# Math 60330: Basic Geometry and Topology
# Problem Set 6

1. Let $\Sigma_g$ be an oriented genus $g$ surface with a basepoint $p \in \Sigma_g$. Assume that $g \geqslant 2$. Prove that $\pi_1(\Sigma_g, p)$ is not abelian. Hint : find a surjective homomorphism from $\pi_1(\Sigma_g, p)$ to the dihedral group of order 8.

2. Let $X$ be a connected graph with vertex set $V(X)$ and edge set $E(X)$. Assume that both $V(X)$ and $E(X)$ are finite sets.

   (a) If $X$ is a tree, prove that $|V(X)| - |E(X)| = 1$.

   (b) For $p \in V(X)$, prove that $\pi_1(X, p)$ is a free group of rank $r$ with $|V(X)| - |E(X)| = 1 - r$.

   (c) Let $F_n$ be a free group of rank $n$ and let $G \subset F_n$ be a subgroup. As we showed in class, $G$ is a free group; let $m$ be its rank. Assume that $r = [F_n : G]$ is finite. Find a formula for $m$ in terms of $n$ and $r$.

3. Let $X \subset \mathbb{R}^n$ be a finite set of $k$ points and let $p \in \mathbb{R}^n \backslash X$.

   (a) If $n = 2$, then calculate $\pi_1(\mathbb{R}^n \backslash X, p)$.

   (b) If $n \geqslant 3$, then prove that $\pi_1(\mathbb{R}^n \backslash X, p) = 1$.

4. Let $T^2 = S^1 \times S^1$ and let $X$ be the quotient of $T^2 \sqcup T^2$ that identifies the circles $S^1 \times 1$ in both tori homeomorphically. Calculate the fundamental group of $X$.

5. Let $W = S^1 \vee S^1$ and let $p \in W$ be the wedge point. Identify $\pi_1(W, p)$ with the free group on $a$ and $b$, where $a$ goes around one $S^1$ and $b$ goes around the other one. Construct three connected 4-fold covers of $W$ that are distinct up to covering space equivalence, including at least 1 irregular cover. For each of these three covers, describe the covering map, say whether or not the cover is regular, and give a free basis (in terms of $a$ and $b$) for the corresponding subgroup of $\pi_1(W, p)$.

6. Let $\{p_1, \ldots, p_n\}$ be a set of $n$ distinct points in $S^2$ and let $X$ be the quotient space of $S^2$ that identifies all the $p_i$ to a single point. Let $q \in X$ be a basepoint. Calculate $\pi_1(X, q)$.