Math 60330: Basic Geometry and Topology
Problem Set 7

1. Prove that the two atlases for S^n on pages 2 and 3 of the notes are equivalent.

2. Recall that $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^1 + \cdots + x_{n+1}^2 = 1\}$. Define a function $f : S^n \to \mathbb{R}$ via the formula

$$f(x_1, \ldots, x_{n+1}) = \frac{x_{n+1}^2}{7 + e^{x_1}}.$$

   Prove that f is smooth directly (that is, by determining its behavior on the charts in an atlas).

3. Define two different smooth atlases on $\mathbb{R}$:

   - The atlas $\mathcal{A}$ has a single chart $\phi : U \to V$ with $U = V = \mathbb{R}$ and $\phi(x) = x$.
   - The atlas $\mathcal{A}$ has a single chart $\phi' : U' \to V'$ with $U' = V' = \mathbb{R}$ and $\phi'(x) = x^3$.

   This gives two different smooth manifolds $(\mathbb{R}, \mathcal{A})$ and $(\mathbb{R}, \mathcal{A}')$ whose underlying set is $\mathbb{R}$. Prove the following things:

   (a) The identity map $i : \mathbb{R} \to \mathbb{R}$ is a smooth homeomorphism from $(\mathbb{R}, \mathcal{A})$ to $(\mathbb{R}, \mathcal{A}')$, but is not a diffeomorphism (here recall that a diffeomorphism is a smooth homeomorphism whose inverse is also smooth).

   (b) Prove that exists some $j : \mathbb{R} \to \mathbb{R}$ which is a smooth diffeomorphism from $(\mathbb{R}, \mathcal{A})$ to $(\mathbb{R}, \mathcal{A}')$.

   Remark 0.1. In fact, one can prove that any two smooth atlases on $\mathbb{R}$ yields diffeomorphic smooth manifolds. Even more is true: for any manifold of dimension at most 3, any two smooth atlases yield diffeomorphic smooth manifolds (one says that there are no "exotic" smooth structures in these dimensions). Remarkably, this starts to fail in dimension 4; in fact, there exist uncountably many non-diffeomorphic smooth structures on $\mathbb{R}^4$.