Math 60330: Basic Geometry and Topology
Problem Set 8

1. (a) Prove that the inclusion map $S^n \hookrightarrow \mathbb{R}^{n+1}$ is an embedding (the most important thing to check is that the induced map on tangent spaces in injective).

   (b) Prove that under the embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$, the image of the tangent bundle TS^n consists of

   $$\{(x, \vec{v}) \in S^n \times \mathbb{R}^{n+1} \mid \text{the vector from 0 to } x \text{ is orthogonal to } \vec{v}\} \subset T\mathbb{R}^{n+1} = \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}.$$

   Of course, this is the tangent bundle to S^n you learned about in multivariable calculus!

2. Let $f : M_1^{n_1} \to M_2^{n_2}$ be a submersion.

   (a) Prove that if $U \subset M_1^{n_1}$ is open, then $f(U)$ is open.

   (b) If $M_1^{n_1}$ is compact and $M_2^{n_2}$ is connected, then prove that f is surjective.

3. A *standard projection* of $\mathbb{R}^m$ onto an n-dimensional subspace is a linear map $\pi : \mathbb{R}^m \to \mathbb{R}^n$ that can be written in the form $\pi(x_1, \ldots, x_m) = (x_{i_1}, \ldots, x_{i_n})$ for some $1 \leqslant i_1 < \cdots < i_n \leqslant m$. Problem: For an embedding $f : M^n \to \mathbb{R}^m$ and a point $p \in M^n$, prove that there exists a chart $\phi : U \to V$ such that $p \in U$ and $\phi = \pi \circ (f|_U)$ for some standard projection $\pi : \mathbb{R}^m \to \mathbb{R}^n$. HINT: Prove that one of the standard projections is a local diffeomorphisms using the tangent space criterion for local diffeomorphisms.

   Remark 0.1. We remark that each chart in the system of charts we gave for S^n on the first day of class is of this form.

4. A polynomial $f(x_1, \ldots, x_k)$ is *homogeneous* of degree m if

   $$f(tx_1, \ldots, tx_k) = t^m f(x_1, \ldots, x_k) \qquad (t, x_1, \ldots, x_k \in \mathbb{R}).$$

   Fix some polynomial $f(x_1, \ldots, x_k)$ which is homogeneous of degree $m \geqslant 1$.

   (a) Prove *Euler's Identity*:

   $$mf = \sum_{i=1}^{k} x_i \frac{\partial f}{\partial x_i}.$$

   (b) Prove that all nonzero numbers $a \in \mathbb{R}$ are regular values of $f(x_1, \ldots, x_k)$, and hence that $f^{-1}(a)$ is a smooth submanifold of $\mathbb{R}^n$ of dimension $(n-1)$.

   (c) Prove that if $a, b > 0$, then the manifolds $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic, and similarly if $a, b < 0$.

5. Fix some real numbers $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{n+1}$. Regarding S^n as a subspace of $\mathbb{R}^{n+1}$, define a map $f : S^n \to \mathbb{R}$ via the formula

$$f(x_1, \ldots, x_{n+1}) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_{n+1} x_{n+1}^2 \qquad \text{for } (x_1, \ldots, x_{n+1}) \in S^n \subset \mathbb{R}^{n+1}.$$

(a) Prove that the regular values of f are exactly the set $\mathbb{R} \backslash \{\lambda_1, \ldots, \lambda_{n+1}\}$.

(b) Consider $a \in \mathbb{R}$ such that $\lambda_k < a < \lambda_{k+1}$ for some $1 \leqslant k \leqslant n$. Define $X = f^{-1}(a)$. Prove that X is diffeomorphic to $S^{k-1} \times S^{n-k}$.