# Math 60330: Basic Geometry and Topology
# Problem Set 1

1. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and let $f: X \to Y$ be a function. Prove that the following two definitions of $f$ being continuous are equivalent.

   - The function $f$ is continuous if for all open sets $U \subset Y$, the preimage $f^{-1}(U) \subset X$ is open.
   - The function $f$ is continuous if for all $\epsilon > 0$ and all $p \in X$, there exists some $\delta > 0$ such that if $q \in X$ satisfy $d_X(p, q) < \delta$, then $d_Y(f(p), f(q)) < \epsilon$.

2. Let $X$ be a topological space and let $\sim$ be an equivalence relation on $X$. In class, we constructed the *quotient space* of $X$ by $\sim$, which is a topological space $Y$ equipped with a projection map $\pi: X \to Y$ with the following two properties.

   - For all $p, q \in X$ such that $p \sim q$, we have $\pi(p) = \pi(q)$.
   - If $Z$ is a topological space and $f: X \to Z$ is a continuous function such that $f(p) = f(q)$ for all $p, q \in X$ satisfying $p \sim q$, then there exists a **unique** continuous function $\overline{f}: Y \to Z$ such that $f = \overline{f} \circ \pi$. In other words, the diagram

     $$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ & \searrow^{f} & \downarrow{\scriptstyle \overline{f}} \\ & & Z \end{array}$$

     commutes.

   **Problem**: Let $Y'$ be another topological space equipped with a projection $\pi': X \to Y'$ satisfying the above conditions (with $Y'$ and $\pi'$ swapped for $Y$ and $\pi$). Prove that $Y$ is homeomorphic to $Y'$. Hint: First use the above conditions to construct functions $\overline{f}: Y \to Y'$ and $\overline{f}': Y' \to Y$. Next, use the above conditions again to prove that $\overline{f} \circ \overline{f}' = \text{id}$ and $\overline{f}' \circ \overline{f} = \text{id}$. This second application of the above conditions will use the uniqueness rather than the existence part.

   *Remark* 0.1. The above condition is called a *universal mapping property*, and the argument in this problem is used throughout mathematics to show that spaces satisfying universal mapping properties are unique up to an appropriate notion of isomorphism.

3. (a) Let $\sim$ be the following equivalence relation on $\mathbb{R}^2$:
   $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 + y_1^2 = x_2 + y_2^2$.
   The quotient of $\mathbb{R}^2$ by $\sim$ is a familiar space. Which one is it?

   (b) Repeat part a for the following equivalence relation:
   $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1^2 + y_1^2 = x_2^2 + y_2^2$.

4. Let $\sim$ be the following equivalence relation on $X = [-1, 1] \times \mathbb{R} \subset \mathbb{R}^2$:

   $(x_1, y_1) \sim (x_2, y_2)$ if and only if one of the following hold:

- $x_1 = x_2 = 1$, or
- $x_1 = x_2 = -1$, or
- $-1 < x_1, x_2 < 1$ and $(x_1^2 - 1)e^{y_1} = (x_2^2 - 1)e^{y_2}$.

(a) Draw the equivalence classes for $\sim$.

(b) Prove that the quotient of $X$ by $\sim$ is not Hausdorff.

5. Let $\sim$ be the equivalence relation on $\mathbb{D}^n$ where $p \sim q$ whenever $\|p\| = \|q\| = 1$. Prove that $\mathbb{D}^n/\sim$ is homeomorphic to $S^n$. Hint: construct continuous maps $f\colon \mathbb{D}^n/\sim \to S^n$ and $g\colon S^n \to \mathbb{D}^n/\sim$ such that $f \circ g = \text{id}$ and $g \circ f = \text{id}$. The map $f$ should take a line segment through the origin connecting two boundary points of $\mathbb{D}^n$ to the great circle on $S^n$ based at the north pole (hence the two boundary points go to the north pole).

6. The Jordan curve theorem is as follows.

   **Theorem 0.2.** *Let $f : [0,1] \to \mathbb{R}^2$ be a continuous map such that $f(0) = f(1)$ and such that $f|_{(0,1)}$ is injective. Then $\mathbb{R}^2 \backslash \text{Image}(f)$ has exactly 2 connected components.*

   This is not an easy theorem to prove; however, in this problem you will prove a special case of it. Say that a function $f : [0,1] \to \mathbb{R}^2$ is *piecewise linear* if there exist $0 = a_1 < a_2 < \cdots < a_k = 1$ such that $f|_{[a_i, a_{i+1}]}$ is linear (i.e. there exist constants $c, c', d, d' \in \mathbb{R}$ such that for $x \in [a_i, a_{i+1}]$ we have $f(x) = (cx + d, c'x + d')$).

   Now let $f : [0,1] \to \mathbb{R}^2$ be a continuous piecewise linear map such that $f(0) = f(1)$ and such that $f|_{(0,1)}$ is injective. Let $0 = a_1 < a_2 < \cdots < a_k = 1$ be the subdivision coming from the piecewise linearity of $f$.

   (a) Prove that $\mathbb{R}^2 \backslash \text{Image}(f)$ has at most 2 path components. Hint: Let $x, y \in \mathbb{R}^2 \backslash \text{Image}(f)$ be two points that are close to each other and to $\text{Image}(f)$, but on opposite sides of one of the line segments forming $\text{Image}(f)$. Show that any point $p \in \mathbb{R}^2 \backslash \text{Image}(f)$ can be connected by a path to either $x$ or $y$.

   (b) Prove that $\mathbb{R}^2 \backslash \text{Image}(f)$ has at least 2 path components. Here's an outline. It is enough to find disjoint open sets $U$ and $V$ such that $\mathbb{R}^2 \backslash \text{Image}(f) = U \cup V$. Consider $p \in \mathbb{R}^2 \backslash \text{Image}(f)$. Say that a ray emanating from $p$ is *generic* if it intersects $\text{Image}(f)$ in finitely many places and does not intersect $f(a_i)$ for any $1 \leqslant i \leqslant k$. Prove first that every $p \in \mathbb{R}^2 \backslash \text{Image}(f)$ has generic rays emanating from it. Let $r$ be a generic ray emanating from $p$. Say that $p$ is even if $r$ intersects $\text{Image}(f)$ in an even number of points and odd otherwise. Prove next that this is independent of $r$. Now define $U = \{q \in \mathbb{R}^2 \backslash \text{Image}(f) \mid q \text{ is even}\}$ and $V = \{q \in \mathbb{R}^2 \backslash \text{Image}(f) \mid q \text{ is odd}\}$. Prove that $U$ and $V$ are disjoint open sets whose union is $\mathbb{R}^2 \backslash \text{Image}(f)$.