Math 60330: Basic Geometry and Topology
Problem Set 11

1. Let $\omega \in \mathcal{A}^n(\mathbb{R}^n)$ and let M be an $n \times n$ matrix. Prove that for all $\vec{v}_1, \ldots, \vec{v} \in \mathbb{R}^n$, we have
$$\omega(M(\vec{v}_1), \ldots, M(\vec{v}_n)) = \det(M)\omega(\vec{v}_1, \ldots, \vec{v}_n).$$

2. Let V be a vector space and let $\omega_1, \ldots, \omega_k \in V^* = \mathcal{A}^1(V)$. Prove that the ω_i are linearly independent if and only if $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k = 0$.

3. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets and let $f\colon U \to V$ be a smooth map. Consider $\omega \in \Omega^k(V)$. Let $x_1, \ldots, x_n$ be the coordinates on $\mathbb{R}^n$ and let $y_1, \ldots, y_m$ be the coordinates on $\mathbb{R}^m$. Set
$$\mathcal{I} = \{(i_1, \ldots, i_k) \mid 1 \leqslant i_1 < \cdots < i_k \leqslant n\}$$
and
$$\mathcal{J} = \{(j_1, \ldots, j_k) \mid 1 \leqslant j_1 < \cdots < j_k \leqslant m\}$$
Write
$$\omega = \sum_{J \in \mathcal{J}} g_J \mathrm{dy}_J \quad \text{and} \quad f^*(\omega) = \sum_{I \in \mathcal{I}} h_I \mathrm{dx}_I.$$
State and prove a relationship between f, the g_J, and the h_I.

4. (a) Letting $x_1, \ldots, x_{n+1}$ be the coordinate functions on $\mathbb{R}^{n+1}$, give an explicit formula in terms of the dx_i's for an n-form on $\mathbb{R}^{n+1}$ that restricts to a volume form ω on S^n.

   (b) Let $\phi\colon S^n \backslash \{(0,0,1)\} \to \mathbb{R}^n$ be stereographic projection. Letting $y_1, \ldots, y_n$ be the coordinate functions on $\mathbb{R}^n$, write down the expression for ω in the local coordinates $\phi\colon S^n \backslash \{(0,0,1)\} \to \mathbb{R}^n$.

5. If M^n is a smooth manifold, then a *symplectic form* on M^n is a 2-form ω with the following two properties:

   - ω is closed, i.e. $d\omega = 0$.
   - ω is non-degenerate in the sense that for all points $p \in M^n$ and all nonzero $\vec{v} \in T_p(M^n)$, there exists some $\vec{w} \in T_p(M^n)$ such that $\omega(\vec{v}, \vec{w}) = 1$.

   Do the following problems.

   (a) Let $x_1, \ldots, x_{2n}$ be the coordinate functions on $\mathbb{R}^{2n}$. Prove that
$$\omega = \mathrm{dx}_1 \wedge \mathrm{dx}_2 + \mathrm{dx}_3 \wedge \mathrm{dx}_4 + \cdots + \mathrm{dx}_{2n-1} \wedge \mathrm{dx}_{2n}$$
   is a symplectic form on $\mathbb{R}^{2n}$.

(b) Let M^n be an arbitrary smooth manifold and let $T^*(M^n)$ be its cotangent bundle. Construct a symplectic form on $T^*(M^n)$. Hint: consider a chart $\phi\colon U \to V$ for M^n. Let $x_1, \ldots, x_n$ be the coordinate functions on V. We then get coordinate functions $x_1, \ldots, x_n, y_1, \ldots, y_n$ on $T^*V = V \times (\mathbb{R}^*)^*$ where $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ corresponds to the point

$$((x_1, \ldots, x_n), (y_1 \mathrm{dx}_1 + \cdots + y_n \mathrm{dx}_n)) \in T^*V.$$

In these coordinates, your symplectic form should be $\mathrm{dx}_1 \wedge \mathrm{dy}_1 + \cdots + \mathrm{dx}_n \wedge \mathrm{dy}_n$. What you have to prove is that these expressions in different charts "glue up" to give a well-defined global 2-form ω, and that this ω is closed and nondegenerate.