# Math 60330: Basic Geometry and Topology
# Problem Set 2

1. Let $X$ be a CW complex.
   (a) Prove that if $X$ has finitely many cells, then $X$ is compact.
   (b) Let $C \subset X$ be a compact subset (not necessarily a subcomplex). Prove that $C$ only intersects finitely many cells of $X$.

2. Construct CW complex structures on the following spaces.
   (a) An $n$-dimensional torus.
   (b) Letting $\{p_1, \ldots, p_n\}$ be $n$ distinct points on $S^2$, the quotient space of $S^2$ that identifies all the $p_i$ to a single point.

3. (a) Carefully prove that the following are covering spaces. Recall that $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$.
      i. The map $\pi \colon \mathbb{C} \to \mathbb{C}^*$ defined by $\pi(z) = e^z$.
      ii. For $n \in \mathbb{Z}\backslash\{0\}$, the map $\pi \colon \mathbb{C}^* \to \mathbb{C}^*$ defined by $\pi(z) = z^n$.
   (b) Prove that the map $\pi \colon \mathbb{C} \to \mathbb{C}$ defined by $\pi(z) = z^2$ is not a covering space.

4. Let $\pi \colon \widetilde{X} \to X$ be a covering space such that $\pi^{-1}(p)$ is finite and nonempty for all $p \in X$. Prove that $X$ is compact Hausdorff if and only if $\widetilde{X}$ is compact Hausdorff.

5. Let $X$ be a Hausdorff space and $G$ be a group acting on $X$. Assume the following two conditions hold.
   - The action is *free*, i.e. the stabilizer of every point in $X$ is trivial.
   - The action is *properly discontinuous*, i.e. for all $x \in X$, there exists a neighborhood $U$ of $x$ such that the set $\{g \in G \mid g(U) \cap U \neq \varnothing\}$ is finite.

   Prove that the action of $G$ on $X$ is a covering space action.

   *Remark* 0.1. The second condition is immediate if $G$ is finite, so this implies that all free actions of finite groups on Hausdorff spaces are covering space actions.

6. Let $\pi \colon \widetilde{X} \to X$ be a covering space and let $f \colon Y \to X$ be an arbitrary continuous map. Define

$$f^*(Y) = \{(y, \widetilde{x}) \in Y \times \widetilde{X} \mid f(y) = \pi(\widetilde{x})\} \subset Y \times \widetilde{X}$$

   and let

$$f^*(\pi) \colon f^*(Y) \to Y$$

   be the restriction of the projection $Y \times \widetilde{X} \to Y$ to the first factor. Prove that $f^*(\pi) \colon f^*(Y) \to Y$ is a covering map.

7. Set $X = \mathbb{R}^2 \backslash \{0\}$. Define an action of the additive group $\mathbb{Z}$ on $X$ via the formula

$$n \cdot (x, y) = (2^n x, 2^{-n} y) \qquad (n \in \mathbb{Z}, (x, y) \in X).$$

(a) Prove that this is a covering space action.

(b) Prove that the quotient $X/\mathbb{Z}$ is not Hausdorff.

(c) Explain how $X/\mathbb{Z}$ is the union of four subspaces homeomorphic to $S^1 \times \mathbb{R}$ coming from the complementary components of the x-axis and the y-axis.