Math 60330: Basic Geometry and Topology
Problem Set 10

1. Let M^n be a smooth manifold, let $\gamma\colon [a,b] \to M^n$ be a smooth path, and let $\omega \in \Omega^1(M^n)$. Finally, let $h\colon [a,b] \to [a,b]$ be a smooth map such that $h(a) = a$ and $h(b) = b$. Define $\gamma_2 = \gamma \circ h$. Prove that $\int_\gamma \omega = \int_{\gamma_2} \omega$.

2. Let M^n be a smooth manifold and let $\omega \in \Omega^1(M^n)$ be such that $\omega = \mathrm{df}$ for some smooth function $f\colon M^n \to \mathbb{R}$.

   (a) If $\gamma\colon [a,b] \to M^n$ is a smooth path, prove that $\int_\gamma \omega = f(\gamma(a)) - f(\gamma(b))$.

   (b) If $\gamma\colon [a,b] \to M^n$ is a closed path (i.e. a path such that $\gamma(a) = \gamma(b)$), prove that $\int_\gamma \omega = 0$.

3. Define a 1-form ω on $M^2 = \mathbb{R}^2 \backslash \{0\}$ via the formula

$$\omega = \left(\frac{-y}{x^2+y^2}\right)\mathrm{dx} + \left(\frac{x}{x^2+y^2}\right)\mathrm{dy}.$$

   (a) Let $\gamma\colon [0,1] \to M^2$ be a circle of radius $r > 0$ around $(0,0)$. Calculate $\int_\gamma \omega$.

   (b) Prove that there does not exist some smooth function $f\colon M^2 \to \mathbb{R}$ such that $\omega = \mathrm{df}$.

   (c) Define $M_2^2 = \{(x,y) \mid x > 0\}$. Construct an explicit function $f\colon M_2^2 \to \mathbb{R}$ such that $\omega = \mathrm{df}$.

4. Let M^n be a smooth connected manifold and let $\omega \in \Omega^1(M^n)$. Assume that for all closed paths $\gamma\colon [a,b] \to M^n$, we have $\int_\gamma \omega = 0$. The goal of this problem is to prove the converse of Problem 2, i.e. that there exists some smooth function $f\colon M^n \to \mathbb{R}$ such that $\omega = \mathrm{df}$.

   (a) Let $\gamma_1\colon [0,1] \to M^n$ and $\gamma_2\colon [0,1] \to M^n$ be two smooth paths such that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$. Prove that $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$. Warning: you have to be careful; it is not necessarily the case that the path $\delta\colon [0,2] \to M^n$ defined via the formula

$$\delta(t) = \begin{cases} \gamma_1(t) & \text{if } 0 \leqslant t \leqslant 1, \\ \gamma_2(t-1) & \text{if } 1 \leqslant t \leqslant 2 \end{cases}$$

   is smooth. The problem occurs at $t = 1$. Try to reparameterize your paths such that this is smooth (c.f. Problem 1).

   (b) Fix some basepoint $x_0 \in M^n$. Define a function $f\colon M^n \to \mathbb{R}$ by letting $f(p) = \int_\gamma \omega$, where $\gamma\colon [0,1] \to M^n$ is a path such that $\gamma(0) = x_0$ and $\gamma(1) = p$ (this is well-defined by part a). Prove that $\mathrm{df} = \omega$.