Covering Spaces and the Fundamental Group

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Examples of spaces

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Basic properties and examples of covering spaces

The main topic of the first part of this book is as follows.

Definition 2.1. Let X be a topological space. A *covering space* of X (or just *cover* for short) is a space $\widetilde{X}$ equipped with a continuous map $f\colon \widetilde{X} \to X$ such that the following holds for all $x \in X$.

- There exists a neighborhood U of x such that $f^{-1}(U)$ is the disjoint union of open sets $\widetilde{U}_\alpha$ for which the restriction $f|_{\widetilde{U}_\alpha}\colon \widetilde{U}_\alpha \to U$ is a homeomorphism.

We will call U a *trivialized neighborhood* of x and the sets $\widetilde{U}_\alpha$ will be called the *sheets* lying above U. We will also call X the *base* of the cover and $\widetilde{X}$ the *total space* of the cover. □

Here are two easy examples.

Example 2.2. The identity map $X \to X$ is a obviously a covering space of X; indeed, in this example we can take the entire space X to be our trivialized neighborhood of any $x \in X$. More generally, if F is a discrete set, then the projection $X \times F \to X$ is a covering space of X. We will call $X \times F \to X$ a *trivial cover*. Every covering space looks locally like a trivial cover. □

Example 2.3. The map $f\colon \mathbb{R} \to S^1$ defined via the formula

$$f(t) = (\cos(2\pi t), \sin(2\pi t))$$

is a covering space. To see this, consider $x \in S^1$. Letting $p \in \mathbb{R}$ be any point such that $f(p) = x$, we have $f^{-1}(x) = \{p + k \mid k \in \mathbb{Z}\}$. Letting $U \subset S^1$ be a small open arc of S^1 surrounding x, the preimage $f^{-1}(U)$ consists of a disjoint union of small intervals surrounding the points $p + k$ for $k \in \mathbb{Z}$; see Figure 2.1. It follows that U is a trivialized neighborhood of x. □

The first basic property of covers is as follows.

Lemma 2.4. *Let $f\colon \widetilde{X} \to X$ be a covering space such that X is connected. Then for all $x_1, x_2 \in X$ we have $|f^{-1}(x_1)| = |f^{-1}(x_2)|$.*

Proof. Define a function $\phi\colon X \to \mathbb{Z} \cup \{\infty\}$ via the formula

$$\phi(x) = |f^{-1}(x)| \qquad (x \in X).$$

We must show that ϕ is constant. Since X is connected, it is enough to show that ϕ is locally constant. Consider $x \in X$ and let $U \subset X$ be a trivialized neighborhood of x. Let $\{\widetilde{U}_\alpha\}_{\alpha \in I}$ be the sheets over $f^{-1}(U)$. Every $x \in U$ has precisely 1 preimage in each $\widetilde{U}_\alpha$ for all $\alpha \in I$, and thus

$$\phi(x) = |I| \qquad (x \in U),$$

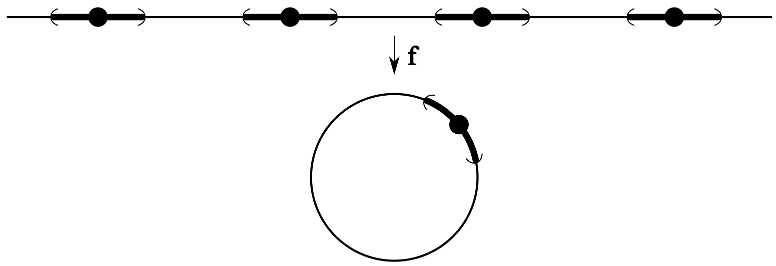

FIGURE 2.1. The cover $f\colon \mathbb{R} \to S^1$ discussed in Example 2.3.

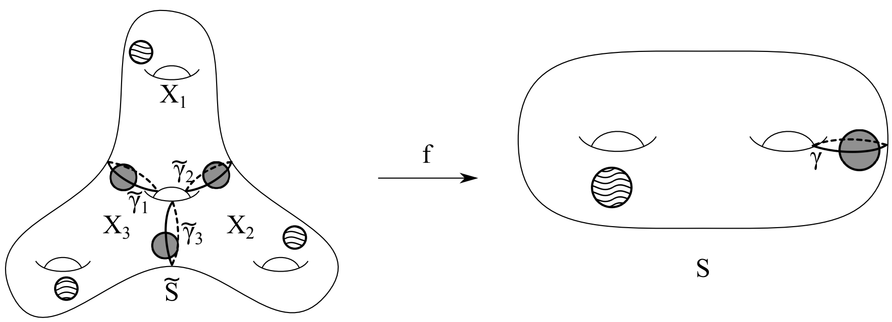

FIGURE 2.2. The map $f\colon \widetilde{\mathrm{S}} \to S$ that takes each $\widetilde{\gamma}_i$ to γ and each X_i to the complement of γ is a degree 3 covering map; see Example 2.8. The shaded circles depict two trivializing neighborhoods and the sheets above them.

as desired. □

This allows us to make the following definition.

DEFINITION 2.5. Let $f\colon \widetilde{\mathrm{X}} \to X$ be a covering space. For $x \in X$, the preimage $f^{-1}(x)$ is the *fiber* of x. If X is connected, the cardinality of $f^{-1}(x)$ is called the *degree* of the cover. Lemma 2.4 implies that this does not depend on x. □

We now give a sequence of important examples of covering spaces.

EXAMPLE 2.6. Pick $n \in \mathbb{Z}$ nonzero. Viewing S^1 as a subset of $\mathbb{C}$, the map $f\colon S^1 \to S^1$ defined via the formula $f(z) = z^n$ is a covering space of degree $|n|$. For $x \in S^1$, a small open arc U of S^1 around x is a trivializing neighborhood of x; the sheets over U consist of $|n|$ open arcs of S^1 evenly spaced around the circle. □

EXAMPLE 2.7. Recall that $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. The map $f\colon S^n \to \mathbb{RP}^n$ that takes $p \in S^n$ to the line through the origin passing through p is a covering space of degree 2. For $x \in \mathbb{RP}^n$, the fiber $f^{-1}(x)$ consists of two antipodal points p_1 and p_2 in S^n. Let $\widetilde{\mathrm{U}}_1$ and $\widetilde{\mathrm{U}}_2$ be disjoint open hemispheres such that $p_1 \in \widetilde{\mathrm{U}}_1$ and $p_2 \in \widetilde{\mathrm{U}}_2$. We then have $f(\widetilde{\mathrm{U}}_1) = f(\widetilde{\mathrm{U}}_2)$; the set $U = f(\widetilde{\mathrm{U}}_1) = f(\widetilde{\mathrm{U}}_2)$ is a trivializing neighborhood of x and the $\widetilde{\mathrm{U}}_i$ are the sheets over U. □

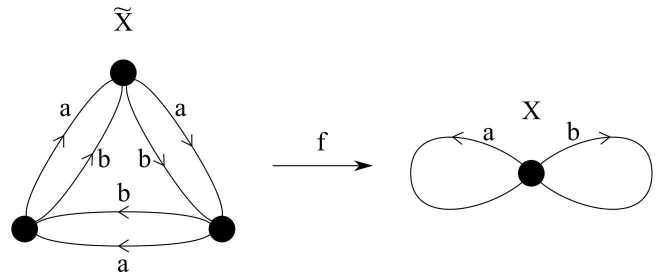

FIGURE 2.3. The map $f\colon \widetilde{\mathrm{X}} \to X$ that takes all the vertices of $\widetilde{\mathrm{X}}$ to the vertex of X and takes the interiors of the edges labeled a and b homeomorphically to the interiors of the edges labeled a and b (in the indicated direction) is a degree 3 covering map; see Example 2.9.

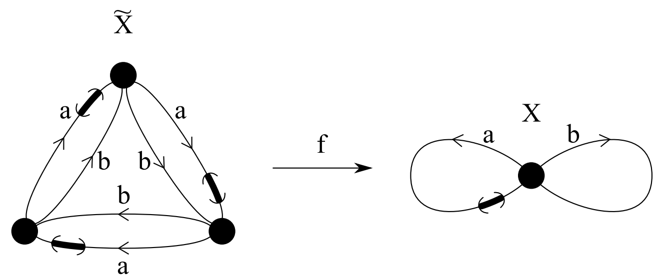

FIGURE 2.4. Finding a trivialized neighborhood of a point in the interior of an edge in Example 2.9.

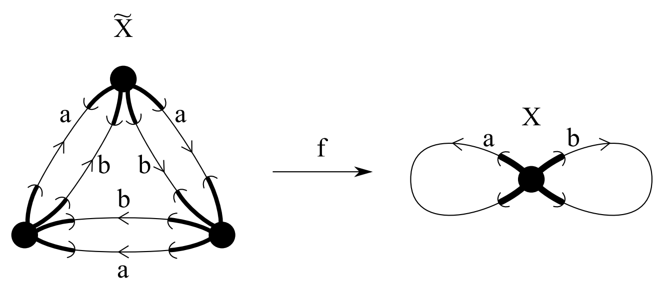

FIGURE 2.5. Finding a trivialized neighborhood of the vertex in Example 2.9.

EXAMPLE 2.8. Let S and $\widetilde{\mathrm{S}}$ be the surfaces in Figure 2.2. We can define a covering space $f\colon \widetilde{\mathrm{S}} \to S$ of degree 3 in the following way. The curves $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$, and $\widetilde{\gamma}_3$ divide $\widetilde{\mathrm{S}}$ into three subsurfaces X_1, X_2, and X_3. Each X_i is a genus 2 surface with 2 boundary components. The map f then takes the interiors of each X_i homeomorphically onto the complement of the curve γ in S and takes each $\widetilde{\gamma}_i$ homeomorphically onto γ. See Figure 2.2 for pictures of the trivializing neighborhoods in this cover. $\square$

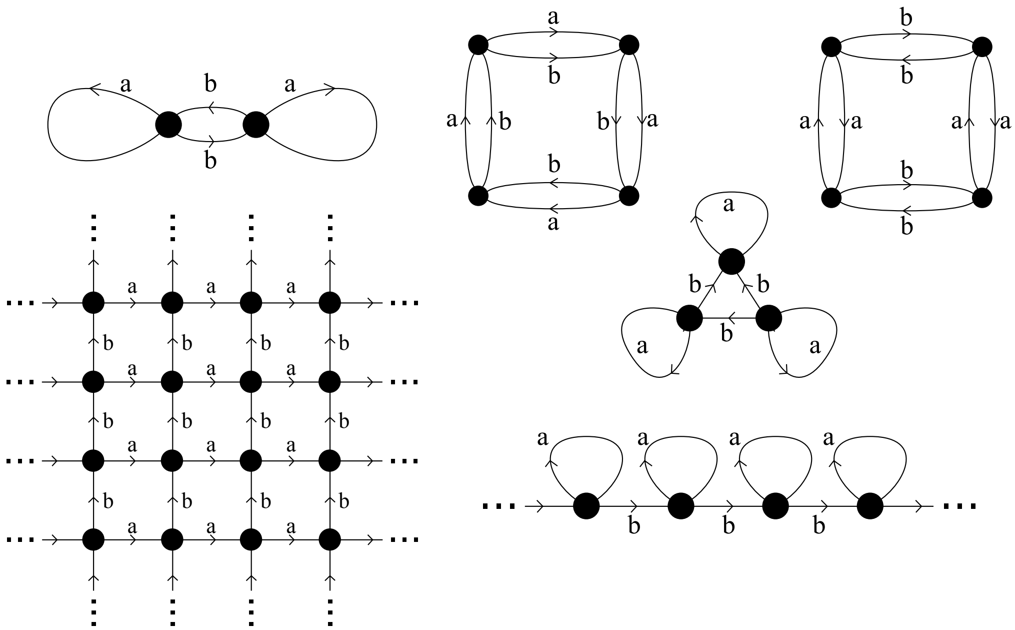

FIGURE 2.6. Each of these graphs yields a covering space of the graph X in Figure 2.3; see Example 2.10.

EXAMPLE 2.9. Let $\widetilde{X}$ and X be the graphs depicted in Figure 2.3. The arrows and labels on the edges are not part of the topological space, but will rather help us to construct a map $f\colon \widetilde{X} \to X$. The map f is defined as follows.

- The map f takes all of the vertices of $\widetilde{X}$ to the single vertex of X.
- The map f takes the interiors of the edges of $\widetilde{X}$ that are labeled a homeomorphically to the interior of the edge of X labeled a, and similarly for the edges labeled b. The arrows on the edges indicate the direction they should be traversed: as a point p moves along an edge of $\widetilde{X}$ in the indicated direction, the point $f(p)$ moves along the appropriate edge of X in the indicated direction.

This map $f\colon \widetilde{X} \to X$ is a covering map of degree 3. Indeed, consider a point $x \in X$. There are two cases.

- If x is in the interior of an edge, then for a trivializing neighborhood we can take a small open arc of that edge surrounding x; see Figure 2.4.
- If x is the vertex, then for a trivializing neighborhood we can take x together with a small open segment of each of the four half-edges coming out of x; see Figure 2.5. □

The key property of the graph $\widetilde{X}$ in Example 2.9 is that around each vertex of $\widetilde{X}$, there is precisely one a-edge coming in and one a-edge going out, and similarly for the b-edges. This is what allowed us to find a trivializing neighborhood around the vertex of X. We can generalize this example as follows.

EXAMPLE 2.10. Let X be the graph depicted in Figure 2.3 and let $\widetilde{X}$ be any graph whose edges are oriented and labeled with a's and b's. Assume that each

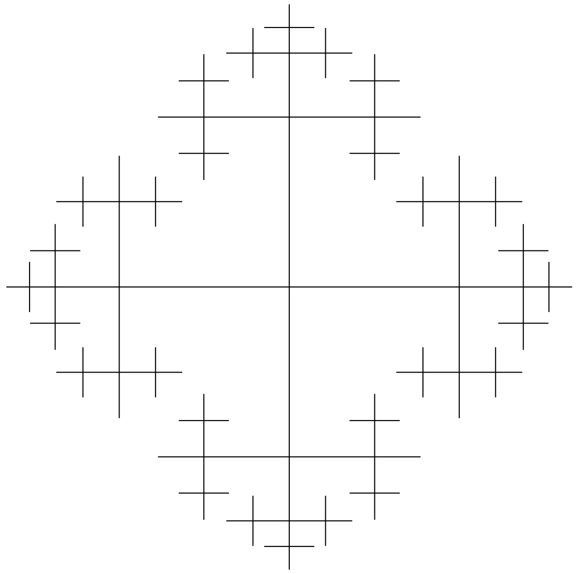

FIGURE 2.7. Part of an infinite 4-valent tree. Each crossing is a vertex, each horizontal edge is oriented from left to right and labeled a, and each vertical edge is oriented from bottom to top and labeled b.

vertex of $\widetilde{\mathrm{X}}$ has valence 4 and has precisely one a-edge (resp. b-edge) coming in and one a-edge (resp. b-edge) coming out. Just like in Example 2.9, we can construct a covering map $f\colon \widetilde{\mathrm{X}} \to X$. The degree of this cover equals the number of vertices of $\widetilde{\mathrm{X}}$. This produces a rich collection of covering spaces of X; see Figure 2.6 for a few examples. $\square$

Our final example is a very special cover of a graph that will show up several times.

EXAMPLE 2.11. Let X be the graph depicted in Figure 2.3 and let T be the oriented labeled infinite 4-valent tree depicted in Figure 2.7. As above, there is a covering map $f\colon T \to X$. You will show in the exercises that if $\widetilde{\mathrm{X}}$ is any other connected graph of the form used in Example 2.10, there is a covering map $T \to \widetilde{\mathrm{X}}$. We will later prove that these are all covers of X with connected total space. Using terminology that we will introduce later in this book, this means that T is the *universal cover* of X. $\square$

EXERCISE 2.12. Carefully prove that the following are covering spaces. Recall that $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

(1) The map $\pi\colon \mathbb{C} \to \mathbb{C}^*$ defined by $\pi(z) = e^z$.
(2) For $n \in \mathbb{Z} \setminus \{0\}$, the map $\pi\colon \mathbb{C}^* \to \mathbb{C}^*$ defined by $\pi(z) = z^n$. $\square$

EXERCISE 2.13. Prove that the map $\pi\colon \mathbb{C} \to \mathbb{C}$ defined by $\pi(z) = z^2$ is not a covering space. $\square$

EXERCISE 2.14. Let $f\colon \widetilde{\mathrm{X}} \to X$ and $g\colon \widetilde{\mathrm{Y}} \to Y$ be covering spaces. Define $h\colon \widetilde{\mathrm{X}} \times \widetilde{\mathrm{Y}} \to X \times Y$ via the formula $h(p,q) = (f(p), g(q))$. Prove that $h\colon \widetilde{\mathrm{X}} \times \widetilde{\mathrm{Y}} \to X \times Y$ is a covering space. $\square$

EXERCISE 2.15. Let $f\colon \widetilde{\mathrm{X}} \to X$ be a cover and let $X' \subset X$ be a subspace. Define $\widetilde{\mathrm{X}}' = f^{-1}(X')$ and $f' = f|_{\widetilde{\mathrm{X}}'}$. Prove that $f'\colon \widetilde{\mathrm{X}}' \to X'$ is a cover. We will call this the *restriction* of f to X'. $\square$

EXERCISE 2.16. Let T be the infinite 4-valent tree from Example 2.11 and let $\widetilde{\mathrm{X}}$ be any connected graph which is oriented and labeled as in Example 2.10. Also, let X be the graph from Figure 2.3, so we have covering maps $f\colon T \to X$ and $g\colon \widetilde{\mathrm{X}} \to X$. Let v be any vertex of T and let w be any vertex of $\widetilde{\mathrm{X}}$. Prove that there exists a covering map $h\colon T \to \widetilde{\mathrm{X}}$ such that $f = g \circ h$. $\square$

EXERCISE 2.17. Let $\pi\colon \widetilde{\mathrm{X}} \to X$ be a covering space such that $\pi^{-1}(p)$ is finite and nonempty for all $p \in X$. Prove that X is compact Hausdorff if and only if $\widetilde{\mathrm{X}}$ is compact Hausdorff. $\square$

CHAPTER 3

Lifts, morphisms, and deck groups

To unlock the structure of the set of all covers of a space X, we must study the following objects.

DEFINITION 3.1. Let $f\colon \widetilde{X} \to X$ be a covering space and let $g\colon Y \to X$ be a continuous map. A *lift* of g to $\widetilde{X}$ is a continuous map $\widetilde{g}\colon Y \to \widetilde{X}$ such that $g = f \circ \widetilde{g}$, i.e. such that the diagram

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "f"] \\ Y \arrow[ur, "\widetilde{g}"] \arrow[r, "g"] & X \end{tikzcd}$$

commutes. □

Lifts are not unique, but the following lemma says that if two agree at a single point, then they must be equal.

LEMMA 3.2. *Let $f\colon \widetilde{X} \to X$ be a covering space, let $g\colon Y \to X$ be a continuous map, and let $\widetilde{g}_1, \widetilde{g}_2\colon Y \to \widetilde{X}$ be two lifts of g to $\widetilde{X}$. Assume that Y is connected and that $\widetilde{g}_1(y_0) = \widetilde{g}_2(y_0)$ for some $y_0 \in Y$. Then $\widetilde{g}_1(y) = \widetilde{g}_2(y)$ for all $y \in Y$.*

PROOF. Set $\Lambda = \{y \in Y \mid \widetilde{g}_1(y) = \widetilde{g}_2(y)\}$. Our goal is to show that $\Lambda = Y$. By assumption $y_0 \in \Lambda$, so Λ is nonempty. Since Y is connected, it is therefore enough to prove that Λ is both open and closed. The fact that Λ is closed is immediate from the fact that $\widetilde{g}_1$ and $\widetilde{g}_2$ are continuous, so it remains to prove that Λ is open. Consider a point $y_1 \in \Lambda$. Let $U \subset X$ be a trivializing neighborhood of $g(y_1)$ and let $\widetilde{U}$ be the subset of $f^{-1}(Y)$ such that $f|_{\widetilde{U}}$ is a homeomorphism that contains $\widetilde{g}_1(y_1) = \widetilde{g}_2(y_1)$. Define

$$V = \widetilde{g}_1^{-1}(\widetilde{U}) \cap \widetilde{g}_2^{-1}(\widetilde{U}),$$

so V is an open neighborhood of y_1 in Y. By construction, the map $f|_{\widetilde{U}}\colon \widetilde{U} \to U$ is a homeomorphism and

$$g|_V = (f|_{\widetilde{U}}) \circ (\widetilde{g}_1|_V) = (f|_{\widetilde{U}}) \circ (\widetilde{g}_2|_V).$$

We deduce that

$$\widetilde{g}_1|_V = \widetilde{g}_2|_V = (f|_{\widetilde{U}})^{-1} \circ (g|_V),$$

and thus that $V \subset \Lambda$. This implies that Λ is open, as desired. □

We now introduce the appropriate notion of morphisms between covers, and in particular give a precise definition of what it means for two covers to be the same.

Definition 3.3. Let $f_1\colon \widetilde{X}_1 \to X$ and $f_2\colon \widetilde{X}_2 \to X$ be covering spaces. A *covering space map* from $f_1\colon \widetilde{X}_1 \to X$ to $f_2\colon \widetilde{X}_2 \to X$ is a continuous map $\phi\colon \widetilde{X}_1 \to \widetilde{X}_2$ such that $f_1 = f_2 \circ \phi$, i.e. such that the diagram

$$\begin{tikzcd} \widetilde{X}_1 \arrow[rr, "\phi"] \arrow[dr, "f_1"'] & & \widetilde{X}_2 \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

commutes. If ϕ is a homeomorphism, then we say that ϕ is a *covering space isomorphism* and that $\widetilde{X}_1$ and $\widetilde{X}_2$ are *isomorphic* covers of X. $\square$

Just like for many other objects in mathematics, it is enlightening to study the self-isomorphisms of a cover.

Definition 3.4. Let $f\colon \widetilde{X} \to X$ be a covering space. An *automorphism* or *deck transformation* of $f\colon \widetilde{X} \to X$ is a covering space isomorphism from $f\colon \widetilde{X} \to X$ to itself. The *deck group* of $f\colon \widetilde{X} \to X$, denoted $\mathrm{Deck}(f\colon \widetilde{X} \to X)$ (or $\mathrm{Deck}(\widetilde{X})$ if the map f is clear), is the set of all deck transformations of $f\colon \widetilde{X} \to X$. The set $\mathrm{Deck}(\widetilde{X})$ forms a group under composition. $\square$

The following lemma will allow us to identify the deck groups of many covers.

Lemma 3.5. *Let $f\colon \widetilde{X} \to X$ be a covering space and let $x_0 \in X$. The following then hold.*

(1) The group $\mathrm{Deck}(\widetilde{X})$ acts on the set $f^{-1}(x_0)$.
(2) Assume that $\widetilde{X}$ is connected and consider $\phi_1, \phi_2 \in \mathrm{Deck}(\widetilde{X})$. If $\phi_1(p) = \phi_2(p)$ for some $p \in f^{-1}(x_0)$, then $\phi_1 = \phi_2$.

Proof. To see that $\mathrm{Deck}(\widetilde{X})$ acts on $f^{-1}(x_0)$, observe that if $\phi \in \mathrm{Deck}(\widetilde{X})$ and $p \in f^{-1}(x_0)$, then the condition $f = f \circ \phi$ implies that

$$x_0 = f(p) = f(\phi(p)),$$

and thus $\phi(p) \in f^{-1}(x_0)$. The second conclusion follows from Lemma 3.2 using the fact that both ϕ_1 and ϕ_2 are lifts of $f\colon \widetilde{X} \to X$ to $\widetilde{X}$. $\square$

We now give a sequence of examples of deck groups.

Example 3.6. Consider the covering space $f\colon \mathbb{R} \to S^1$ defined by the formula $f(t) = (\cos(2\pi t), \sin(2\pi t))$. The deck group of this cover is $\mathbb{Z}$, which acts on $\mathbb{R}$ by integer translations. To see this, observe that it is clear that $\mathbb{Z} \subset \mathrm{Deck}(\mathbb{R})$. Moreover, if $\phi \in \mathrm{Deck}(\mathbb{R})$ satisfies $\phi(0) = n \in \mathbb{Z}$, then by Lemma 3.5 the element ϕ must equal translation by n, and thus $\mathbb{Z} = \mathrm{Deck}(\mathbb{R})$. $\square$

Example 3.7. Consider the covering space $f\colon S^n \to \mathbb{RP}^n$. By a reasoning similar to that in Example 3.6, the deck group of this cover is $\mathbb{Z}/2$, which acts on S^n as multiplication by -1. $\square$

Example 3.8. Let $f_1\colon \widetilde{X}_1 \to X$ be the cover depicted in Figure 3.1. By a reasoning similar to that in Example 3.6, the deck group of this cover is $\mathbb{Z}$, which acts on $\widetilde{X}_1$ by horizontal translations. $\square$

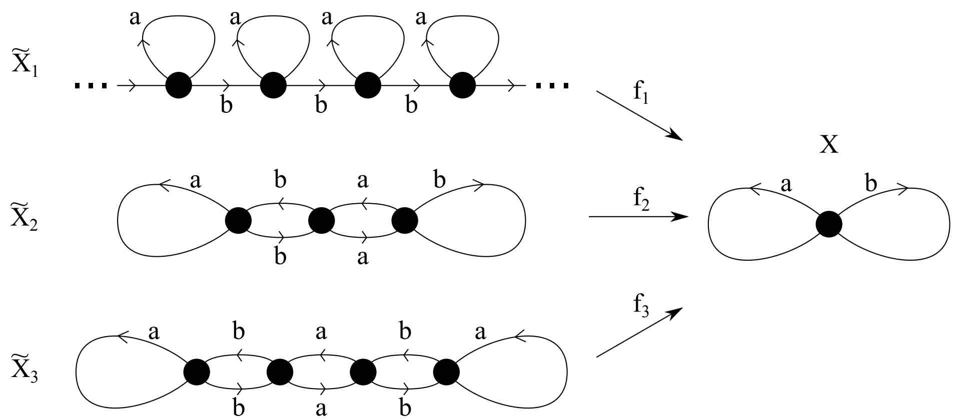

FIGURE 3.1. $\mathrm{Deck}(\widetilde{X}_1) = \mathbb{Z}$ and $\mathrm{Deck}(\widetilde{X}_2) = 1$ and $\mathrm{Deck}(\widetilde{X}_3) = \mathbb{Z}/2$; see Examples 3.8, 3.9, and 3.10

EXAMPLE 3.9. Let $f_2\colon \widetilde{X}_2 \to X$ be the cover depicted in Figure 3.1. Then $\mathrm{Deck}(\widetilde{X}_2) = 1$; indeed, any element of $\mathrm{Deck}(\widetilde{X}_2)$ must preserve the central vertex since it is the only vertex that is not adjacent to a loop, and thus by Lemma 3.5 must be the identity. □

EXAMPLE 3.10. Let $f_3\colon \widetilde{X}_3 \to X$ be the cover depicted in Figure 3.1. Then $\mathrm{Deck}(\widetilde{X}_3) = \mathbb{Z}/2$, which acts on $\widetilde{X}_3$ by a 180 degree rotation. Indeed, this rotation ρ clearly is an order 2 element of $\mathrm{Deck}(\widetilde{X}_3)$, so $\mathbb{Z}/2 \subset \mathrm{Deck}(\widetilde{X}_3)$. Any element ϕ of $\mathrm{Deck}(\widetilde{X}_3)$ must either fix or flip the outermost two vertices since they are the only vertices that are adjacent to loops, so by Lemma 3.5 we either have $\phi = \mathrm{id}$ or $\phi = \rho$. We conclude that in fact $\mathbb{Z}/2 = \mathrm{Deck}(\widetilde{X}_3)$. □

The following class of covering spaces will be very important.

DEFINITION 3.11. A covering space $f\colon \widetilde{X} \to X$ is *regular* if for all $x_0 \in X$, the group $\mathrm{Deck}(\widetilde{X})$ acts transitively on $f^{-1}(x_0)$. This means that for $p, q \in f^{-1}(x_0)$, there exists some $\phi \in \mathrm{Deck}(\widetilde{X})$ such that $\phi(p) = q$. □

EXAMPLE 3.12. The covering spaces in Examples 3.6, 3.7, and 3.8 are regular, while the covering spaces in Examples 3.9 and 3.10 are not regular. □

Our next goal is to characterize regular covers. We begin with the following definition.

DEFINITION 3.13. Let G be a group acting continuously on a space Z. This action is a *covering space action* if for all $p \in Z$, there exists a neighborhood U of p such that

$$\{g \in G \mid g(U) \cap U \neq \emptyset\} = \{\mathrm{id}\}.$$

□

The following lemma says that deck groups of covers provide examples of cover space actions.

LEMMA 3.14. *Let $f\colon \widetilde{X} \to X$ be a covering space. Then the action of $\mathrm{Deck}(\widetilde{X})$ on $\widetilde{X}$ is a covering space action.*

PROOF. Consider $p \in \widetilde{\mathrm{X}}$. Let $U \subset X$ be a trivializing neighborhood of $f(p)$ and let $\{\widetilde{\mathrm{U}}_\alpha\}_{\alpha \in I}$ be the disjoint subsets making up $f^{-1}(U)$ such that $f|_{\widetilde{\mathrm{U}}_\alpha} \colon \widetilde{\mathrm{U}}_\alpha \to U$ is a homeomorphism for all $\alpha \in I$. Let $\beta \in I$ be such that $p \in \widetilde{\mathrm{U}}_\beta$. We will prove that $\widetilde{\mathrm{U}}_\beta$ satisfies the condition in the definition of a covering space action. The group $\mathrm{Deck}(\widetilde{\mathrm{X}})$ permutes the $\widetilde{\mathrm{U}}_\alpha$. If $\phi \in \mathrm{Deck}(\widetilde{\mathrm{X}})$ is such that

$$\phi(\widetilde{\mathrm{U}}_\beta) \cap \widetilde{\mathrm{U}}_\beta \neq \emptyset,$$

then we must in fact have $\phi|_{\widetilde{\mathrm{U}}_\beta} = \mathrm{id}$. Lemma 3.5 then implies that $\phi = \mathrm{id}$. The lemma follows. □

LEMMA 3.15. *Let G be a group acting on a space X via covering space action. Then the projection map $f \colon X \to X/G$ is a regular cover with deck group G.*

PROOF. Consider a point $p \in X/G$. Write $p = f(x)$, let $U \subset X$ be an open neighborhood of x such that

$$\{g \in G \mid g(U) \cap U \neq \emptyset\} = \{\mathrm{id}\}, \tag{1}$$

and let $\overline{\mathrm{U}} = f(U)$. The set $\overline{\mathrm{U}}$ is thus an open neighborhood of p and

$$f^{-1}(U) = \bigcup_{g \in G} g(U).$$

For distinct $g_1, g_2 \in G$, we have

$$g_1(U) \cap g_2(U) = g_1(U \cap g_1^{-1} g_2(U)) = g_1(\emptyset) = \emptyset;$$

here the second equality uses (1). By construction, f takes $g(U)$ homeomorphically to $\overline{\mathrm{U}}$. It follows that $f \colon X \to X/G$ is a covering space. It is clear that $G \subset \mathrm{Deck}(f \colon X \to X/G)$; since G permutes the elements in any given fiber of f transitively, we see that $f \colon X \to X/G$ is a regular cover. Moreover, we can apply Lemma 3.5 to deduce that $G = \mathrm{Deck}(f \colon X \to X/G)$, and we are done. □

EXERCISE 3.16. Let $f \colon \widetilde{\mathrm{X}} \to X$ be a degree 2 cover. Prove that $\widetilde{\mathrm{X}}$ is a regular cover. □

EXERCISE 3.17. Let $f \colon \widetilde{\mathrm{X}} \to X$ be a covering space.

(1) If $g \colon Y \to X$ is a continuous map, then define

$$g^*(\widetilde{\mathrm{X}}) = \{(y, p) \mid g(y) = f(p)\} \subset Y \times \widetilde{\mathrm{X}}.$$

Also, let $g^*(f) \colon g^*(\widetilde{\mathrm{X}}) \to Y$ be the restriction of of the projection $Y \times \widetilde{\mathrm{X}} \to Y$ onto the first factor. Prove that $g^*(f) \colon g^*(\widetilde{\mathrm{X}}) \to Y$ is a covering space.

(2) If $g \colon X \to X$ is the identity map, then prove that $g^*(\widetilde{\mathrm{X}})$ is isomorphic to $\widetilde{\mathrm{X}}$.

(3) If X' is a subspace of X and $g \colon X' \to X$ is the inclusion of X' into X, prove that $g^*(f) \colon g^*(\widetilde{\mathrm{X}}) \to X'$ is isomorphic to the restriction of f to X'. □

(4) If $\widetilde{\mathrm{X}}$ is a trivial cover of X and $g \colon Y \to X$ is a continuous map, prove that $g^*(\widetilde{\mathrm{X}})$ is a trivial cover of Y.

(5) If $g \colon Y \to X$ and $h \colon Z \to Y$ are continuous maps, prove that the cover $(g \circ h)^*(\widetilde{\mathrm{X}})$ of Z is isomorphic to $h^*(g^*(\widetilde{\mathrm{X}}))$ of Z.

(6) Let $g\colon Y \to X$ is the constant map that takes every point of Y to a fixed point $p_0 \in X$, prove that $g^*(\widetilde{\mathrm{X}})$ is a trivial cover of Y. Hint: You can prove this directly, but it is better to deduce it from the last two parts of the exercise.

CHAPTER 4

Lifting paths

The following theorem will play a fundamental role in our analysis of covering spaces.

THEOREM 4.1. *Let $f\colon \widetilde{\mathrm{X}} \to X$ be a covering space, let $\gamma\colon [a,b] \to X$ be a path, and let $p \in \widetilde{\mathrm{X}}$ be such that $f(p) = \gamma(a)$. Then there exists a unique lift $\widetilde{\gamma}\colon [a,b] \to \widetilde{\mathrm{X}}$ of γ such that $\widetilde{\gamma}(a) = p$.*

Before we prove Theorem 4.1, we point out a corollary.

COROLLARY 4.2. *All covers of an interval $[a,b]$ are trivial.*

PROOF. Let $f\colon \widetilde{\mathrm{X}} \to [a,b]$ be a cover. Set $\widetilde{\mathrm{X}}_a = f^{-1}(a)$, so $\widetilde{\mathrm{X}}_a$ is a discrete set. Define a map $g\colon \widetilde{\mathrm{X}}_a \times [a,b] \to \widetilde{\mathrm{X}}$ as follows. For $p \in \widetilde{\mathrm{X}}_a$, Theorem 4.1 says that there is a unique lift $\widetilde{\gamma}_p\colon [a,b] \to \widetilde{\mathrm{X}}$ of the identity map $[a,b] \to [a,b]$ such that $\widetilde{\gamma}_p(a) = p$. We define

$$g(p,t) = \widetilde{\gamma}_p(t) \qquad (p \in \widetilde{\mathrm{X}}_a, t \in [a,b].$$

By construction, g is a map of covering spaces of $[a,b]$. We must prove that it is an isomorphism of covering spaces, i.e. that g is a homeomorphism.

We will construct an inverse to g. We begin by constructing a map $\phi\colon \widetilde{\mathrm{X}} \to \widetilde{\mathrm{X}}_a$ as follows. Consider a point $q \in \widetilde{\mathrm{X}}$. Set $t_q = f(q) \in [a,b]$. By Theorem 4.1, there is a unique lift $\widetilde{\delta}_q\colon [a,t_q] \to \widetilde{\mathrm{X}}$ of the inclusion $[a,t_q] \hookrightarrow [a,b]$ such that $\widetilde{\delta}_q(t_q) = q$. Define $\phi(q) = \widetilde{\delta}_q(0)$. To see that ϕ is continuous at q, let $U \subset [a,b]$ be a trivializing neighborhood for f of $f(q)$ and let $\widetilde{\mathrm{U}}$ be the sheet of f over U containing q. Then it is clear from the construction that $\phi(q') = \phi(q)$ for all $q' \in \widetilde{\mathrm{U}}$, which implies that ϕ is continuous at q.

Define $h\colon \widetilde{\mathrm{X}} \to \widetilde{\mathrm{X}}_a \times [a,b]$ via the formula

$$h(q) = (\phi(q), f(q)) \qquad (q \in \widetilde{\mathrm{X}}).$$

For all $q \in \widetilde{\mathrm{X}}$, the uniqueness in Theorem 4.1 implies that $\widetilde{\delta}_q = \widetilde{\gamma}_{p_q}|_{[a,t_q]}$. This implies that h is an inverse to g, as desired. □

We now turn to the proof of Theorem 4.1. We will need the following lemma from point-set topology.

LEMMA 4.3. *Let (M,d) be a compact metric space and let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of M. Then there exists some $\epsilon > 0$ (called the* Lebesgue number *of the covering) such that for all subsets $X \subset M$ with* $\operatorname{diam}(X) < \epsilon$*, there exists some $\alpha \in I$ such that $X \subset U_\alpha$.*

Proof. Assume that this is false. This implies that for $n \geq 1$ there exists a subset K_n of M such that $\operatorname{diam}(K_n) < 1/n$ and such that K_n is not contained in any U_α. For all $n \geq 1$, pick an arbitrary point $p_n \in K_n$. Since M is compact, there exists a limit point $x_0 \in M$ of the set $\{p_n\}_{n \geq 1}$. Pick $\alpha \in I$ such that $x_0 \in U_\alpha$. Since U_α is open, there exists some $\epsilon > 0$ such that the ball $B_{2\epsilon}(x_0)$ of radius 2ϵ around x_0 is contained in U_α. Let $N \geq 1$ be such that $1/N < \epsilon$. We can find some $n \geq N$ such that $p_n \in B_\epsilon(x_0)$. Since

$$\operatorname{diam}(K_n) < 1/n \leq 1/N < \epsilon,$$

it follows that

$$K_n \subset B_{2\epsilon}(x_0) \subset U_\alpha,$$

a contradiction. $\square$

Proof of Theorem 4.1. To simplify our notation, we will deal with the special case $[a, b] = [0, 1]$; the general case is done in a similar way. Let $\{U(\alpha)\}_{\alpha \in I}$ be an open cover of X by trivializing neighborhoods for $f\colon \widetilde{\mathrm{X}} \to X$. For $\alpha \in I$, set $V(\alpha) = \gamma^{-1}(U_\alpha)$. The set $\{V(\alpha)\}_{\alpha \in I}$ is thus an open cover of $[0, 1]$. Let $\epsilon > 0$ be a Lebesgue number for $\{V(\alpha)\}_{\alpha \in I}$ as in Lemma 4.3 and let $n \geq 1$ be such that $1/n < \epsilon$. We then define $\widetilde{\gamma}\colon [0, 1] \to \widetilde{\mathrm{X}}$ as follows. First, set $\widetilde{\gamma}(0) = p$. Next, assume that for some $0 \leq k < n - 1$ the lift $\widetilde{\gamma}$ has been defined on $[0, k/n] \subset [0, 1]$. We will show how to extend $\widetilde{\gamma}$ to $[0, (k + 1)/n]$. Since $1/n < \epsilon$, there exists some $\alpha_0 \in I$ such that $[k/n, (k+1)/n] \subset V(\alpha_0)$. This implies that $\gamma([k/n, (k+1)/n]) \subset U(\alpha_0)$. Let $\{\widetilde{\mathrm{U}}_\beta(\alpha_0)\}_{\beta \in J}$ be the sheets of $\widetilde{\mathrm{X}}$ lying over $U(\alpha_0)$ and let $\beta_0 \in J$ be such that $\widetilde{\gamma}(k/n) \in \widetilde{\mathrm{U}}_{\beta_0}(\alpha_0)$. The map $f|_{\widetilde{\mathrm{U}}_{\beta_0}(\alpha_0)}\colon \widetilde{\mathrm{U}}_{\beta_0}(\alpha_0) \to U(\alpha_0)$ is a homeomorphism, and we extend $\widetilde{\gamma}$ to $[0, (k + 1)/n]$ by letting

$$\widetilde{\gamma}|_{[k/n,(k+1)/n]} = (f|_{\widetilde{\mathrm{U}}_{\beta_0}(\alpha_0)})^{-1} \circ (\gamma|_{[k/n,(k+1)/n]}).$$

Repeating this process, we can define $\widetilde{\gamma}$ on all of $[0, 1]$. The uniqueness of $\widetilde{\gamma}$ is clear from the construction. $\square$

CHAPTER 5

Homotopies and their lifts

The following equivalence relation on functions is fundamental in algebraic topology.

Definition 5.1. Two continuous maps $g_0 \colon Y \to X$ and $g_1 \colon Y \to X$ are *homotopic* if there exists a continuous map $G \colon Y \times [0,1] \to X$ such that

$$g_0(y) = G(y,0) \quad \text{and} \quad g_1(y) = G(y,1)$$

for $y \in Y$. For $t \in [0,1]$, define $g_t \colon Y \to X$ via the formula $g_t(y) = G(y,t)$ for $y \in Y$. The family of continuous maps $g_t \colon Y \to X$ will be called a *homotopy* from g_0 to g_1. $\square$

Example 5.2. Let X be a convex subset of $\mathbb{R}^n$. Then any two continuous maps $g_0 \colon Y \to X$ and $g_1 \colon Y \to X$ are homotopic via the "straight-line" homotopy $g_t \colon Y \to X$ defined via the formula

$$g_t(y) = (1-t)g_0(y) + tg_1(y) \qquad (y \in Y). \qquad \square$$

Just like paths, homotopies can be lifted to covers.

Theorem 5.3. *Let $f \colon \widetilde{\mathrm{X}} \to X$ be a covering space, let $g_t \colon Y \to X$ be a homotopy, and let $\widetilde{\mathrm{g}}_0 \colon Y \to \widetilde{\mathrm{X}}$ be a lift of g_0 to $\widetilde{\mathrm{X}}$. Then $\widetilde{\mathrm{g}}_0$ lies in a unique homotopy $\widetilde{\mathrm{g}}_t \colon Y \to \widetilde{\mathrm{X}}$ such that each $\widetilde{\mathrm{g}}_t$ is a lift of g_t.*

Remark 5.4. Of course, Theorem 5.3 would also hold if we allowed the time parameter t in our homotopies to range over a general interval $[a,b]$ rather than just $[0,1]$. Using this, Theorem 4.1 is actually the special case of Theorem 5.3 where Y is a single point. $\square$

Before we prove Theorem 5.3, we point out a corollary.

Corollary 5.5. *Let $f \colon \widetilde{\mathrm{X}} \to X \times [0,1]$ be a covering space. Define $g \colon \widetilde{\mathrm{X}}_0 \to X$ to be the restriction of $\widetilde{\mathrm{X}}$ to $X \times \{0\} = X$. Then $\widetilde{\mathrm{X}}$ is isomorphic to the product covering space $\widetilde{\mathrm{X}}_0 \times [0,1]$ of $X \times [0,1]$.*

Proof. The proof is identical to that of Corollary 4.2, which says that all covers of an interval $[a,b]$ are trivial. Merely substitute Theorem 5.3 for Theorem 4.1. $\square$

Proof of Theorem 5.3. For $y \in X$, let $\gamma_y \colon [0,1] \to X \times [0,1]$ be defined via the formula

$$\gamma_y(t) = g_t(y) \qquad (t \in [0,1]).$$

Theorem 4.1 says that there exists a unique lift $\widetilde{\gamma}_y(t) \colon Y \to \widetilde{\mathrm{X}}$ of γ_y such that $\widetilde{\gamma}_y(0) = \widetilde{\mathrm{g}}_0(y)$. For $t \in [0,1]$, we then define $\widetilde{\mathrm{g}}_t \colon Y \to \widetilde{\mathrm{X}}$ via the formula

$$\widetilde{\mathrm{g}}_t(y) = \widetilde{\gamma}_y(t) \qquad (y \in Y).$$

It is easy to see that $\widetilde{g}_t$ is a continuous homotopy lifting g_t. $\square$

CHAPTER 6

Covers of contractible spaces

The main theorem of this section can be informally stated as saying that if a space can be continuously deformed to a one-point space, then it has no non-trivial covers. To make this precise, we begin we the following definition.

DEFINITION 6.1. A space X is *contractible* if for some point $p \in X$, the identity map $X \to X$ is homotopic to the constant map $X \to X$ that takes every point in X to p. The associated homotopy will be called a *contraction* of X. $\square$

EXAMPLE 6.2. If $A \subset \mathbb{R}^n$ is convex, then A is contractible. Indeed, letting $p \in A$ be any point, the identity map $A \to A$ is homotopic to the constant map $A \to A$ taking every point to p via the straight-line homotopy discussed in Example 5.2. $\square$

Our main theorem then is as follows.

THEOREM 6.3. *Every covering space of a contractible space is trivial.*

This theorem will be a consequence of the following important proposition. In it, we will use the notion of pullbacks of coverss from Exercise 3.17.

PROPOSITION 6.4. *Let $f\colon \widetilde{\mathrm{X}} \to X$ be a covering map and let $g_0, g_1\colon Y \to X$ be homotopic continuous maps. Then the covering spaces $g_0^*(\widetilde{\mathrm{X}})$ and $g_1^*(\widetilde{\mathrm{X}})$ of Y are isomorphic.*

PROOF. Let $G\colon Y \times [0,1] \to X$ be a continuous function such that

$$g_0(y) = G(y,0) \quad \text{and} \quad g_1(y) = G(y,1) \qquad (y \in Y).$$

Also, for $i = 0,1$ define $\iota_i\colon Y \to Y \times [0,1]$ via the formula $\iota_i(y) = (y,i)$. We then have $g_0 = G \circ \iota_0$ and $g_1 = G \circ \iota_1$. Defining $\widetilde{\mathrm{Z}} \to Y \times [0,1]$ to be the cover $G^*(\widetilde{\mathrm{X}})$ of $Y \times [0,1]$, this implies that

$$g_0^*(\widetilde{\mathrm{X}}) = \iota_0^*(\widetilde{\mathrm{Z}}) \quad \text{and} \quad g_1^*(\widetilde{\mathrm{X}}) = \iota_1^*(\widetilde{\mathrm{Z}});$$

see part 5 of Exercise 3.17. Letting $\widetilde{\mathrm{Z}}_0 \to Y$ be the restriction of $\widetilde{\mathrm{Z}}$ to $Y \times \{0\} = Y$, Corollary 5.5 implies that $\widetilde{\mathrm{Z}}$ is isomorphic to the cover $\widetilde{\mathrm{Z}}_0 \times [0,1]$ of $Y \times [0,1]$. This implies that $\iota_0^*(\widetilde{\mathrm{Z}})$ and $\iota_1^*(\widetilde{\mathrm{Z}})$ are both isomorphic to $\widetilde{\mathrm{Z}}_0$, and the proposition follows. $\square$

PROOF OF THEOREM 6.3. Recall that the statement of the theorem is as follows. Let X be a contractible space and let $\widetilde{\mathrm{X}}$ be a cover of X. Then we must prove that $\widetilde{\mathrm{X}}$ is a trivial cover. Let $g_t\colon X \to X$ be a homotopy such that $g_0 = \mathrm{id}$ and such that g_t is the constant map taking every point of X to some fixed $p \in X$. Proposition 6.4 implies that $g_0^*(\widetilde{\mathrm{X}})$ and $g_1^*(\widetilde{\mathrm{X}})$ are isomorphic covers. Part 2 of Exercise 3.17 says that g_0^* is isomorphic to $\widetilde{\mathrm{X}}$ and part 6 of Exercise 3.17 says that $g_1^*(\widetilde{\mathrm{X}})$ is a trivial cover. The theorem follows. $\square$

CHAPTER 7

Equivalent paths and their lifts

Let X be a topological space. As notation, set $I = [0,1]$.

Definition 7.1. A *path* in X from $p \in X$ to $q \in X$ is a continuous function $\gamma\colon I \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$. □

Definition 7.2. Let $f\colon X \to Y$ be a continuous map and let γ be a path in X. Define $f_*(\gamma)$ to be the path $f \circ \gamma\colon I \to Y$ in Y. □

We will study paths in a space up to homotopies that fix the endpoints of the paths. The proper definition is as follows.

Definition 7.3. Let $\gamma\colon I \to X$ and $\gamma'\colon I \to X$ be paths in X. We will say that γ is *equivalent* to γ' if γ and γ' both go from the same point $p \in X$ to the same point $q \in X$ and there exists a continuous function $F\colon I^2 \to X$ such that

$$F(t,0) = \gamma(t) \quad \text{and} \quad F(t,1) = \gamma'(t) \qquad (t \in I)$$

and

$$F(0,s) = p \quad \text{and} \quad F(1,s) = q \qquad (s \in I).$$

If γ is equivalent to γ', then we will write $\gamma \sim \gamma'$. This is clearly an equivalence relation on paths; given a path $\gamma\colon I \to X$, we will write $[\gamma]$ for its $\sim$-equivalence class. □

Lemma 7.4. *Let $f\colon X \to Y$ be a continuous map and let γ_1 and γ_2 be paths in X such that $\gamma_1 \sim \gamma_2$. Then $f_*(\gamma_1) \sim f_*(\gamma_2)$.*

Proof. If $F\colon I^2 \to X$ is a homotopy witnessing the fact that $\gamma_1 \sim \gamma_2$, then $f \circ F$ is a homotopy witnessing the fact that $f_*(\gamma_1) \sim f_*(\gamma_2)$. □

It thus makes sense to write $f_*([\gamma])$ for a map $f\colon X \to Y$ and a path γ in X.

The importance of paths for the study of covering spaces comes from the following important theorem.

Theorem 7.5. *Let $f\colon \widetilde{\mathrm{X}} \to X$ be a covering space and let $x_0 \in X$ be a point. Pick $\widetilde{\mathrm{x}}_0 \in \widetilde{\mathrm{X}}$ such that $f(\widetilde{\mathrm{x}}_0) = x_0$. There is then a bijection between*

$$\{[\gamma] \mid \gamma \textit{ path in } X \textit{ with } \gamma(0) = x_0\}$$

and

$$\{[\widetilde{\gamma}] \mid \widetilde{\gamma} \textit{ path in } \widetilde{\mathrm{X}} \textit{ with } \widetilde{\gamma}(0) = \widetilde{\mathrm{x}}_0\}.$$

Proof. Given a path γ in X starting at x_0, Theorem 4.1 says that there is a unique lift $\widetilde{\gamma}\colon I \to \widetilde{\mathrm{X}}$ with $\widetilde{\gamma}(0) = \widetilde{\mathrm{x}}_0$. Conversely, if $\widetilde{\gamma}\colon I \to \widetilde{\mathrm{X}}$ is a path starting at $\widetilde{\mathrm{x}}_0$, then $f_*(\widetilde{\gamma})$ is a path in X starting at x_0. This establishes a bijection between

$$\{\gamma \mid \gamma \text{ path in } X \text{ with } \gamma(0) = x_0\}$$

and

$$\{\widetilde{\gamma} \mid \widetilde{\gamma} \text{ path in } \widetilde{\mathrm{X}} \text{ with } \widetilde{\gamma}(0) = \widetilde{\mathrm{x}}_0\}.$$

To see that this descends to a bijection between homotopy classes of paths, we must prove that if γ_0 and γ_1 are equivalent paths in X starting at x_0 and $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ are their lifts to $\widetilde{\mathrm{X}}$ starting at $\widetilde{\mathrm{x}}_0$, then $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ are equivalent paths. Let x_1 be the common endpoint of γ_0 and γ_1. Since γ_0 and γ_1 are equivalent paths, there exists a homotopy γ_t of paths such that

$$\gamma_t(0) = x_0 \quad \text{and} \quad \gamma_t(1) = x_1$$

for all $t \in [0,1]$. Theorem 5.3 says that we can lift γ_t to a homotopy $\widetilde{\gamma}'_t$ of paths in X such that $\widetilde{\gamma}'_0 = \widetilde{\gamma}_0$. We will prove that $\widetilde{\gamma}'_t$ is a homotopy fixing the endpoints between $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$. Observe first that

$$f(\widetilde{\gamma}'_t(0)) = \gamma_t(0) = x_0$$

for all $t \in [0,1]$. Since the fiber $f^{-1}(x_0)$ is discrete, we deduce that $\widetilde{\gamma}'_t(0)$ is a constant function of t, i.e. that

$$\widetilde{\gamma}'_t(0) = \widetilde{\gamma}'_0(0) = \widetilde{\mathrm{x}}_0$$

for all $t \in [0,1]$. In a similar way, we see that $\widetilde{\gamma}'_t(1)$ is a constant function of t.

Summarizing, $\widetilde{\gamma}'_t$ is a homotopy fixing the endpoints between $\widetilde{\gamma}_0$ and some other path $\widetilde{\gamma}'_1$. The path $\widetilde{\gamma}'_1$ is a lift of γ_1 starting at $\widetilde{\mathrm{x}}_0$; by the uniqueness of path-lifting, we conclude that $\widetilde{\gamma}'_1 = \widetilde{\gamma}_1$, as desired. □

CHAPTER 8

Operations on paths

Theorem 7.5 suggests that understanding the structure of equivalence classes of paths in a space will help us unlock the structure of covering spaces. In this section, we begin to explore the algebraic structure hiding in these equivalence classes.

We begin with the following lemma, which says that reparameterizing a path does not change its equivalence class.

LEMMA 8.1. *Let $\gamma\colon I \to X$ be a path in X from $p \in X$ to $q \in X$ and let $\phi\colon I \to I$ be a continuous function such that $\phi(0) = 0$ and $\phi(1) = q$. Then $[\gamma] = [\gamma \circ \phi]$.*

PROOF. Define $F\colon I^2 \to X$ via the formula

$$F(t,s) = \gamma((1-s)t + s\phi(t)) \qquad (t \in I, s \in I).$$

We then have

$$F(t,0) = \gamma(t) \quad \text{and} \quad F(t,1) = \gamma(\phi(t)) \qquad (t \in I)$$

and

$$F(0,s) = \gamma(s\phi(0)) = \gamma(0) = p \qquad (s \in I)$$

and

$$F(1,s) = \gamma((1-s) + s\phi(1)) = \gamma(1-s+s) = \gamma(1) = q \qquad (s \in I).$$

The lemma follows. □

We now define a sort of "multiplication" of paths; however, the product $\gamma \cdot \gamma'$ is only defined if the ending point of γ is the same as the starting point of γ'. The precise definition is as follows.

DEFINITION 8.2. Let γ be a path in X from $p \in X$ to $q \in X$ and let γ' be a path in X from $q \in X$ to $r \in X$. We then define $\gamma \cdot \gamma'$ to be the path in X from $p \in X$ to $r \in X$ that is defined via the formula

$$(\gamma \cdot \gamma')(t) = \begin{cases} \gamma(2t) & \text{if } 0 \le t \le 1/2, \\ \gamma'(2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$

for $t \in I$. □

This multiplication respects our equivalence relation in the following sense.

LEMMA 8.3. *Let γ_1 and γ_2 be paths in X from $p \in X$ to $q \in X$ such that $[\gamma_1] = [\gamma_2]$. Also, let γ_1' and γ_2' be paths in X from $q \in X$ to $r \in X$ such that $[\gamma_1'] = [\gamma_2']$. Then $[\gamma_1 \cdot \gamma_1'] = [\gamma_2 \cdot \gamma_2']$.*

PROOF. Trivial. □

It thus makes sense to multiply equivalence classes of paths. The multiplication of paths is not itself associative, but as the following lemma shows that it becomes associative when we pass to equivalence classes.

LEMMA 8.4. *Let γ be a path in X from $p \in X$ to $q \in X$, let γ' be a path in X from $q \in X$ to $r \in X$, and let γ'' be a path in X from $r \in X$ to $s \in X$. Then $[(\gamma \cdot \gamma') \cdot \gamma''] = [\gamma \cdot (\gamma' \cdot \gamma'')]$.*

PROOF. Observe that for $t \in I$, we have

$$((\gamma \cdot \gamma') \cdot \gamma'')(t) = \begin{cases} \gamma(4t) & \text{if } 0 \leq t \leq 1/4, \\ \gamma'(4t-1) & \text{if } 1/4 \leq t \leq 1/2, \\ \gamma''(2t-1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

and

$$(\gamma \cdot (\gamma' \cdot \gamma''))(t) = \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq 1/2, \\ \gamma'(4t-2) & \text{if } 1/2 \leq t \leq 3/4, \\ \gamma''(4t-3 & \text{if } 3/4 \leq t \leq 1. \end{cases}$$

Thus

$$(\gamma \cdot (\gamma' \cdot \gamma'')) = ((\gamma \cdot \gamma') \cdot \gamma'') \circ \phi,$$

where $\phi\colon I \to I$ is defined via the formula

$$\phi(t) = \begin{cases} t/2 & \text{if } 0 \leq t \leq 1/2, \\ t - 1/2 & \text{if } 1/2 \leq t \leq 3/4, \\ 2t-1 & \text{if } 3/4 \leq t \leq 1. \end{cases}$$

The lemma now follows from Lemma 8.1. $\square$

We now dicusss identity elements for this multiplication. Since paths can only be multiplied if their endpoints match up, we will need a different identity element for every point of X.

DEFINITION 8.5. For $p \in X$, let $e_p\colon I \to X$ be the constant path $e_p(t) = p$. $\square$

LEMMA 8.6. *Let γ be a path in X from $p \in X$ to $q \in X$. Then $[e_p \cdot \gamma] = [\gamma \cdot e_q] = [\gamma]$.*

PROOF. We have

$$(e_p \cdot \gamma)(t) = \begin{cases} p & \text{if } 0 \leq t \leq 1/2, \\ \gamma(2t-1) & \text{if } 1/2 \leq t \leq 1 \end{cases} \qquad (t \in I).$$

This implies that $e_p \cdot \gamma = \gamma \circ \phi$, where $\phi\colon I \to I$ is defined via the formula

$$\phi(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1/2, \\ 2t-1 & \text{if } 1 \leq t \leq 1 \end{cases} \qquad (t \in I).$$

Lemma 8.1 then implies that $[e_p \cdot \gamma] = [\gamma]$. A similar argument shows that $[\gamma \cdot e_q] = [\gamma]$. $\square$

We now discuss inverses.

DEFINITION 8.7. Let γ be a path in X from $p \in X$ to $q \in X$. Define $\overline{\gamma}$ to be the path in X from q to p defined via the formula $\overline{\gamma}(t) = \gamma(1-t)$. $\square$

LEMMA 8.8. *If γ is a path in X from $p \in X$ to $q \in X$, then $[\gamma \cdot \overline{\gamma}] = [e_p]$ and $[\overline{\gamma} \cdot \gamma] = [e_q]$.*

PROOF. Since $\overline{\overline{\gamma}} = \gamma$, it is enough to prove that $[\gamma \cdot \overline{\gamma}] = [e_p]$. Define $F\colon I^2 \to X$ via the formula

$$F(t,s) = \begin{cases} \gamma(2t) & \text{if } 0 \le t \le s/2, \\ \gamma(s) & \text{if } s/2 \le t \le 1 - s/2, \\ \gamma(2-2t) & \text{if } 1 - s/2 \le t \le 1 \end{cases} \qquad (s, t \in I).$$

For a fixed $s_0 \in I$, the path $t \mapsto F(t, s_0)$ starts at p, then goes along γ until it hits $\gamma(s_0)$, then waits for a while, and then goes back along $\overline{\gamma}$ to p. We have

$$F(0,s) = \gamma(0) = p \quad \text{and} \quad F(1,s) = \gamma(1) = p$$

for $s \in I$ and

$$F(t,0) = \gamma(0) = p \quad \text{and} \quad F(t,1) = (\gamma \cdot \overline{\gamma})(t)$$

for $t \in I$. It follows that $[\gamma \cdot \overline{\gamma}] = [e_p]$, as desired. $\square$

CHAPTER 9

The fundamental group

We now begin to do stuff that is well-covered in Hatcher. We first define the fundamental group as on page 27 of Hatcher. We then discussed how to change the basepoint as on page 28. We then jumped ahead and proved Proposition 1.31 in Hatcher, which says that if $f\colon (Y,y) \to (X,x)$ is a based covering space, then the induced map $f_*\colon \pi_1(Y,y) \to \pi_1(X,x)$ is injective. We then proved Proposition 1.39 (in a slightly different language), which says that if $f\colon (Y,y) \to (X,x)$ is a based regular G-cover with Y path-connected, then there is an induced homomorphism $\psi\colon \pi_1(X,x) \to G$ fitting into a short exact sequence

$$1 \longrightarrow \pi_1(Y,y) \xrightarrow{f_*} \pi_1(X,x) \xrightarrow{\psi} G \longrightarrow 1.$$

This allows us to compute the fundamental group of many spaces (X,x) by the following procedure:

- Find a based regular G-cover $f\colon (Y,y) \to (X,x)$ such that Y is path-connected and $\pi_1(Y,y) = 1$ (these two conditions are often summarized by saying that Y is *simply-connected*). Then $\pi_1(X,x) \cong G$ via the above homomorphism ψ.

Here are three basic examples of this.

(1) You can see that $\pi_1(S^1,1) = \mathbb{Z}$ using the cover $\mathbb{R} \to S^1$.
(2) You can see that $\pi_1(\mathbb{RP}^n,p) = \mathbb{Z}/2$ using the cover $S^n \to \mathbb{RP}^n$.
(3) Let X be the wedge of two circles with wedge point p. Then you can see that $\pi_1(X,p)$ is the free group F_2 on two letters using the cover associated to the infinite 4-valent tree.

After this, we prove the lifting criterion for covers (Proposition 1.33 in Hatcher) and give the following corollaries:

- If $f\colon (Y_1,y_1) \to (X,x)$ and $g\colon (Y_2,y_2) \to (X,x)$ are such that each Y_i is path connected and locally path connected and satisfy $\mathrm{Im}(f_*) \subset \mathrm{Im}(g_*)$, then there exists a unique covering map $h\colon (Y_1,y_1) \to (Y_2,y_2)$ such that $f = g \circ h$. This is not quite in Hatcher, but Proposition 1.37 is a special case (where $\mathrm{Im}(f_*) = \mathrm{Im}(g_*)$), and the general case is proved in an identical way.
- If $f\colon (Y_1,y_1) \to (X,x)$ and $g\colon (Y_2,y_2) \to (X,x)$ are such that each Y_i is path connected and locally path connected and satisfy $\mathrm{Im}(f_*) = \mathrm{Im}(g_*)$, then there is an isomorphism of covers from (Y_1,y_1) to (Y_2,y_2). This is precisely Proposition 1.37 of Hatcher.
- A cover $f\colon (Y,y) \to (X,x)$ with Y path-connected and locally path-connected is regular if and only if $\mathrm{Im}(f_*)$ is a normal subgroup of $\pi_1(X,x)$. This is Proposition 1.39 of Hatcher.

The next thing we proved is that if (X, x) is path-connected and semilocally simply-connected, then for every subgroup H of $\pi_1(X, x)$ there is a (unique by the above) cover $f\colon (Y, y) \to (X, x)$ such that $\mathrm{Im}(f_*) = H$. This is Proposition 1.36 in Hatcher; the key is constructing the universal cover.

All of this can be summarized by saying that if (X, x) is path-connected and semilocally simply-connected, then there is a bijection between

$$\{\text{subgroups of } \pi_1(X, x)$$

and

$$\{\text{based covers } f\colon (Y, y) \to (X, x) \text{ with } Y \text{ path-connected}\}.$$

This bijection associates to a based cover $f\colon (Y, y) \to (X, x)$ the subgroup $\mathrm{Im}(f_*)$ of $\pi_1(X, x)$. It restricts to a bijection between normal subgroups and regular covers.

We then proved the Seifert-Van Kampen theorem. For examples of how this is used, see Chapter 1.2 of Hatcher. For the proof we gave, see the document

`http://www.nd.edu/~andyp/notes/SeifertVanKampen.pdf`