Math 60330: Basic Geometry and Topology
Problem Set 9

1. Recall that in class we proved the fundamental theorem of algebra using the notion of regular values. In this problem, you will fill in two details from that proof. Recall the setup: $f(z)$ is a nonconstant polynomial and $g\colon S^2 \to S^2$ is defined as follows. Let $\infty = (0,0,1) \in S^2$ and $\phi\colon S^2\backslash\{\infty\} \to \mathbb{C}$ be stereographic projection. Then g is defined via the formula

$$g(p) = \begin{cases} \infty & \text{if } p = \infty, \\ \phi^{-1}(f(\phi(p))) & \text{if } p \neq \infty. \end{cases}$$

   You should prove the following two facts.

   (a) Prove that g is a smooth function (the only issue is proving that it is smooth at ∞).

   (b) Prove that the only critical points of g are the images under ϕ of the zeros of $f'(z)$ together with possibly the point ∞.

2. Let M^n be a smooth manifold and let $p \in M^n$ be a point. Let $C^\infty(M^n)$ be the ring of smooth functions $f\colon M^n \to \mathbb{R}$. A *derivation* of $C^\infty(M^n)$ at p is an $\mathbb{R}$-linear map

$$\delta\colon C^\infty(M^n) \to \mathbb{R}$$

   such that

$$\delta(fg) = g(p)\cdot\delta(f) + f(p)\cdot\delta(g) \qquad (f, g \in C^\infty(M^n)).$$

   Let $\operatorname{Der}_p M^n$ be the vector space of derivations of $C^\infty(M^n)$ at p.

   (a) If $\vec{v} \in T_pM^n$, then prove that the directional derivative in the direction of $\vec{v}$ is a derivation at p.

   (b) If $f \in C^\infty(M^n)$ is a constant function and $\delta \in \operatorname{Der}_p M^n$, then prove that $\delta(f) = 0$.

   (c) If $f, g \in C^\infty(M^n)$ are such that there exists a neighborhood U of p with $f|_U = g|_U$ and $\delta \in \operatorname{Der}_p M^n$, then prove that $\delta(f) = \delta(g)$. Hint: The different $f - g$ vanishes on U, and thus is unchanged if you multiply it by a function that equals 1 on $M^n\backslash U$.

   (d) Define a map $\Psi\colon T_pM^n \to \operatorname{Der}_p M^n$ by letting $\Psi(\vec{v})$ be the directional derivative in the direction of $\vec{v}$ (see part a). Prove that Ψ is injective (hint: this is a computation in local coordinates!).

   (e) Prove that the map Ψ in the previous step is surjective (hint: You will use parts c-d together with the versions of Taylor's theorem that asserts that if $f\colon \mathbb{R}^n \to \mathbb{R}$ is a smooth function, then you can find $c \in \mathbb{R}$ and smooth functions $f_1, \ldots, f_n\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$f(x_1, \ldots, x_n) = c + \sum_{i=1}^{n} x_1 f_1(x_1, \ldots, x_n).$$

Remark 0.1. Because of the previous problem, the tangent space of a manifold is often defined as the set of derivations at the point.

3. Let G be a Lie group, that is, a smooth manifold that is also a group such that the product map $G \times G \to G$ and the inversion map $G \to G$ are all smooth. Let $e \in G$ be the identity. A vector field V on G is *left invariant* if it satisfies the following property for all $g \in G$. Let $\tau_g \colon G \to G$ be defined via the formula $\tau_g(h) = gh$. Then we require that $(D_h \tau_g)(V(h)) = V(gh)$ for all $h \in G$.

   (a) Let V_1 and V_2 be left invariant vector fields on G such that $V_1(e) = V_2(e)$. Prove that $V_1 = V_2$.

   (b) For $\vec{v} \in T_e G$, prove that there exists a left-invariant vector field V on G such that $V(e) = \vec{v}$.