X \rho_i\omega,$$

as needed. Now suppose that $\{\rho'_j\}$ is another suitable partition of unity. Then from what we have just observed, for each i,

$$\int_X \rho_i\omega = \sum_j \int_X \rho'_j\rho_i\omega;$$

similarly, for each j,

$$\int_X \rho'_j\omega = \sum_i \int_X \rho_i\rho'_j\omega.$$

Then

$$\sum_i \int_X \rho_i\omega = \sum_i \sum_j \int_X \rho'_j\rho_i\omega = \sum_j \sum_i \int_X \rho_i\rho'_j\omega = \sum_j \int_X \rho'_j\omega,$$

showing that $\int_X \omega$ is the same when calculated with either partition.

It is trivial to check that $\int_X$ has the standard linearity properties:

$$\int_X (\omega_1 + \omega_2) = \int_X \omega_1 + \int_X \omega_2 \quad \text{and} \quad \int_X c\omega = c\int_X \omega \qquad \text{if } c \in \mathbf{R}.$$

We also let you verify that our generalized integration theory continues to behave properly when domains are changed.

Theorem. If $f\colon Y \longrightarrow X$ is an orientation-preserving diffeomorphism, then

$$\int_X \omega = \int_Y f^*\omega$$

for every compactly supported, smooth k-form on X ($k = \dim X = \dim Y$).

You might also convince yourself that this transformation property absolutely determines our theory. We have constructed the only linear operation on compactly supported forms that transforms naturally and that reduces to usual integration in Euclidean space.

Although we can only integrate k-forms over the k-dimensional X, we can integrate other forms over submanifolds. If Z is an oriented submanifold of X and ω is a form on X, our abstract operations give us a natural way of "restricting" ω to Z. Let $i\colon Z \hookrightarrow X$ be the inclusion map, and define the *restriction* of ω to Z to be the form $i^*\omega$. It is obvious that when ω is a 0-form, $i^*\omega$ is just the usual restriction of the function ω to Z. Now if $\dim Z = l$ and ω is an l-form whose support intersects Z in a compact set, we define the integral of ω over Z to be the integral of its restriction,

$$\int_Z \omega = \int_Z i^*\omega.$$

Let's work out some specific examples. Suppose that

$$\omega = f_1\,dx_1 + f_2\,dx_2 + f_3\,dx_3$$

is a smooth 1-form on $\mathbf{R}^3$, and let $\gamma: I \longrightarrow \mathbf{R}^3$ be a simple curve, a diffeomorphism of the unit interval $I = [0, 1]$ onto $C = \gamma(I)$ a compact one-manifold with boundary. Then

$$\int_C \omega = \int_I \gamma^* \omega.$$

If

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t)),$$

then

$$\gamma^* \, dx_i = d\gamma_i = \frac{d\gamma_i}{dt} \, dt,$$

so we obtain

$$\int_C \omega = \sum_{i=1}^{3} \int_0^1 f_i[\gamma(t)] \frac{d\gamma_i}{dt}(t) \, dt.$$

If we define $\vec{F}$ to be the vector field (f_1, f_2, f_3) in $\mathbf{R}^3$, then the right side is usually called the *line integral* of $\vec{F}$ over C and denoted $\oint \vec{F} \, d\gamma$.

Next on $\mathbf{R}^3$, consider a compactly supported 2-form

$$\omega = f_1 \, dx_2 \wedge dx_3 + f_2 \, dx_3 \wedge dx_1 + f_3 \, dx_1 \wedge dx_2.$$

We integrate ω over a surface S, which, for simplicity, we assume to be the graph of a function. $G: \mathbf{R}^2 \longrightarrow \mathbf{R}$, $x_3 = G(x_1, x_2)$. (See Figure 4-1.) (This assumption is not really restrictive because, locally, any surface may be written as the graph of a function, although one must sometimes write x_1 or x_2 as a function of the other two coordinates rather than the usual x_3.)

What is $\int_S \omega$? We can choose for S the parametrization $h: \mathbf{R}^2 \longrightarrow S$ defined by

$$h(x_1, x_2) = (x_1, x_2, G(x_1, x_2)).$$

Compute:

$$h^* \, dx_1 \wedge dx_2 = dx_1 \wedge dx_2;$$

$$h^* \, dx_2 \wedge dx_3 = dx_2 \wedge dG = dx_2 \wedge \left(\frac{\partial G}{\partial x_1} dx_1 + \frac{\partial G}{\partial x_2} dx_2\right)$$

$$= -\frac{\partial G}{\partial x_1} dx_1 \wedge dx_2;$$

and, similarly,

$$h^* \, dx_3 \wedge dx_1 = -\frac{\partial G}{\partial x_2} dx_1 \wedge dx_2.$$

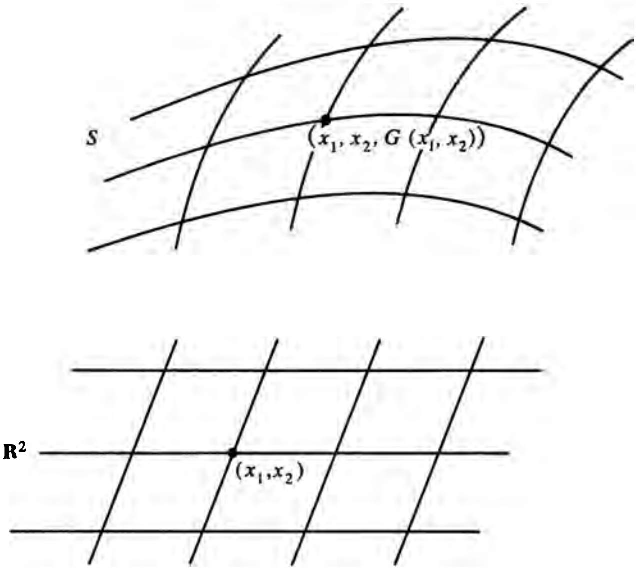


Figure 4-1

We emerge with the formula

$$\int_S \omega = \int_{\mathbf{R}^2} (n_1 f_1 + n_2 f_2 + n_3 f_3)\, dx_1\, dx_2,$$

where

$$\vec{n} = (n_1, n_2, n_3) = \left(-\frac{\partial G}{\partial x_1}, -\frac{\partial G}{\partial x_2}, 1\right).$$

Check that at any point $x = (x_1, x_2, G(x_1, x_2))$ in S, the vector $\vec{n}(x)$ is normal to the surface S; that is, $\vec{n}(x) \perp T_x(S)$. We can rewrite the integral in a form you probably learned in second-year calculus. Let $\vec{u} = \vec{n}/|\vec{n}|$ be the unit normal vector, let $\vec{F} = (f_1, f_2, f_3)$, and define a smooth 2-form $dA = |\vec{n}|\, dx_1 \wedge dx_2$. Then

$$\int_S \omega = \int_{\mathbf{R}^2} (\vec{F} \cdot \vec{u})\, dA.$$

The form dA is usually called the *area form* of the surface S, a title you may motivate for yourself with a little exercise. First show that if ΔS is a small rectangle in $\mathbf{R}^3$ and $\Delta S'$ is its projection onto the $x_1 x_2$ plane, then

$$\text{Area}\,(\Delta S) = (\sec \theta) \cdot \text{area}\,(\Delta S'),$$

where θ is the angle between the normal to ΔS and the x_3 axis. [First slide ΔS on its plane until one edge is parallel to the intersection of this plane with the x_1x_2 plane; then the formula is easy. Now show that for our surface

$$|\vec{n}| = \sqrt{1 + \left(\frac{\partial G}{\partial x_1}\right)^2 + \left(\frac{\partial G}{\partial x_2}\right)^2}$$

equals the secant of the angle θ between the normal vector to S and the x_3 axis. Finally, note that the area of $\Delta S'$ is the integral of $dx_1 \wedge dx_2$ over $\Delta S'$.] (See Figure 4-2.)

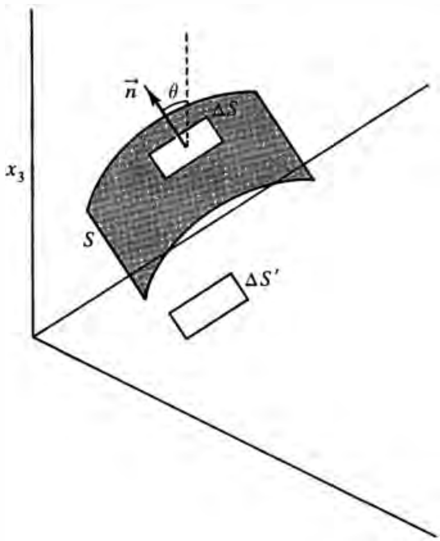


Figure 4-2

EXERCISES

1. Let Z be a finite set of points in X, considered as a 0-manifold. Fix an orientation of Z, an assignment of orientation numbers $\sigma(z) = \pm 1$ to each $z \in Z$. Let f be any function on X, considered as a 0-form, and check that

$$\int_Z f = \sum_{z \in Z} \sigma(z) \cdot f(z).$$

2. Let X be an oriented k-dimensional manifold with boundary, and ω a compactly supported k-form on X. Recall that $-X$ designates the oriented manifold obtained simply by reversing the orientation on X.

Check that

$$\int_{-X} \omega = -\int_X \omega.$$

3. Let $c: [a, b] \longrightarrow X$ be a smooth curve, and let $c(a) = p$, $c(b) = q$. Show that if ω is the differential of a function on X, $\omega = df$, then

$$\int_a^b c^*\omega = f(q) - f(p).$$

4. Let $c: [a,b] \longrightarrow X$ be a smooth curve, and let $f: [a_1, b_1] \longrightarrow [a, b]$ be a smooth map with $f(a_1) = a$ and $f(b_1) = b$. Show that the integrals

$$\int_a^b c^*\omega \quad \text{and} \quad \int_{a_1}^{b_1} (c \circ f)^*\omega$$

are the same (i.e., $\int_c \omega$ is independent of orientation-preserving reparametrization of c).

*5. A *closed curve* on a manifold X is a smooth map $\gamma: S^1 \longrightarrow X$. If ω is a 1-form on X, define the *line integral* of ω around γ by

$$\oint_\gamma \omega = \int_{S^1} \gamma^*(\omega).$$

For the case $X = \mathbf{R}^k$, write $\oint_\gamma \omega$ explicitly in terms of the coordinate expressions of γ and ω.

6. Let $h: \mathbf{R}^1 \longrightarrow S^1$ be $h(t) = (\cos t, \sin t)$. Show that if ω is any 1-form on S^1, then

$$\int_{S^1} \omega = \int_0^{2\pi} h^*\omega.$$

*7. Suppose that the 1-form ω on X is the differential of a function, $\omega = df$. Prove that $\oint_\gamma \omega = 0$ for all closed curves γ on X. [HINT: Exercises 3 and 5.]

*8. Define a 1-form ω on the punctured plane $\mathbf{R}^2 - \{0\}$ by

$$\omega(x, y) = \left(\frac{-y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy.$$

(a) Calculate $\int_C \omega$ for any circle C of radius r around the origin.

(b) Prove that in the half-plane $\{x > 0\}$, ω is the differential of a function. [HINT: Try arctan (y/x) as a random possibility.]
(c) Why isn't ω the differential of a function globally on $\mathbf{R}^2 - \{0\}$?

***9.** Prove that a 1-form ω on S^1 is the differential of a function if and only if $\int_{S^1} \omega = 0$. [HINT: "Only if" follows from Exercise 6. Now let h be as in Exercise 5, and define a function g on $\mathbf{R}$ by

$$g(t) = \int_0^t h^*\omega.$$

Show that if $\int_{S^1} \omega = 0$, then $g(t + 2\pi) = g(t)$. Therefore $g = f \circ h$ for some function f on S^1. Check $df = \omega$.]

***10.** Let ν be any 1-form on S^1 with nonzero integral. Prove that if ω is any other 1-form, then there exists a constant c such that $\omega - c\nu = df$ for some function f on S^1.

11. Suppose that ω is a 1-form on the connected manifold X, with the property that $\oint_\gamma \omega = 0$ for all closed curves γ. Then if $p, q \in X$, define $\int_p^q \omega$ to be $\int_0^1 c^*\omega$ for a curve $c\colon [0, 1] \longrightarrow X$ with $c(0) = p$, $c(1) = q$. Show that this is well defined (i.e., independent of the choice of c). [HINT: You can paste any two such curves together to form a closed curve, using a trick first to make the curves constant in neighborhoods of zero and one. For this last bit, use Exercise 4.]

***12.** Prove that any 1-form ω on X with the property $\oint_\gamma \omega = 0$ for all closed curves γ is the differential of a function, $\omega = df$. [HINT: Show that the connected case suffices. Now pick $p \in X$ and define $f(x) = \int_p^x \omega$. Check that $df = \omega$ by calculating f in a coordinate system on a neighborhood U of x. Note that you can work entirely in U by picking some $p' \in U$, for $f(x) = f(p') + \int_{p'}^x \omega$.]

13. Let S be an oriented two-manifold in $\mathbf{R}^3$, and let $\vec{n}(x) = (n_1(x), n_2(x), n_3(x))$ be the outward unit normal to S at x. (See Exercise 19 of Chapter 3, Section 2 for definition.) Define a 2-form dA on S by

$$dA = n_1\, dx_2 \wedge dx_3 + n_2\, dx_3 \wedge dx_1 + n_3\, dx_1 \wedge dx_2.$$

(Here each dx_i is restricted to S.) Show that when S is the graph of a function $F: \mathbf{R}^2 \longrightarrow \mathbf{R}$, with orientation induced from $\mathbf{R}^2$, then this dA is the same as that defined in the text.

14. Let

$$\omega = f_1\, dx_2 \wedge dx_3 + f_2\, dx_3 \wedge dx_1 + f_3\, dx_1 \wedge dx_2$$

be an arbitrary 2-form in $\mathbf{R}^3$. Check that the restriction of ω to S is the form $(\vec{F} \cdot \vec{n})\, dA$, where

$$\vec{F}(x) = (f_1(x), f_2(x), f_3(x)).$$

[HINT: Check directly that if $u, v \in T_x(S) \subset \mathbf{R}^3$, then $\omega(x)(u, v)$ equals one half the determinant of the matrix

$$\begin{pmatrix} \vec{F}(x) \\ u \\ v \end{pmatrix}.$$

If $\vec{F}(x) \in T_x(S)$, this determinant is zero, so only the normal component of $\vec{F}(x)$ contributes.]

§5 Exterior Derivative

Forms cannot only be integrated, they can also be differentiated. We have already seen how to do this in general for 0-forms, obtaining from a smooth function f the 1-form df. In Euclidean space, it is obvious how to continue. If $\omega = \sum a_I\, dx_I$ is a smooth p-form on an open subset of R^k, we simply differentiate its coefficient functions. Define the *exterior derivative* of ω to be the $(p + 1)$ form $d\omega = \sum da_I \wedge dx_I$. The following theorem lists the most important properties of this definition.

Theorem. The exterior differentiation operator d, defined on smooth forms on the open $U \subset \mathbf{R}^k$ (or H^k), possesses the following three properties:

1. Linearity:

$$d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2.$$

2. The Multiplication Law:

$$d(\omega \wedge \theta) = (d\omega) \wedge \theta + (-1)^p \omega \wedge d\theta$$

if ω is a p-form.

3. The Cocycle Condition:

$$d(d\omega) = 0.$$

Furthermore, this is the only operator that exhibits these properties and agrees with the previous definition of df for smooth functions f.

Proof. Part (a) is obvious; parts (b) and (c) are computational. We will write out (c), leaving (b) as an exercise. If $\omega = \sum_I a_I\, dx_I$, then

$$d\omega = \sum_I da_I \wedge dx_I = \sum_I \left(\sum_i \frac{\partial a_I}{\partial x_i} dx_i\right) \wedge dx_I.$$

Then

$$d\,(d\omega) = \sum_I \sum_i \left(\sum_j \frac{\partial^2 a_I}{\partial x_i \partial x_j} dx_j\right) \wedge dx_i \wedge dx_I.$$

Using the fact that

$$\frac{\partial^2 a_I}{\partial x_i \partial x_j} = \frac{\partial^2 a_I}{\partial x_j \partial x_i},$$

but

$$dx_j \wedge dx_i = -dx_i \wedge dx_j,$$

we cancel terms in the sum two by two, showing $d(d\omega) = 0$.

Uniqueness follows easily. Suppose that D were another operator satisfying (a), (b), (c), and such that $Df = df$ for functions. Then $D\,(dx_I) = 0$. For by (b),

$$D(dx_{i_1} \wedge \cdots \wedge dx_{i_p}) = \sum_j \pm\, dx_{i_1} \wedge \cdots \wedge D\, dx_{i_j} \wedge \cdots \wedge dx_{i_p},$$

but

$$D(dx_{i_j}) = D(Dx_{i_j}) = 0.$$

Now let $\omega = \sum_I a_I dx_I$ be any p-form. Then by (a) and (b),

$$D\omega = \sum_I [D(a_I) \wedge dx_I + a_I D\,(dx_I)].$$

Since $D\,(dx_I) = 0$ and $D(a_I) = da_I$, $D\omega = d\omega$. Q.E.D.

Corollary. Suppose that $g\colon V \longrightarrow U$ is a diffeomorphism of open sets of $\mathbf{R}^k$ (or H^k). Then for every form ω on U, $d(g^*\omega) = g^*(d\omega)$.

Proof. Just check that the operator $D = (g^{-1})^* \circ d \circ g^*$ satisfies (a), (b), and (c). You proved the corollary earlier for functions, so D and d agree for functions on U. Consequently, $D = d$, or $d \circ g^* = g^* \circ d$. Q.E.D.

Just as the natural transformation law

$$\int_V g^*\omega = \int_U \omega$$

allowed us to define integration on manifolds, so the relation $d \circ g^* = g^* \circ d$ permits us to differentiate forms on manifolds. Suppose that ω is a p-form on X, a manifold with boundary. We define its exterior derivative $d\omega$ locally. If $\phi: U \longrightarrow X$ is any local parametrization, define $d\omega$ on the image set $\phi(U)$ to be $(\phi^{-1})^* d(\phi^*\omega)$. If $\psi: V \longrightarrow X$ is another parametrization with overlapping image, then on the overlap we know

$$(\phi^{-1})^* d(\phi^*\omega) = (\psi^{-1})^* d(\psi^*\omega).$$

For set $g = \phi^{-1} \circ \psi$. By the corollary,

$$g^* d(\phi^*\omega) = d(g^*\phi^*\omega) = d(\psi^*\omega),$$

so

$$(\psi^{-1})^* d(\psi^*\omega) = (\psi^{-1})^* g^* d(\phi^*\omega) = (\phi^{-1})^* d(\phi^*\omega).$$

As every point of X lies inside the image of some parametrization, $d\omega$ is a well defined $(p + 1)$ form globally defined on X. We leave you to translate the properties of the exterior derivative on Euclidean space into the general setting, proving the next theorem.

Theorem. The exterior differentiation operator defined for forms on arbitrary manifolds with boundary has the following properties:

1. $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$.
2. $d(\omega \wedge \theta) = (d\omega) \wedge \theta + (-1)^p\omega \wedge d\theta$, where ω is a p-form.
3. $d\,(d\omega) = 0$.
4. If f is a function, df agrees with the earlier definition.
5. If $g: Y \longrightarrow X$ is a diffeomorphism, then $d \circ g^* = g^* \circ d$.

The two Euclidean operations, $\int$ and d, extend to manifolds for analogous reasons: both transform naturally under coordinate change. However, there is actually a great difference in the depth of the two transformation properties. The Change of Variables Theorem for integration (which we invoked in the last section but did not prove) is rather subtle, requiring a precise analysis of the way Euclidean volume is distorted by diffeomorphisms. In contrast, the fact that d commutes with pullback, $d \circ g^* = g^* \circ d$, is a simple consequence of the definition. Moreover, g need not be a diffeomorphism; the formula is valid for arbitrary maps.

Theorem. Let $g: Y \longrightarrow X$ be any smooth map of manifolds with boundary. Then for every form ω on X, $d(g^*\omega) = g^*(d\omega)$.

Proof. When ω is a 0-form, you proved the formula at the end of Section 3. It follows also when $\omega = df$ is the differential of a 0-form, for $d\omega = 0$ implies $g^*(d\omega) = 0$, and

$$d(g^*\omega) = d(g^*df) = d\,(dg^*f) = 0.$$

Furthermore, part (2) of the previous theorem shows that if this theorem holds for some ω and θ, then it is also valid for $\omega \wedge \theta$. But locally, every form on X is expressible as a wedge product of a 0-form and a number of differentials of 0-forms, since in Euclidean space every form is $\sum a_I\, dx_I$. Because the theorem is local (the two forms $d(g^*\omega)$ and $g^*(d\omega)$ are equal if they are equal in a neighborhood of every point), we are done. Q.E.D.

Before closing this section, let us calculate completely the operator d in $\mathbf{R}^3$. In effect, it is probably quite well known to you, although expressed in terms of vector fields rather than forms.

0-forms. If f is a function on $\mathbf{R}^3$, then

$$df = g_1\, dx_1 + g_2\, dx_2 + g_3\, dx_3,$$

where

$$(g_1, g_2, g_3) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right) = \overrightarrow{\text{grad}}\,(f),$$

the *gradient* vector field of f.

1-forms. Let

$$\omega = f_1\, dx_1 + f_2\, dx_2 + f_3\, dx_3.$$

Then

$$\begin{aligned} d\omega &= df_1 \wedge dx_1 + df_2 \wedge dx_2 + df_3 \wedge dx_3 \\ &= g_1\, dx_2 \wedge dx_3 + g_2\, dx_3 \wedge dx_1 + g_3\, dx_1 \wedge dx_2, \end{aligned}$$

where

$$g_1 = \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \quad g_2 = \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \quad g_3 = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}.$$

If we define $\vec{F}$ and $\vec{G}$ to be the vector fields (f_1, f_2, f_3) and (g_1, g_2, g_3), then $\vec{G} = \text{curl } \vec{F}$.

2-forms. For

$$\omega = f_1\, dx_2 \wedge dx_3 + f_2\, dx_3 \wedge dx_1 + f_3\, dx_1 \wedge dx_2,$$

$$\begin{aligned} d\omega &= \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3 \\ &= (\text{div } \vec{F})\, dx_1 \wedge dx_2 \wedge dx_3. \end{aligned}$$

3-forms. $d(\text{any 3-form}) = 0$. (Why?)

So the classical operators of vector calculus in 3-space—the gradient, curl, and divergence—are really the d operator in vector field form. Show that the cocycle condition $d^2 = 0$ on R^3 is equivalent to the two famous formulas $\text{curl}\,(\text{grad} f) = 0$ and $\text{div}\,(\text{curl } \vec{F}) = 0$.

EXERCISES

1. Calculate the exterior derivatives of the following forms in $\mathbf{R}^3$:
   (a) $z^2\, dx \wedge dy + (z^2 + 2y)\, dx \wedge dz$.
   (b) $13x\, dx + y^2\, dy + xyz\, dz$.
   (c) $f\, dg$, where f and g are functions.
   (d) $(x + 2y^3)(dz \wedge dx + \frac{1}{2} dy \wedge dx)$.

2. Show that the vector field

$$\left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right)$$

has curl zero, but that it cannot be written as the gradient of any function.

§6 Cohomology with Forms†

A p-form ω on X is *closed* if $d\omega = 0$ and *exact* if $\omega = d\theta$ for some $(p - 1)$ form θ on X. Exact forms are all closed, since $d^2 = 0$, but it may not be true that closed forms are all exact. In fact, whether or not closed forms on X are actually exact turns out to be a purely topological matter. You are probably familiar with this from calculus, although perhaps expressed

†This section is not required reading for subsequent sections, but some later exercises do refer to it.

in vector field language. All gradient vector fields have curl zero, but the converse depends on the domain of definition, as the preceding Exercise 2 shows. In the language of forms, Exercise 2 says that the 1-form

$$\omega = \left(\frac{-y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy$$

is closed but not exact.

In this section we shall make a rudimentary study of the closed versus exact distinction. The subject is quite pretty, so, as in the winding number section, we will leave most of the verifications for you. Hopefully, this will compensate for the dearth of interesting exercises in the last few sections.

In order to measure the failure of the implication closed $\Rightarrow$ exact, we define an equivalence relation on the vector space of closed p-forms on X. Two closed p-forms ω and ω' are called *cohomologous*, abbreviated $\omega \sim \omega'$, if their difference is exact: $\omega - \omega' = d\theta$. (Check that this is indeed an equivalence relation.) The set of equivalence classes is denoted $H^p(X)$, the pth *DeRham cohomology group* of X. $H^p(X)$ is more than a set; it has a natural real vector space structure. For if $\omega_1 \sim \omega_1'$ and $\omega_2 \sim \omega_2'$, then $\omega_1 + \omega_2 \sim \omega_1' + \omega_2'$; also, if $c \in \mathbf{R}$, then $c\omega_1 \sim c\omega_1'$. Thus the vector space operations on closed p-forms naturally define addition and scalar multiplication of cohomology classes. The 0 cohomology class in the vector space $H^p(X)$ is just the collection of exact forms, since $\omega + d\theta \sim \omega$ always.

Suppose that $f\colon X \longrightarrow Y$ is a smooth map, so that f^* pulls p-forms on Y back to p-forms on X. Use the relation $f^* \circ d = d \circ f^*$ to check that f^* pulls back closed forms to closed forms and exact forms to exact forms. In fact, if $\omega \sim \omega'$, then $f^*\omega \sim f^*\omega'$. Thus f^* pulls cohomology classes on Y back to cohomology classes on X; that is, f^* defines a mapping $f^\#\colon H^p(Y) \longrightarrow H^p(X)$. Since f^* is linear, you can easily check that $f^\#$ is linear. (Remember that $f^\#$ *pulls back:* i.e., when $f\colon X \longrightarrow Y$, then $f^\#\colon H^p(Y) \longrightarrow H^p(X)$.)

The numbered statements in the following discussion are for you to prove.

1. If $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $(g \circ f)^\# = f^\# \circ g^\#$.

In some simple cases, we can easily compute $H^p(X)$. For example, $H^p(X) = 0$ for all $p > \dim X$. The next easiest case is

2. The dimension of $H^0(X)$ equals the number of connected components in X. [HINT: There are no exact zero forms. Show that a zero form—that is, a function—is closed if and only if it is constant on each component of X.]

In obtaining information about other cohomology groups, we shall define an operator P on forms. Just like the operators d and $\int$, P is first defined in Euclidean space and then extended to manifolds by local parametrizations. And as with the earlier operators, the reason P may be so extended is that it transforms properly under diffeomorphisms.

Suppose that U is an open set in $\mathbf{R}^k$ and ω is a p-form on $\mathbf{R} \times U$. Then ω may be uniquely expressed as a sum

$$\omega = \sum_I f_I(t, x)\, dt \wedge dx_I + \sum_J g_J(t, x)\, dx_J. \tag{1}$$

Here t is the standard coordinate function on $\mathbf{R}$, $x_1, \ldots, x_k$ are the standard coordinate functions on $\mathbf{R}^k$, I and J are increasing index sequences, respectively of length $p - 1$ and p. The operator P transforms ω into a $p - 1$ form $P\omega$ on $\mathbf{R} \times U$, defined by

$$P\omega(t, x) = \sum_I \left[\int_0^t f_I(s, x)\, ds\right] dx_I.$$

Notice that $P\omega$ does not involve a dt term.

Now let $\phi : V \longrightarrow U$ be a diffeomorphism of open subsets of $\mathbf{R}^k$, and let $\Phi : \mathbf{R} \times V \longrightarrow \mathbf{R} \times U$ be the diffeomorphism $\Phi = \text{identity} \times \phi$. Prove the essential transformation property

3. $\Phi^* P\omega = P\Phi^*\omega$. [HINT: This is not difficult if you avoid writing everything in coordinates. Just note that $\Phi^* dt = dt$ and that Φ^* converts each of the two sums in the expression (1) for ω into sums of the same type.]

Now copy the arguments used to put d and $\int$ on manifolds to prove

4. There exists a unique operator P, defined for all manifolds X, that transforms p-forms on $\mathbf{R} \times X$ into $p - 1$ forms on $\mathbf{R} \times X$ and that satisfies the following two requirements:
   (1) If $\phi : X \longrightarrow Y$ is a diffeomorphism, and $\Phi = \text{identity} \times \phi$, then $\Phi^* \circ P = P \circ \Phi^*$.
   (2) If X is an open subset of $\mathbf{R}^k$, P is as defined above.

The main attraction of this operator is the following marvelous formula. (No doubt it appears anything but marvelous at first, but wait!)

5. Let $\pi : \mathbf{R} \times X \longrightarrow X$ be the usual projection operator and $i_a : X \longrightarrow \mathbf{R} \times X$ be any embedding $x \longrightarrow (a, x)$. Then

$$dP\omega + Pd\omega = \omega - \pi^* i_a^* \omega.$$

[HINT: Essentially this is a question of unraveling notation. For example, if ω is expressed by the sum (1), then $\pi^* i_a^* \omega = \sum_J g_J(x, a)\, dx_J$.]

The first important consequence of this formula is

6. (*Poincaré Lemma*) The maps

$$i_a^{\#}: H^p(\mathbf{R} \times X) \to H^p(X) \quad \text{and} \quad \pi^{\#}: H^p(X) \to H^p(\mathbf{R} \times X)$$

are inverses of each other. In particular, $H^p(\mathbf{R} \times X)$ is isomorphic to $H^p(X)$.
[HINT: $\pi \circ i_a =$ identity, so Exercise 1 implies $i_a^{\#} \circ \pi^{\#} =$ identity. For $\pi^{\#} \circ i_a^{\#}$, interpret Exercise 5 for closed forms ω.]

Take X to be a single point, so $H^p(X) = 0$ if $p > 0$. Now the Poincaré lemma implies by induction:

Corollary. $H^p(\mathbf{R}^k) = 0$ if $k > 0$; that is, every closed p-form on R^k is exact if $p > 0$.

A slightly more subtle consequence is

7. If $f, g: X \to Y$ are homotopic maps, then $f^{\#} = g^{\#}$. [HINT: Let $H: \mathbf{R} \times X \to Y$ be a smooth map such that $H(a, x) = f(x)$ and $H(b, x) = g(x)$. Then

$$f^{\#} = i_a^{\#} \circ H^{\#} \quad \text{and} \quad g^{\#} = i_b^{\#} \circ F^{\#}.$$

But it is clear from Exercise 6 that $i_a^{\#} = i_b^{\#}$.]

Now strengthen the corollary to Exercise 6 by proving

8. If X is contractible, then $H^p(X) = 0$ for all $p > 0$.

We conclude this section with one deeper result.

Theorem. $H^p(S^k)$ is one dimensional for $p = 0$ and $p = k$. For all other p, $H^p(S^k) = 0$ $(k > 0)$.

Here is an inductive approach to the theorem. Assume the theorem for S^{k-1}, and we shall prove it for S^k. Let U_1 be S^k minus the south pole, and let U_2 be S^k minus the north pole. By stereographic projection, both U_1 and U_2 are diffeomorphic to $\mathbf{R}^{k-1}$. Prove

9. $U_1 \cap U_2$ is diffeomorphic to $\mathbf{R} \times S^{k-1}$. [HINT: Stereographic projection shows that $U_1 \cap U_2$ is diffeomorphic to $\mathbf{R}^{k-1} - \{0\}$.]

We shall now apply a classical technique of algebraic topology, the "Mayer-Vietoris argument," to prove the following key fact:

Proposition. For $p > 1$, the vector spaces $H^p(U_1 \cup U_2)$ and $H^{p-1}(U_1 \cap U_2)$ are isomorphic.

Begin with a closed p-form ω on $U_1 \cup U_2 = S^k$. Since U_1 is contractible, Exercise 8 implies that the restriction of ω to U_1 is exact; thus $\omega = d\phi_1$ on U_1. Similarly, $\omega = d\phi_2$ on U_2. Now, consider the $p - 1$ form $\nu = \phi_1 - \phi_2$ on $U_1 \cap U_2$. Since $d\phi_1 = d\phi_2$ on $U_1 \cap U_2$, ν is closed. Thus we have a procedure for manufacturing closed $p - 1$ forms on $U_1 \cap U_2$ out of closed p-forms on $U_1 \cup U_2$.

This procedure is easily reversed. Find functions ρ_1 and ρ_2 on S^k such that ρ_1 vanishes in a neighborhood of the north pole and ρ_2 vanishes in a neighborhood of the south pole, but $\rho_1 + \rho_2 = 1$ everywhere. Now given a closed $p - 1$ form ν on $U_1 \cap U_2$, define the form ϕ_1 on U_1 to be $\rho_1\nu$. Although ν itself may blow up at the north pole, ρ_1 kills it off, so that ϕ_1 is smoothly defined on all of U_1. Similarly, set $\phi_2 = -\rho_2\nu$ on U_2. Note that $\phi_1 - \phi_2 = \nu$ on $U_1 \cap U_2$. Define a p-form ω on $U_1 \cup U_2$ by setting $\omega = d\phi_1$ on U_1 and $\omega = d\phi_2$ on U_2. Since $d\phi_1 - d\phi_2 = d\nu = 0$ on $U_1 \cap U_2$, ω is a well-defined smooth form on all of $U_1 \cup U_2$, and it is certainly closed.

10. Show that these two procedures prove the proposition. (What happens for $p = 1$?)

Now we are almost done.

$$H^p(U_1 \cup U_2) = H^p(S^k),$$

and by Exercise 9 and the Poincaré lemma,

$$H^{p-1}(U_1 \cap U_2) \simeq H^{p-1}(S^{k-1}).$$

Thus

$$H^p(S^k) \simeq H^{p-1}(S^{k-1}) \qquad \text{if } p > 1.$$

Two items are missing. We must start the induction by proving dim $H^1(S^1)$ is 1, and we must supply the missing datum $H^1(S^k) = 0$ for $k > 1$. The latter is an easy consequence of Stokes theorem, so we delay its proof until Exercise 10, Section 7. And you will discover that you have already proved $H^1(S^1)$ to be one dimensional if you simply collect together Exercises 9 and 10 of Section 4 and Exercise 2 of Section 5.

§7 Stokes Theorem

There is a remarkable relationship among the operators $\int$ and d on forms and the operation ∂ which to each manifold with boundary associates its boundary (remarkable because ∂ is a purely geometric operation and d and $\int$ are purely analytic). In one dimension, the relationship is annunciated by the fundamental theorems of calculus, and in two and three dimensions it is the subject of the classical theorems of Green, Gauss, and Stokes. In general, suppose that X is any compact-oriented k-dimensional manifold with boundary, so ∂X is a $k - 1$ dimensional manifold with the boundary orientation.

The Generalized Stokes Theorem. If ω is any smooth $(k - 1)$ form on X, then

$$\int_{\partial X} \omega = \int_X d\omega.$$

Proof. Both sides of the equation are linear in ω, so we may assume ω to have compact support contained in the image of a local parametrization $h: U \to X$, U being an open subset of R^k or H^k.

First, assume U is open in R^k; i.e., $h(U)$ does not intersect the boundary. Then

$$\int_{\partial X} \omega = 0 \quad \text{and} \quad \int_X d\omega = \int_U h^*(d\omega) = \int_U dv,$$

where $v = h^*\omega$. Since v is a $(k - 1)$ form in k-space, we may write it as

$$v = \sum_{i=1}^{k} (-1)^{i-1} f_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k.$$

Here the $\widehat{dx_i}$ means that the term dx_i is omitted from the product. Then

$$dv = \left(\sum_i \frac{\partial f_i}{\partial x_i}\right) dx_1 \wedge \cdots \wedge dx_k$$

and

$$\int_{\mathbf{R}^k} dv = \sum_i \int_{\mathbf{R}^k} \frac{\partial f_i}{\partial x_i} dx_1 \cdots dx_k. \tag{2}$$

The integral over $\mathbf{R}^k$ is computed as usual by an iterated sequence of integrals over $\mathbf{R}^1$, which may be taken in any order (Fubini's theorem). Integrate the ith term first with respect to x_i:

$$\int_{\mathbf{R}^{k-1}} \left(\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x_i} dx_i\right) dx_1 \cdots \widehat{dx_i} \cdots dx_k.$$

Of course,

$$\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x_i} dx_i$$

is the function of $x_1, \ldots, \hat{x}_i, \ldots, x_k$ that assign to any $(k-1)$ tuple $(b_1, \ldots, \hat{b}_i, \ldots, b_k)$ the number $\int_{-\infty}^{\infty} g'(t)\,dt$, where $g(t) = f_i(b_1, \ldots, t, \ldots, b_k)$. Since ν has compact support, g vanishes outside any sufficiently large interval $(-a, a)$ in $\mathbf{R}^1$. Therefore the Fundamental Theorem of Calculus gives

$$\int_{-\infty}^{\infty} g'(t)\,dt = \int_{-a}^{a} g'(t)\,dt = g(a) - g(-a) = 0 - 0 = 0.$$

Thus $\int_X d\omega = 0$, as desired.

When $U \subset H^k$, the preceding analysis works for every term of (2) except the last. Since the boundary of H^k is the set where $x_k = 0$, the last integral is

$$\int_{\mathbf{R}^{k-1}} \left(\int_0^{\infty} \frac{\partial f_k}{\partial x_k}\,dx_k \right) dx_1 \cdots dx_{k-1}.$$

Now compact support implies that f_k vanishes if x_k is outside some large interval $(0, a)$, but although $f_k(x_1, \ldots, x_{k-1}, a) = 0, f_k(x_1, \ldots, x_{k-1}, 0) \neq 0$. Thus applying the Fundamental Theorem of Calculus as above, we obtain

$$\int_X d\omega = \int_{\mathbf{R}^{k-1}} -f_k(x_1, \ldots, x_{k-1}, 0)\,dx_1 \cdots dx_{k-1}.$$

On the other hand,

$$\int_{\partial X} \omega = \int_{\partial H^k} \nu.$$

Since $x_k = 0$ on ∂H^k, $dx_k = 0$ on ∂H^k as well. Consequently, if $i < k$, the form $(-1)^{i-1} f_i\,dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx^k$ restricts to 0 on ∂H^k. So the restriction of ν to ∂H^k is $(-1)^{k-1} f(x_1, \ldots, x_{k-1}, 0)\,dx_1 \wedge \cdots \wedge dx_{k-1}$, whose integral over ∂H^k is therefore $\int_{\partial X} \omega$.

Now ∂H^k is naturally diffeomorphic to R^{k-1} under the obvious map $(x_1, \ldots, x_{k-1}) \longrightarrow (x_1, \ldots, x_{k-1}, 0)$, but this diffeomorphism does not always carry the usual orientation of $\mathbf{R}^{k-1}$ to the boundary orientation of ∂H^k. Let $e_1, \ldots, e_k$ be the standard ordered basis for $\mathbf{R}^k$, so $e_1, \ldots, e_{k-1}$ is the standard ordered basis for $\mathbf{R}^{k-1}$. Since H^k is the upper half-space, the outward unit normal to ∂H^k is $-e_k = (0, \ldots, 0, -1)$. Thus in the boundary orientation of ∂H^k, the sign of the ordered basis $\{e_1, \ldots, e_{k-1}\}$ is defined to be the sign of the ordered basis $\{-e_k, e_1, \ldots, e_{k-1}\}$ in the standard orientation of H^k. The latter sign is easily seen to be $(-1)^k$, so the usual diffeomorphism $\mathbf{R}^k \longrightarrow \partial H^k$ alters orientation by this factor $(-1)^k$.

The result is the formula

$$\int_{\partial X} \omega = \int_{\partial H^k} (-1)^{k-1} f_k(x_1, \dots, x_{k-1}, 0)\, dx_1 \cdots dx_{k-1}$$
$$= (-1)^k \int_{\mathbf{R}^{k-1}} (-1)^{k-1} f_k(x_1, \dots, x_{k-1}, 0)\, dx_1 \cdots dx_{k-1}.$$

Since $(-1)^k(-1)^{k-1} = -1$, it is exactly the formula we derived for $\int_X d\omega$.

Q.E.D.

The classical versions of Stokes theorem are included in the exercises, along with some typical applications. But, for us, the essential value of Stokes theorem is that it provides a fundamental link between analysis and topology. The final sections of the book are devoted to exploring this link.

EXERCISES

1. Show that Stokes theorem for a closed interval $[a, b]$ in $\mathbf{R}^1$ is just the Fundamental Theorem of Calculus. (See Exercise 1, Section 4.)

2. Prove the classical Green's formula in the plane: let W be a compact domain in $\mathbf{R}^2$ with smooth boundary $\partial W = \gamma$. Prove

$$\int_\gamma f\,dx + g\,dy = \int_W \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy.$$

3. Prove the Divergence Theorem: let W be a compact domain in $\mathbf{R}^3$ with smooth boundary, and let $\vec{F} = (f_1, f_2, f_3)$ be a smooth vector field on W. Then

$$\int_W (\operatorname{div} \vec{F})\, dx\,dy\,dz = \int_{\partial W} (\vec{n} \cdot \vec{F})\, dA.$$

(Here $\vec{n}$ is the outward normal to ∂W. See Exercises 13 and 14 of Section 4 for dA, and page 178 for div $\vec{F}$.)

4. Prove the classical Stokes theorem: let S be a compact oriented two-manifold in $\mathbf{R}^3$ with boundary, and let $\vec{F} = (f_1, f_2, f_3)$ be a smooth vector field in a neighborhood of S. Prove

$$\int_S (\operatorname{curl} \vec{F} \cdot \vec{n})\, dA = \int_{\partial S} f_1\, dx_1 + f_2\, dx_2 + f_3\, dx_3.$$

(Here $\vec{n}$ is the outward normal to S. For dA, see Exercises 13 and 14 of Section 4, and for curl $\vec{F}$, see page 178.)

5. The Divergence Theorem has an interesting interpretation in fluid dynamics. Let D be a compact domain in $\mathbf{R}^3$ with a smooth boundary $S = \partial D$. We assume that D is filled with an incompressible fluid whose density at x is $\rho(x)$ and whose velocity is $\vec{v}(x)$. The quantity

$$\int_S \rho(\vec{v} \cdot \vec{n})\, dA$$

measures the amount of fluid flowing out of S at any fixed time. If $x \in D$ and B_ϵ is the ball of radius ϵ about x, the "infinitesimal amount" of fluid being added to D at x at any fixed time is

$$(*)\quad \lim_{\epsilon \to 0} \int_{\partial B_\epsilon} \frac{\rho(\vec{v} \cdot \vec{n})\, dA}{\operatorname{vol}(B_\epsilon)}.$$

Show that $(*) = \operatorname{div} \rho\vec{v}$, and deduce from the Divergence Theorem that the amount of fluid flowing out of D at any fixed time equals the amount being added.

6. The Divergence Theorem is also useful in electrostatics. Let D be a compact region in $\mathbf{R}^3$ with a smooth boundary S. Assume $0 \in \operatorname{Int}(D)$. If an electric charge of magnitude q is placed at 0, the resulting force field is $q\vec{r}/r^3$, where $\vec{r}(x)$ is the vector to a point x from 0 and $r(x)$ is its magnitude. Show that the amount of charge q can be determined from the force on the boundary by proving Gauss's law:

$$\int_S \vec{F} \cdot \vec{n}\, dA = 4\pi q.$$

[HINT: Apply the Divergence Theorem to a region consisting of D minus a small ball around the origin.]

***7.** Let X be compact and boundaryless, and let ω be an exact k-form on X, where $k = \dim X$. Prove that $\int_X \omega = 0$. [HINT: Apply Stokes theorem. Remember that X is a manifold with boundary, even though ∂X is empty.]

***8.** Suppose that $X = \partial W$, W is compact, and $f: X \longrightarrow Y$ is a smooth map. Let ω be a closed k-form on Y, where $k = \dim X$. Prove that if f extends to all of W, then $\int_X f^*\omega = 0$.

***9.** Suppose that $f_0, f_1: X \longrightarrow Y$ are homotopic maps and that the compact, boundaryless manifold X has dimension k. Prove that for all closed k-

forms ω on Y,

$$\int_X f_0^*\omega = \int_X f_1^*\omega.$$

[HINT: Previous exercise.]

10. Show that if X is a simply connected manifold, then $\oint_\gamma \omega = 0$ for all closed 1-forms ω on X and all closed curves γ in X. [HINT: Previous exercise.]

11. Prove that if X is simply connected, then all closed 1-forms ω on X are exact. (See Exercise 11, Section 4.)

12. Conclude from Exercise 11 that $H^1(S^k) = 0$ if $k > 1$.

13. Suppose that Z_0 and Z_1 are compact, cobordant, p-dimensional submanifolds of X. Prove that

$$\int_{Z_1} \omega = \int_{Z_2} \omega$$

for every closed p-form ω on X.

14. (a) Suppose that ω_1 and ω_2 are cohomologous p-forms on X, and Z is a compact oriented p-dimensional submanifold. Prove that

$$\int_Z \omega_1 = \int_Z \omega_2.$$

(b) Conclude that integration over Z defines a map of the cohomology group $H^p(X)$ into $\mathbf{R}$, which we denote by

$$\int_Z : H^p(X) \to \mathbf{R}.$$

Check that $\int_Z$ is a linear functional on $H^p(X)$.

(c) Suppose that Z bounds; specifically, assume that Z is the boundary of some compact oriented $p + 1$ dimensional submanifold-with-boundary in X. Show that $\int_Z$ is zero on $H^p(X)$.

(d) Show that if the two compact oriented manifolds Z_1 and Z_2 in X are cobordant, then the two linear functionals $\int_{Z_1}$ and $\int_{Z_2}$ are equal.

§8 Integration and Mappings

Our primary application of Stokes theorem is the following theorem, which relates the analytic operation of integration to the topological behavior of mappings.

Degree Formula. Let $f: X \longrightarrow Y$ be an arbitrary smooth map of two compact, oriented manifolds of dimension k, and let ω be a k-form on Y. Then

$$\int_X f^*\omega = \deg(f) \int_Y \omega.$$

Thus the mapping f alters the integral of every form by an integer multiple, that integer being the purely topological invariant $\deg(f)$. The theorem has numerous applications, one simply being the argument principle described in the chapter introduction. Let

$$p(z) = z^m + a_1 z^{m-1} + \cdots + a_m$$

be a complex polynomial, and Ω a region in the plane whose smooth boundary contains no zeros of p. We wish to prove that the number of zeros of p inside Ω, counting multiplicities, is given by the integral

$$\frac{1}{2\pi} \int_{\partial\Omega} d \arg p(z).$$

We had better interpret the meaning of the integral. Recall that each nonzero complex number w may be written as $re^{i\theta}$, where r is the norm of w and θ is, by definition, its *argument*. But $\theta = \arg(w)$ is not really a well-defined function; since $e^{i\theta} = e^{i(\theta + 2\pi)}$, the values $\theta + 2\pi n$ all qualify to be $\arg(w)$, where n is any integer. Happily, the ambiguity can be dispelled by passing to the exterior derivative. For in a suitable neighborhood of any point, we can always choose values for $\arg(w)$ at every point so as to obtain a smooth function of w; let's call it $\arg_0(w)$. Then every other smooth function $\phi(w)$ in that neighborhood which satisfies the necessary formula $w = |w| e^{i\phi(w)}$ equals $\arg_0(w) + 2\pi n$ for some integer n. Since ϕ and $\arg_0$ differ by a constant, $d\phi = d \arg_0$. This smooth 1-form, defined on $\mathbf{C} - \{0\}$, is what we call $d \arg$. Although the notation suggests that it is the differential of a function, we see that this assumption is misleading; only locally do suitable functions exist.

In the argument principal, the integrand is the 1-form

$$z \longrightarrow d \arg p(z) = p^*(d \arg).$$

This integrand is defined and smooth on the complex plane minus the zeros of p. Since we have already proven that the zeros of p in Ω are counted by the degree of the map $f\colon \partial\Omega \longrightarrow S^1$, where

$$f(z) = \frac{p(z)}{|p(z)|} = e^{i \arg p(z)},$$

we must identify the integral of $d \arg p(z)$ with $2\pi \deg(f)$. Apply the degree formula to the restriction of the 1-form $d \arg$ to S^1. If

$$w = f(z) = e^{i \arg p(z)},$$

then $\arg(w) = \arg p(z)$. Since $\arg p(z)$ is actually a smooth function, at least locally, we get

$$d \arg p(z) = d[f^* \arg(w)] = f^* \, d \arg(w).$$

Thus

$$\int_{\partial\Omega} d \arg p(z) = \deg(f) \int_{S^1} d \arg(w).$$

Calculating the integral over S^1 is absolutely trivial. Obviously, removal of a single point, say $w = 1$, from S^1 will not change the integral. But we may parametrize $S^1 - \{1\}$ by $\theta \longrightarrow e^{i\theta}$, $\theta \in (0, 2\pi)$. Arg (w) is a smooth function on $S^1 - \{1\}$ that pulls back to the identity $\theta \longrightarrow \theta$ on $(0, 2\pi)$. Therefore

$$\int_{S^1} d \arg(w) = \int_0^{2\pi} d\theta = 2\pi,$$

and we are done.

At the heart of the degree formula lies the following theorem, which should remind you strongly of a fundamental property of degree.

Theorem. If $X = \partial W$ and $f\colon X \longrightarrow Y$ extends smoothly to all of W, then $\int_X f^*\omega = 0$ for every k-form ω on Y. (Here X, Y, W are compact and oriented and $k = \dim X = \dim Y$.)

Proof. Let $F\colon W \longrightarrow Y$ be an extension of f. Since $F = f$ on X,

$$\int_X f^*\omega = \int_{\partial W} F^*\omega = \int_W F^* \, d\omega.$$

But ω is a k-form on a k-dimensional manifold, so $d\omega = 0$. (All $k + 1$ forms on k-dimensional manifolds are automatically 0.) Q.E.D.

Corollary. If $f_0, f_1 : X \longrightarrow Y$ are homotopic maps of compact oriented k-dimensional manifolds, then for every k-form ω on Y

$$\int_X f_0^*\omega = \int_X f_1^*\omega.$$

Proof. Let $F : I \times X \longrightarrow Y$ be a homotopy. Now

$$\partial(I \times X) = X_1 - X_0,$$

so

$$0 = \int_{\partial(I\times X)} (\partial F)^*\omega = \int_{X_1} (\partial F)^*\omega - \int_{X_0} (\partial F)^*\omega$$

(0 according to the theorem). But when we identify X_0 and X_1 with X, ∂F becomes f_0 on X_0 and f_1 on X_1. Q.E.D.

A local version of the degree formula around regular values is very easily established, and its proof shows most concretely the reason why the factor deg (f) appears.

Lemma. Let y be a regular value of the map $f: X \longrightarrow Y$ between oriented k-dimensional manifolds. Then there exists a neighborhood U of y such that the degree formula

$$\int_S f^*\omega = \deg(f)\int_Y \omega$$

is valid for every k-form ω with support in U.

Proof. Because f is a local diffeomorphism at each point in the preimage $f^{-1}(y)$, y has a neighborhood U such that $f^{-1}(U)$ consists of disjoint open sets $V_1, \ldots, V_N$, and $f: V_i \longrightarrow U$ is a diffeomorphism for each $i = 1, \ldots, N$ (Exercise 7, Chapter 1, Section 4). If ω has support in U, then $f^*\omega$ has support in $f^{-1}(U)$; thus

$$\int_X f^*\omega = \sum_{i=1}^{N} \int_{V_i} f^*\omega.$$

But since $f: V_i \longrightarrow U$ is a diffeomorphism, we know that

$$\int_{V_i} f^*\omega = \sigma_i \int_U \omega,$$

the sign σ_i being ± 1, depending on whether $f: V_i \longrightarrow U$ preserves or reverses orientation. Now, by definition, deg $(f) = \sum \sigma_i$, so we are done. Q.E.D.

Finally, we prove the degree formula in general. Choose a regular value y for $f\colon X \longrightarrow Y$ and a neighborhood U of y as in the lemma. By the Isotopy Lemma of Chapter 3, Section 6, for every point $z \in Y$ we can find a diffeomorphism $h\colon Y \longrightarrow Y$ that is isotopic to the identity and that carries y to z. Thus the collection of all open sets $h(U)$, where $h : Y \longrightarrow Y$ is a diffeomorphism isotopic to the identity, covers Y. By compactness, we can find finitely many maps $h_1, \ldots, h_n$ such that $Y = h_1(U) \cup \cdots \cup h_n(U)$. Using a partition of unity, we can write any form ω as a sum of forms, each having support in one of the sets $h_i(U)$; therefore, since both sides of the degree formula

$$\int_X f^*\omega = \deg(f) \int_Y \omega$$

are linear in ω, it suffices to prove the formula for forms supported in some $h(U)$.

So assume that ω is a form supported in $h(U)$. Since $h \sim$ identity, then $h \circ f \sim f$. Thus the corollary above implies

$$\int_X f^*\omega = \int_X (h \circ f)^*\omega = \int_X f^*h^*\omega.$$

As $h^*\omega$ is supported in U, the lemma implies

$$\int_X f^*(h^*\omega) = \deg(f) \int_Y h^*\omega.$$

Finally, the diffeomorphism h is orientation preserving; for $h \sim$ identity implies $\deg(h) = +1$. Thus the change of variables property gives

$$\int_Y h^*\omega = \int_Y \omega,$$

and

$$\int_X f^*\omega = \deg(f) \int_Y \omega,$$

as claimed.

EXERCISES

1. Check that the 1-form $d \arg$ in $\mathbf{R}^2 - \{0\}$ is just the form

$$\frac{-y}{x^2 + y^2}\, dx + \frac{y}{x^2 + y^2}\, dy$$

discussed in earlier exercises. [HINT: $\theta = \arctan(y/x)$.] (This form is also often denoted $d\theta$.) In particular, you have already shown that $d\arg$ is closed but not exact.

2. Let γ be a smooth closed curve in $\mathbf{R}^2 - \{0\}$ and ω any closed 1-form on $\mathbf{R}^2 - \{0\}$. Prove that

$$\oint_\gamma \omega = W(\gamma, 0) \int_{S^1} \omega,$$

where $W(\gamma, 0)$ is the *winding number* of γ with respect to the origin. $W(\gamma, O)$ is defined just like $W_2(\gamma, 0)$, but using degree rather than degree mod 2; that is, $W(\gamma, O) = \deg(\gamma/|\gamma|)$. In particular, conclude that

$$W(\gamma, 0) = \frac{1}{2\pi} \oint_\gamma d\arg.$$

3. We can easily define *complex valued forms* on X. The forms we have used heretofore are *real forms*. Create imaginary p-forms by multiplying any real form by $i = \sqrt{-1}$. Then complex forms are sums $\omega_1 + i\omega_2$, where ω_1 and ω_2 are real. Wedge product extends in the obvious way, and d and $\int$ are defined to commute with multiplication by i:

$$d\omega = d\omega_1 + i\,d\omega_2, \qquad \int_X \omega = \int_X \omega_1 + i \int_X \omega_2.$$

Stokes theorem is valid for complex forms, for it is valid for their real and imaginary parts. We can now use our apparatus to prove a fundamental theorem in complex variable theory: the Cauchy Integral Formula.

(a) Let z be the standard complex coordinate function on $\mathbf{C} = \mathbf{R}^2$. Check that $dz = dx + i\,dy$.

(b) Let $f(z)$ be a complex valued function on an open subset U of $\mathbf{C}$. Prove that the 1-form $f(z)\,dz$ is closed if and only if $f(z) = f(x, y)$ satisfies the Cauchy-Riemann equation

$$\frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x}.$$

Express f in terms of its real and imaginary parts $f = f_1 + if_2$, and show that the Cauchy-Riemann equation is actually two real equations. If $f(z)\,dz$ is closed, the function f is called *holomorphic* in U.

(c) Show that the product of two holomorphic functions is holomorphic.

(d) Check that the complex coordinate function z is holomorphic. Conclude that every complex polynomial is holomorphic.

(e) Suppose that f is holomorphic in U and γ_1, γ_2 are two homotopic closed curves in U. Prove that

$$\oint_{\gamma_1} f(z)\,dz = \oint_{\gamma_2} f(z)\,dz.$$

[HINT: Use Exercise 9, Section 7.]

(f) If f is holomorphic in a simply connected region U, show that $\oint_{\gamma} f(z)\,dz = 0$ for every closed curve γ in U. [HINT: Part (e).]

(g) Prove that the function $f(z) = 1/z$ is holomorphic in the punctured plane $\mathbf{C} - \{0\}$. Similarly, $1/(z - a)$ is holomorphic in $\mathbf{C} - \{a\}$.

(h) Let C_r be a circle of radius r around the point $a \in \mathbf{C}$. Prove that

$$\int_{C_r} \frac{1}{z - a}\,dz = 2\pi i,$$

by direct calculation.

(i) Suppose that $f(z)$ is a holomorphic function in U and C_r is the circle of radius r around $a \in U$. Prove that

$$\int_{C_r} \frac{f(z)}{z - a}\,dz = 2\pi i \cdot f(a).$$

[HINT: By part (e), this does not depend on r. Note that $|f(z) - f(a)| < \epsilon_r$ on C_r, where $\epsilon_r \longrightarrow 0$ as $r \longrightarrow 0$. So let $r \longrightarrow 0$ and use a simple continuity argument.]

(j) Prove the *Cauchy Integral Formula*: If f is holomorphic in U and γ is a closed curve in U not passing through $a \in U$, then

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - a}\,dz = W(\gamma, a) \cdot f(a).$$

[HINT: Use part (i) and Exercise 2.]

4. Construct a k-form on S^k with nonzero integral. [HINT: Construct a compactly supported k-form in $\mathbf{R}^k$ with nonzero integral, and project stereographically.]

5. (a) Prove that a closed k-form ω on S^k is exact if and only if $\int_{S^k} \omega = 0$.
[HINT: dim $H^k(S^k) = 1$. Now use previous exercise.]

(b) Conclude that the linear map $\int_{S^k} : H^k(S^k) \longrightarrow \mathbf{R}$ is an isomorphism.

6. Prove that a compactly supported k-form ω on $\mathbf{R}^k$ is the exterior derivative of a compactly supported $k-1$ form if and only if $\int_{\mathbf{R}^k} \omega = 0$. [HINT: Use stereographic projection to carry ω to a form ω' on S^k. By Exercise 5, $\omega' = dv$. Now dv is zero in a contractible neighborhood U of the north pole N. Use this to find a $k-2$ form μ on S^k such that $\nu = d\mu$ near N. Then $\nu - d\mu$ is zero near N, so it pulls back to a compactly supported form on $\mathbf{R}^k$.]

7. Show that on any compact oriented k-dimensional manifold X, the linear map $\int_X : H^k(X) \longrightarrow \mathbf{R}$ is an isomorphism. In particular, show $\dim H^k(X) = 1$. [HINT: Let U be an open set diffeomorphic to $\mathbf{R}^k$, and let ω be a k-form compactly supported in U with $\int_X \omega = 1$. Use Exercise 6 to show that every compactly supported form in U is cohomologous to a scalar multiple of ω. Now choose open sets $U_1, \ldots, U_N$ covering X, each of which is deformable into U by a smooth isotopy. Use Exercise 7 of Section 6 and a partition of unity to show that any k-form θ on X is cohomologous to $c\omega$ for some $c \in \mathbf{R}$. Indentify c.]

8. Let $f: X \longrightarrow Y$ be a smooth map of compact oriented k-manifolds. Consider the induced map on the top cohomology groups, $f^\#: H^k(Y) \longrightarrow H^k(X)$. Integration provides canonical isomorphisms of both $H^k(Y)$ and $H^k(X)$ with $\mathbf{R}$, so under these isomorphisms the linear map $f^\#$ must correspond to multiplication by some scalar. Prove that this scalar is the degree of f. In other words, the following square commutes:

$$\begin{tikzcd} H^k(Y) \arrow[r, "f^\#"] \arrow[d, "\int_Y"'] & H^k(X) \arrow[d, "\int_X"] \\ \mathbf{R} \arrow[r, "\text{multiplication by}", "\deg(f)"'] & \mathbf{R} \end{tikzcd}$$

§9 The Gauss-Bonnet Theorem

We begin this section with a discussion of volume. Suppose that X is a compact oriented k-dimensional manifold in $\mathbf{R}^N$. For each point $x \in X$, let $v_X(x)$ be the volume element on $T_x(X)$, the alternating k-tensor that has value $1/k!$ on each positively oriented orthonormal basis for $T_x(X)$. (See Exercise 10, Section2.) It is not hard to show that the k-form v_X on X is smooth; it is called the *volume form* of X. For example, the volume form on

$\mathbf{R}^k$ is just $dx_1 \wedge \cdots \wedge dx_k$. The ungraceful $1/k!$ is necessitated by the way we normalized the definition of wedge product. (Compare with the footnote on page 156). The integral $\int_X \mathfrak{v}_X$ is defined to be the *volume* of X.

The volume form is valuable because it provides a means of integrating functions. If g is a function on X, then $g\mathfrak{v}_x$ is a k-form on X, thus we can define $\int_X g$ to be $\int_X g\mathfrak{v}_X$. (When X is $\mathbf{R}^k$, $\mathfrak{v}_X = dx_1 \wedge \cdots \wedge dx_k$, so that $\int_{\mathbf{R}^k} g$ is the usual integral.)

Of course, you realize that the volume form is geometrical, not topological; it depends strongly on the precise manner in which the manifold sits inside Euclidean space. Consequently, integration of functions is not a natural topological operation; it does not transform properly under diffeomorphism. (For most diffeomorphisms $h\colon Y \longrightarrow X$, that is, $\int_Y h^*g \neq \int_X g$.)

The reason that integration of functions transforms improperly, as we explained in Section 4, is that diffeomorphisms distort volume. We can quantitatively measure the distortion with the aid of volume forms. In fact, if $f\colon X \longrightarrow Y$ is any smooth map of two k-dimensional manifolds with boundary, the pullback $f^*\mathfrak{v}_Y$ is a k-form on X. At every point $x \in X$, $\mathfrak{v}_X(x)$ is a basis for $\Lambda^k[T_x(X)^*]$, so $(f^*\mathfrak{v}_Y)(x)$ must be a scalar multiple of $\mathfrak{v}_X(x)$. This scalar is called the *Jacobian* of f at x and is denoted $J_f(x)$. For motivation, note that the tensor $\mathfrak{v}_X(x)$ assigns to the k-tuple $(v_1, \ldots, v_k)$ plus or minus the volume of the parallelopiped it spans in the vector space $T_x(X)$, multiplied by that awkward factor $1/k!$ (Compare with Exercise 11, Section 2.). $(f^*\mathfrak{v}_Y)(x)$ assigns $\pm$ the volume of the parallelopiped spanned by $df_x(v_1), \ldots, df_x(v_k)$ in $T_{f(x)}(Y)$, multiplied by the same factor. Thus the magnitude of $J_f(x)$ is the factor by which df_x expands or contracts volume; its sign reflects whether df_x preserves or reverses orientation. In this sense, J_f measures everywhere the infinitesimal change of volume and orientation effected by f.

We apply these generalities to study the geometry of *hypersurfaces*, k-dimensional submanifolds of $\mathbf{R}^{k+1}$. Now, the hypersurface X is oriented if and only if we can smoothly choose between the two unit normal vectors to X at every point (Exercise 18, Section 2, Chapter 3). If, in particular, X is a compact hypersurface, we know from the Jordan-Brouwer Separation Theorem that X is orientable as the boundary of its "inside"; so we can just choose $\vec{n}(x)$ as the outward pointing normal.

The map $g : X \longrightarrow S^k$, defined by $g(x) = \vec{n}(x)$, is called the *Gauss map* of the oriented hypersurface X, and its Jacobian $J_g(x) = \kappa(x)$ is called the *curvature* of X at x. For example, if X is a k-sphere of radius r, then $\kappa(x) = 1/r^k$ everywhere (Exercise 6.) As the radius increases, the curvature decreases, for large spheres are flatter than small ones. Of course, when $X = \mathbf{R}^k$, then $\kappa = 0$, since g is constant.

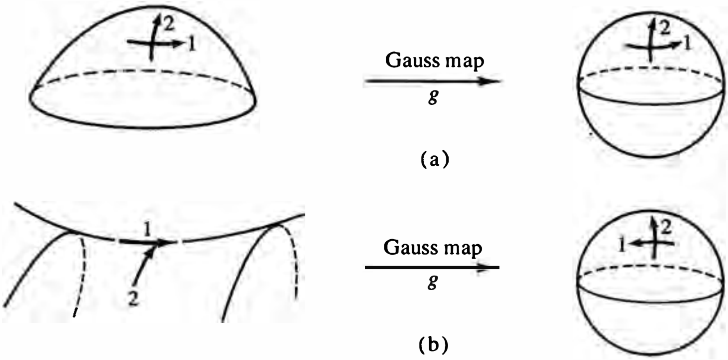


Figure 4-3

Thus the magnitude of $\kappa(x)$, in some sense, measures how curved X is at x; the more curved the space, the faster the normal vector turns. For surfaces, the sign of $\kappa(x)$, indicating whether or not the Gauss map is disorienting, serves to distinguish between local convex appearance and local saddlelike appearance: in part (a) of Figure 4-3, the Gauss map preserves orientation, while in part (b) it reverses orientation.

The curvature is a strictly geometric characteristic of spaces; it obviously is not preserved by topological transformations. But one of the loveliest theorems in mathematics implies that the global integral of the curvature on compact, even-dimensional hypersurfaces is a topological invariant. Thus no matter how we kick, twist, or stretch the space, all the local changes in curvature must exactly cancel. Moreover, since $\int_X \kappa$ is a global topological invariant of X, you will probably not be shocked to learn that it is expressible in terms of (what else?) the Euler characteristic.

The Gauss-Bonnet Theorem. If X is a compact, even-dimensional hypersurface in $\mathbf{R}^{k+1}$, then

$$\int_X \kappa = \tfrac{1}{2}\gamma_k\, \chi(X)$$

where $\chi(X)$ is the Euler characteristic of X and the constant γ_k is the volume of the unit k-sphere S^k.

Of course, when X is odd dimensional, the formula is false, since the Euler characteristic is automatically zero.

The first part of the proof is an application of the degree formula to convert the integral into a topological expression.

$$\int_X \kappa = \int_X J_g v_X = \int_X g^* v_{S^k} = \deg(g) \int_{S^k} v_{S^k} = \deg(g) \cdot \gamma_k.$$

So, in order to prove Gauss-Bonnet, we must show that the degree of the Gauss map equals one-half the Euler characteristic of X: $\deg(g) = \frac{1}{2}\chi(X)$. To do so, we shall use the Poincaré-Hopf theorem.

Choose a unit vector $a \in S^k$ such that both a and $-a$ are regular values of g. (Why can we do so?) Let $\vec{v}$ be the vector field on X whose value at a point x is the projection of the vector $-a$ onto $T_x(X)$:

$$\vec{v}(x) = (-a) - [-a \cdot \vec{n}(x)]\vec{n}(x) = [a \cdot g(x)]g(x) - a.$$

(See Figure 4-4.) A point $z \in X$ is a zero of $\vec{v}$ if and only if $g(z)$ is a multiple of a; that is, $g(z) = \pm a$. Since a and $-a$ are regular values of g, and X is

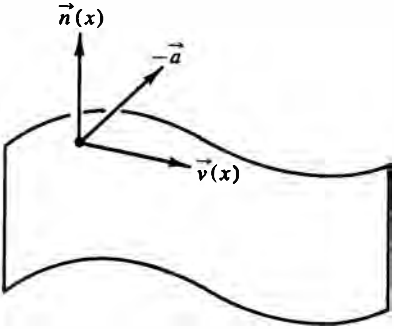


Figure 4-4

compact, $\vec{v}$ has only finitely many zeros. Let us denote the translation map $y \longrightarrow y - a$ in $\mathbf{R}^{k+1}$ by T, so we can write the mapping $\vec{v} : X \longrightarrow \mathbf{R}^{k+1}$ as $\vec{v} = T \circ [a \cdot g]g$.

Lemma. If $g(z) = a$, then $d\vec{v}_z = dT_a \circ dg_z$; and if $g(z) = -a$, then $d\vec{v}_z = -dT_a \circ dg_z$.

Proof. Calculate the derivative at z of the map $f: X \longrightarrow \mathbf{R}^{k+1}$ defined by $f(x) = [a \cdot g(x)]g(x)$: if $w \in T_z(X)$, then the vector $df_z(w) \in \mathbf{R}^{k+1}$ is

$$df_z(w) = [a \cdot g(z)]\, dg_z(w) + [a \cdot dg_z(w)]g(z) \tag{3}$$

[To check this, write w as the tangent to a curve $c(t)$, so $df_z(w)$ is the tangent to the curve $f(c(t))$. Now apply the usual product rule to each coordinate of $f(c(t))$.] Since $a \in S^k$, the product $a \cdot a$ equals 1; thus the first term in Eq. (3) is $dg_z(w)$ if $g(z) = +a$ and $-dg_z(w)$ if $g(z) = -a$. For the second term, differentiate the constant function $g(x) \cdot g(x) = 1$ to show that $g(z) \cdot dg_z(w) = 0$. (This result simply expresses the fact that the image of dg_z is tangent to

S^k at $g(z)$, thus perpendicular to the vector $g(z)$.) Consequently, if $g(z) = \pm a$, $a \cdot dg_z(w) = 0$, and the second term vanishes. Q.E.D.

Corollary. The index of the vector field $\vec{v}$ at its zero z is $+1$ if $g: X \longrightarrow S^k$ preserves orientation at z, and -1 if g reverses orientation at z.

Proof. According to Exercises 3 and 5 of Chapter 3, Section 5, the derivative $d\vec{v}_z: T_z(X) \longrightarrow \mathbf{R}^{k+1}$ actually carries $T_z(X)$ into itself; furthermore, if $d\vec{v}_z$ is an isomorphism of $T_z(X)$, then $\operatorname{ind}_z(\vec{v})$ equals the sign of the determinant of $d\vec{v}_z: T_z(X) \longrightarrow T_z(X)$. Now the linear map $dT_a: \mathbf{R}^{k+1} \longrightarrow \mathbf{R}^{k+1}$ is just the identity, so, considered as linear maps of $T_z(X)$ into $\mathbf{R}^{k+1}$,

$$d\vec{v}_z = \pm dg_z.$$

First, suppose that $g(z) = +a$, so $d\vec{v}_z = +dg_z$. Since

$$d\vec{v}_z: T_z(X) \longrightarrow T_z(X) \quad \text{and} \quad dg_z: T_z(X) \longrightarrow T_a(S^k),$$

the two subspaces $T_z(X)$ and $T_a(S^k)$ must be identical. Moreover, they have the same orientation, for the outward unit normal to X at z is $\vec{n}(z) = g(z) = a$, while the outward unit normal to S^k at a is a itself. Since a is a regular value of g, the linear map $dg_z = d\vec{v}_z$ is an isomorphism. Since $\det(d\vec{v}_z) = \det(dg_z)$, the exercise implies that $\operatorname{ind}_z(\vec{v}) = +1$ if dg_z preserves orientation and -1 if dg_z reverses orientation.

If $g(z) = -a$, then $d\vec{v}_z = -dg_z$. Again, we conclude that the two subspaces $T_z(X)$ and $T_{-a}(S^k)$ in $\mathbf{R}^{k+1}$ are identical, including their orientations. The regularity of $-a$ implies that $d\vec{v}_z$ is an isomorphism, so the exercise applies. Here

$$\det(d\vec{v}_z) = \det(-dg_z) = (-1)^k \det(dg_z),$$

which still equals $\det(dg_z)$, because k is even. So again, $\operatorname{ind}_z(\vec{v}) = +1$ if dg_z preserves orientation and -1 if dg_z reverses orientation. Q.E.D.

The Gauss-Bonnet theorem follows directly. For by the Poincaré-Hopf theorem, the sum of the indices of $\vec{v}$ equals $\chi(X)$. By the corollary, if we first add the indices at zeros where $g(z) = +a$, we get $I(g, \{a\}) = \deg(g)$; similarly, adding the indices at zeros where $g(z) = -a$ gives $I(g, \{-a\}) = \deg(g)$. Therefore $\chi(X) = 2 \deg(g)$, completing the proof. (*Question:* Why does the proof fail for odd-dimensional hypersurfaces?). Since this computation is a famous result in its own right, let us record it as a theorem.

Theorem. For even-dimensional manifolds, the Euler characteristic equals twice the degree of the Gauss map.

A final remark: One can give an alternative definition of curvature simply by using metric properties of the manifold X itself, not properties of its embedding in $\mathbf{R}^N$. For two dimensions, this fact was realized by Gauss and is called the *theorem egregium*. Chern has given an "intrinsic" proof of the Gauss-Bonnet theorem, that is, one that completely avoids embedding (see [16]). For an accessible two-dimensional version of this proof, see [17]. (A "piecewise linear" form of the intrinsic Gauss-Bonnet theorem is derived below, in Exercise 11.)

EXERCISES

1. Let $f: \mathbf{R}^2 \longrightarrow \mathbf{R}$ be a smooth function, and let $S \subset \mathbf{R}^3$ be its graph. Prove that the volume form on S is just the form dA described in Section 4.†

2. If S is an oriented surface in $\mathbf{R}^3$ and (n_1, n_2, n_3) is its unit normal vector, prove that the volume form is

$$n_1\, dx_2 \wedge dx_3 + n_2\, dx_3 \wedge dx_1 + n_3\, dx_2 \wedge dx_3.$$

In particular, show that the volume form on the unit sphere is

$$x_1\, dx_2 \wedge dx_3 + x_2\, dx_3 \wedge dx_1 + x_3\, dx_1 \wedge dx_2.$$

[HINT: Show that the 2-form just described is the 2-form defined by: $(v, w) \longrightarrow \frac{1}{2} \det \begin{pmatrix} v \\ w \\ n \end{pmatrix}$ for pairs of vectors v and w in $\mathbf{R}^3$.]

3. Let $A : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ be a rotation (i.e., an orthogonal linear mapping). Show that the map of S^{n-1} onto S^{n-1} induced by A preserves the volume form (that is, A^* applied to the volume form gives the volume form back again).

4. Let $C : [a, b] \longrightarrow \mathbf{R}^3$ be a parametrized curve in $\mathbf{R}^3$. Show that its "volume" (i.e., the integral over C of the volume form) is just the arc length:

$$\int_a^b \left| \frac{dC}{dt} \right| dt.$$

5. Let f be a smooth map of the interval $[a, b]$ into the positive real numbers. Let S be the surface obtained by rotating the graph of f around the

†Volume form is unfortunate terminology in two dimensions. "Area form" would be better.

x axis in $\mathbf{R}^3$. There is a classical formula which says that the surface area of S is equal to

$$\int_a^b 2\pi f\sqrt{1+(f')^2}\,dt.$$

Derive this formula by integrating the volume form over S. [HINT: Use Exercise 2.]

6. Prove that for the $n-1$ sphere of radius r in $\mathbf{R}^n$, the Gaussian curvature is everywhere $1/r^{n-1}$. [HINT: Show only that the derivative of the Gauss map is everywhere just $1/r$ times the identity.]

7. Compute the curvature of the hyperboloid $x^2+y^2-z^2=1$ at the point $(1,0,0)$. [HINT: The answer is -1.]

8. Let $f=f(x,y)$ be a smooth function on $\mathbf{R}^2$, and let $S\subset\mathbf{R}^3$ be its graph. Suppose that

$$f(0)=\frac{\partial f}{\partial x}(0)=\frac{\partial f}{\partial y}(0)=0.$$

Let κ_1 and κ_2 be the eigenvalues of the Hessian

$$\frac{1}{2}\begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2}, & \dfrac{\partial^2 f}{\partial x\partial y} \\ \dfrac{\partial^2 f}{\partial y\partial x}, & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix} \quad \text{at } 0.$$

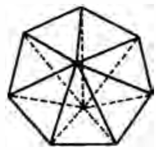
Show that the curvature of S at $(0,0,0)$ is just the product $\kappa_1\kappa_2$. [HINT: It's enough simply to consider the case $f(x,y)=\kappa_1x^2+\kappa_2y^2$. Why?]

9. A surface $S\subset\mathbf{R}^3$ is called "ruled" if through every point of S there passes a straight line contained in S. Show that a ruled surface has Gaussian curvature less than or equal to zero everywhere.

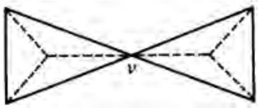
10. Let $T=T_{a,b}$ be the standard torus consisting of points in $\mathbf{R}^3$ that are a distance a from the circle of radius b in the xy plane ($a<b$). At what points is the curvature positive, at what points is it negative, and at what points is it zero?

11. Let $T_1, T_2, \ldots, T_k$ be a collection of closed triangles in $\mathbf{R}^3$. The set $S=T_1\cup\cdots\cup T_k$ is called a polyhedral surface if the following statements are true.† [See part (a) of Figure 4-5.]

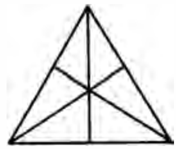
†We learned about this "intrinsic" form of the Gauss-Bonnet theorem for polyhedral surfaces from Dennis Sullivan.



(a)



(b)



(c)

Figure 4-5

(a) Each side of T_i is also the side of exactly one other triangle T_j.
(b) No two triangles have more than one side in common.
(c) If $T_{i_1}, \ldots, T_{i_k}$ are the triangles in the collection having v as a vertex, and s_{i_t} is the side opposite v in T_{i_t}, then $\cup\, s_{i_t}$ is connected. [This disqualifies example (b) in Figure 4-5.]

For each vertex v we define $\kappa(v)$ to be 2π minus the sum of the angles at the vertex. Prove:

(i) If each triangle T_i is subdivided into smaller triangles (for example, as indicated in part (c) of Figure 4-5), the total sum $\sum \kappa(v)$ over all vertices is unchanged.

(ii)
$$\sum_v \kappa(v) = 2\pi \cdot \chi(S),$$

where the Euler characteristic $\chi(S)$ is defined to be number of vertices minus the number of sides plus the number of triangles. (Compare with Chapter 3, Section 7.) [HINT: Every triangle has three sides, and every side is contained in two triangles.]

12. Let X be an oriented $n - 1$ dimensional manifold and $f: X \longrightarrow \mathbf{R}^n$ an immersion. Show that the Gauss map $g: X \longrightarrow S^{n-1}$ is still defined even though X is not, properly speaking, a submanifold of $\mathbf{R}^n$. Prove that when X is even dimensional, the degree of g is one-half the Euler characteristic of X. Show this is not the case for odd-dimensional manifolds by showing that the Gauss map of the immersion

$$S^1 \longrightarrow \mathbf{R}^2, \qquad t \longrightarrow [\cos(nt), \sin(nt)]$$

has degree n.