Math 60330: Basic Geometry and Topology
Problem Set 4

1. Let $f\colon \widetilde{X} \to X$ be a covering space. Assume that X is endowed with the structure of a CW complex. Prove that $\widetilde{X}$ can be endowed with the structure of a CW complex such that f takes the interiors of k-cells in $\widetilde{X}$ homeomorphically to the interiors of k-cells in X. Hint: first construct $\widetilde{X}^{(0)}$, then construct $\widetilde{X}^{(1)}$, then construct $\widetilde{X}^{(2)}$, etc. At each stage, you will need to use the lifting criterion to figure out how to attach cells.

2. Consider covering spaces $f : Y \to X$ with Y and X connected CW complexes, the cells of Y projecting homeomorphically onto cells of X. Restricting f to the 1-skeleton then gives a covering space $Y^{(1)} \to X^{(1)}$ over the 1-skeleton of X. Prove the following.

   (a) Two such covering spaces $Y_1 \to X$ and $Y_2 \to X$ are isomorphic iff the restrictions $(Y_1)^{(1)} \to X^{(1)}$ and $(Y_2)^{(1)} \to X^{(1)}$ are isomorphic.

   (b) $Y \to X$ is a regular covering space iff $Y^{(1)} \to X^{(1)}$ is a regular covering space.

   (c) The groups of deck transformations of the coverings $Y \to X$ and $Y^{(1)} \to X^{(1)}$ are isomorphic, via the restriction map.

3. Let X be a path-connected topological space with **abelian** fundamental group. Fix two points $p, q \in X$. Recall that $\varphi_\gamma : \pi_1(X, q) \to \pi_1(X, p)$ is the homomorphism associated to an equivalence class γ of paths from p to q. Prove that if γ and γ' are two paths from p to q, then $\varphi_\gamma = \varphi_{\gamma'}$.

4. Let X be a topological space, let $p, q \in X$ be two points, and let f and g be two paths from p to q. Prove that f is equivalent to g if and only if $f \cdot \overline{g}$ is equivalent to the constant path e_p.

5. Let X be a topological space and let $p \in X$.

   (a) Construct a bijection between maps $\gamma\colon I \to X$ such that $\gamma(0) = \gamma(1) = p$ and based maps $(S^1, 1) \to (X, p)$ (here we are regarding S^1 as a subset of $\mathbb{C}$).

   (b) Consider two based maps $f, g\colon (S^1, 1) \to (X, p)$. Let $\gamma_f, \gamma_g\colon I \to X$ be the maps associated to f and g under the bijection from part a. Prove that $[\gamma_f] = [\gamma_g]$ if and only if there exists a continuous map $F\colon S^1 \times I \to X$ such that $F(t, 0) = f(t)$ and $F(t, 1) = g(t)$ for all $t \in S^1$ and such that $F(1, s) = p$ for all $s \in [0, 1]$.

   (c) Consider a based map $f\colon (S^1, 1) \to (X, p)$. Let $\gamma_f\colon I \to X$ be the map associated to f under the bijection from part a. Prove that $[\gamma_f] = 1$ (the trivial element of the fundamental group) if and only if there exists a continuous map $G\colon \mathbb{D}^2 \to X$ such that $G|_{S^1} = f$.

(d) Assume that X is path-connected. Prove that the following conditions are all equivalent:

   i. Every map $S^1 \to X$ is homotopic to a constant map.
   ii. For every map $f : S^1 \to X$, there exists a map $g : \mathbb{D}^2 \to X$ such that $g|_{\partial \mathbb{D}^2} = f$.
   iii. $\pi_1(X, p) = 1$.

6. Let G be a topological group. Let $e \in G$ be the identity element. Prove that $\pi_1(G, e)$ is abelian. Hint : in addition to the multiplication of loops $\cdot$ in $\pi_1(G, e)$, the group structure of G gives another way of multiplying loops. Namely, for loops f and g based at e, we can define $f * g$ to be the loop $t \mapsto f(t)g(t)$. The first step is to prove that the loop $f * g$ is equivalent to the loop $g \cdot f$.

7. Let X be a topological space and let $\{U_\alpha\}$ be an open covering of X with the following properties.

   (a) There exists a point $p \in X$ such that $p \in U_\alpha$ for all α.
   (b) Each U_α is *simply-connected*, that is, U_α is path-connected and $\pi_1(U_\alpha, q) = 1$ for all $q \in U_\alpha$.
   (c) For $\alpha \neq \beta$, the set $U_\alpha \cap U_\beta$ is path-connected.

   Prove that X is simply-connected. Hint : consider $\gamma \in \pi_1(X, p)$. Prove that we can write $\gamma = \gamma_1 \cdots \gamma_k$, where $\gamma_i \in \pi_1(X, p)$ can be realized by a loop based at p that lies entirely inside one of the U_α. The notion of the *Lebesgue number* of a covering from point-set topology will be useful here.

8. Using the previous problem, prove that S^n is simply-connected for $n \geqslant 2$.