Math 60330: Basic Geometry and Topology
Final Exam

This exam is due on Friday, December 16th by 5pm. You have unlimited time. You are welcome to consult the course notes and other textbooks. However, you may not talk to anyone other than Prof. Andrew Putman about this exam before it is due; do so will be a violation of the honor code.

1. Define $\mathbb{CP}^n$ to be the set of equivalence classes of $\mathbb{C}^{n+1}\setminus\{0\}$ under the equivalence relation that identifies x and λx for all $x \in \mathbb{C}^{n+1}\setminus\{0\}$ and $\lambda \in \mathbb{C}\setminus\{0\}$. Endow $\mathbb{CP}^n$ with the quotient topology, so a set $U \subset \mathbb{CP}^n$ is open if and only if the preimage of U in $\mathbb{C}^{n+1}\setminus\{0\}$ under the natural surjection $\mathbb{C}^{n+1}\setminus\{0\} \to \mathbb{CP}^n$ is open. Problem: prove that $\mathbb{CP}^n$ is a smooth orientable manifold of dimension $2n$.

2. Let $f : M_1^{n_1} \to M_2^{n_2}$ be a smooth map between smooth manifolds. Also, let $i : X^k \hookrightarrow M_2^{n_2}$ be an embedding of a smooth manifold X^k into $M_2^{n_2}$. Assume that for all $p \in M_1^{n_1}$ and $x \in X^k$ with $f(p) = i(x)$, the span of the images of $D_p f : T_p M_1^{n_1} \to T_{f(p)} M_2^{n_2}$ and $D_x i : T_x X^k \to T_{i(x)} M_2^{n_2}$ is all of $T_{f(p)} M_2^{n_2} = T_{i(x)} M_2^{n_2}$. Prove that the set $f^{-1}(i(X^k)) \subset M_1^{n_1}$ is a smooth $(n_1 - (n_2 - k))$-dimensional submanifold of $M_1^{n_1}$. Hint: One special case of this is where X^k is a single point x and $i(x)$ is a regular value of f. Use the local immersion theorem to reduce this (locally) to this special case.

3. Let M^n be an arbitrary smooth manifold. Prove that the tangent bundle TM^n is orientable.

4. Let G be a Lie group, that is, a group G that is also a smooth manifold such that the multiplication map $M \times M \to M$ taking (x, y) to xy and the inversion map $M \to M$ taking x to x^{-1} are smooth. Letting n be the dimension of G, prove that $TG \cong G \times \mathbb{R}^n$.

5. Let M^3 be a 3-manifold and let $\alpha \in \Omega^1(M^3)$ be a 1-form such that $\alpha \wedge \mathrm{d}\alpha$ is a volume form on M^3. Prove that there does not exist an embedding $f \colon X^2 \to M^3$ with X^2 a 2-manifold such that for all $p \in X^2$ and all $\vec{v} \in T_p X^2$, we have
$$\alpha(f(p))((D_p f)(\vec{v})) = 0.$$

6. Let $f_0 \colon M^k \to X^n$ and $f_1 \colon M^k \to X^n$ be smoothly homotopic maps between smooth manifolds and let $\omega \in \Omega^k(X^n)$ be closed. Assume that M^k is oriented and compact. Prove that
$$\int_{M^k} f_0^*(\omega) = \int_{M^k} f_1^*(\omega).$$
Make sure to be careful about orientations in your proof!