A geometrically-minded introduction to smooth manifolds

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CHAPTER 1

Multivariable calculus

In this chapter, we quickly review the rudiments of multivariable differential calculus.

1.1. Smooth maps and their derivatives

Let $f : V_1 \to V_2$ be a continuous function between open sets $V_1 \subset \mathbb{R}^n$ and $V_2 \subset \mathbb{R}^m$. We say that f is *smooth* if all of its mixed partial derivatives exist. To keep things straight, we will illustrate all the features of f we will discuss with the following running example.

EXAMPLE. Let $V_1 = \mathbb{R}^2$ and $V_2 = \mathbb{R}^3$. Define $f : V_1 \to V_2$ via the formula

$$f(x_1, x_2) = (x_1^2 - 3x_2^3, x_1 x_2, x_2 + 3) \in V_2 \qquad ((x_1, x_2) \in V_1).$$

It is clear that all mixed partial derivatives of f exist, so f is smooth. □

As a first approximation, the derivative of f at a point $p \in V_1$, denoted $D_p f$, is the matrix of first partial derivatives. Thus $D_p f$ is an $m \times n$ matrix whose (i, j)-entry is $\frac{\partial f_i}{\partial x_j}$, where f_i is the i^{th} coordinate function of f.

EXAMPLE. Returning to the above example, if $p = (p_1, p_2)$ then

$$D_p f = \begin{pmatrix} 2p_1 & -9p_2^2 \\ p_2 & p_1 \\ 0 & 1 \end{pmatrix}$$ □

However, this is not quite the correct point of view. In reality, one should view the derivative $D_p f$ as being the linear map

$$\begin{aligned} \mathbb{R}^n &\to \mathbb{R}^m \\ \vec{x} &\mapsto (D_p f) \cdot \vec{x} \end{aligned}$$

which corresponds to the matrix of first partial derivatives we discussed above. But this is potentially confusing since the $\mathbb{R}^n$ and $\mathbb{R}^m$ look like the same places where V_1 and V_2 live, but in reality they should be thought of as something different, namely the spaces of tangent vectors of V_1 and V_2 at the points $p \in V_1$ and $f(p) \in V_2$, respectively. These spaces of tangent vectors will be denoted $T_p V_1$ and $T_{f(p)} V_2$, so $T_p V_1 = \mathbb{R}^n$ and $T_{f(p)} V_2 = \mathbb{R}^m$ and $D_p f$ is a linear map from the vector space $T_p V_1$ to the vector space $T_{f(p)} V_2$. We remark that though all the $T_p V_1$ for $p \in V_1$ equal the vector space $\mathbb{R}^n$, they should *not* be viewed as being the same thing.

EXAMPLE. Returning to the above example, if we write $\vec{x} = (x_1, x_2) \in T_p V_1 = \mathbb{R}^2$, then $D_p f$ is the linear map from $T_p V_1 = \mathbb{R}^2$ to $T_{f(p)} V_2 = \mathbb{R}^3$ defined via the formula

$$(D_pf)(\vec{x}) = \begin{pmatrix} 2p_1 & -9p_2^2 \\ p_2 & p_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2p_1x_1 - 9p_2^2x_2 \\ p_2x_1 + p_1x_2 \\ x_2 \end{pmatrix};$$

here we are regarding $\vec{x}$ as a column vector. □

1.2. The chain rule

One of the most important property of derivatives is the *chain rule*. Let $f : V_1 \to V_2$ and $g : V_2 \to V_3$ be smooth maps, where $V_1 \subset \mathbb{R}^n$ and $V_2 \subset \mathbb{R}^m$ and $V_3 \subset \mathbb{R}^\ell$ are open. We then have the composition $g \circ f : V_1 \to V_3$. For $p \in V_1$, we have linear maps

$$D_pf : T_pV_1 \to T_{f(p)}V_2$$

and

$$D_{f(p)}g : T_{f(p)}V_2 \to T_{g(f(p))}V_3$$

and

$$D_p(g \circ f) : T_pV_1 \to T_{g(f(p))}V_3.$$

The chain rule can be stated as follows.

Theorem 1.1 (Chain Rule I). *Let $V_1 \subset \mathbb{R}^n$ and $V_2 \subset \mathbb{R}^m$ and $V_3 \subset \mathbb{R}^\ell$ be open sets and let $f : V_1 \to V_2$ and $g : V_2 \to V_3$ be smooth maps. Then for all $p \in V_1$ we have*

$$D_p(g \circ f) = (D_{f(p)}g) \circ (D_pf).$$

Example. Let $V_1 = \mathbb{R}^2$ and $V_2 = \mathbb{R}^3$ and $V_3 = \mathbb{R}$. Define $f : V_1 \to V_2$ via the formula

$$f(x_1, x_2) = (x_1^2 - 3x_2^3, x_1x_2, x_2 + 3) \in V_2 \qquad ((x_1, x_2) \in V_1)$$

and $g : V_2 \to V_3$ via the formula

$$g(y_1, y_2, y_3) = (y_1 + 2y_2^2 + 3y_3^3).$$

As we calculated in the previous section, for $p \in V_1$ written as $p = (p_1, p_2)$ the linear map $D_pf : T_pV_1 \to T_{f(p)}V_2$ is represented by the matrix

$$\begin{pmatrix} 2p_1 & -9p_2^2 \\ p_2 & p_1 \\ 0 & 1 \end{pmatrix}.$$

For $q \in V_2$ written as $q = (q_1, q_2, q_3)$, the linear map $D_qg : T_qV_2 \to T_{g(q)}V_3$ is represented by the matrix

$$\begin{pmatrix} 1 & 4q_2 & 9q_3^2 \end{pmatrix}.$$

Let's now check the chain rule. The composition $g \circ f : V_1 \to V_3$ is given via the formula

$$(g \circ f)(p_1, p_2) = ((p_1^2 - 3p_2^3) + 2(p_1p_2)^2 + 3(p_2 + 3)^3) \in \mathbb{R}^1.$$

The derivative $D_p(g \circ f)$ of this at $p = (p_1, p_2)$ is represented by the matrix

$$\begin{pmatrix} 2p_1 + 4(p_1p_2)p_2 & -9p_2^2 + 4(p_1p_2)p_1 + 9(p_2 + 3)^2 \end{pmatrix}.$$

Plugging the equations of $f(p)$ into the above formula for $D_qg : T_qV_2 \to T_{g(q)}V_3$, the linear map $D_{f(p)}g : T_{f(p)}V_2 \to T_{g(f(p))}V_3$ is represented by the matrix

$$\begin{pmatrix} 1 & 4(p_1p_2) & 9(p_2 + 3)^2 \end{pmatrix}$$

The chain rule then asserts that

$$\begin{aligned}&\begin{pmatrix}2p_1 + 4(p_1p_2)p_2 & -9p_2^2 + 4(p_1p_2)p_1 + 9(p_2+3)^2\end{pmatrix}\\ &= \begin{pmatrix}1 & 4(p_1p_2) & 9(p_2+3)^2\end{pmatrix} \cdot \begin{pmatrix}2p_1 & -9p_2^2 \\ p_2 & p_1 \\ 0 & 1\end{pmatrix},\end{aligned}$$

which is easily verified. □

We now globalize all of this. The *tangent bundles* of V_1 and V_2 are defined to be

$$TV_1 = V_1 \times \mathbb{R}^n \quad \text{and} \quad TV_2 = V_2 \times \mathbb{R}^m,$$

respectively. The tangent bundle TV_1 should be viewed as the union of the tangent spaces T_pV_1 as p ranges over V_1; the space $T_pV_1 = \mathbb{R}^n$ is identified with $\{p\} \times \mathbb{R}^n \subset TV_1$. Similarly, TV_2 should be viewed as the union of the tangent spaces $T_qV_2 = \mathbb{R}^m$ as q ranges over V_2. The derivatives D_pf piece together to give a continuous map $Df : TV_1 \to TV_2$ defined via the formula

$$(Df)(p, \vec{x}) = (f(p), (D_pf)(\vec{x})) \in TV_2 = V_2 \times \mathbb{R}^m \qquad ((p, \vec{x}) \in TV_1 = V_1 \times \mathbb{R}^n).$$

Example. Continuing our running example, if $V_1 = \mathbb{R}^n$ and $V_2 = \mathbb{R}^m$ and $f : V_1 \to V_2$ is defined via the formula

$$f(x_1, x_2) = (x_1^2 - 3x_2^3, x_1x_2, x_2 + 3) \in V_2 \qquad ((x_1, x_2) \in V_1),$$

then the map $Df : TV_1 \to TV_2$ is the map defined via the formula

$$Df(p, \vec{x})) = ((p_1^2 - 3p_2^3, p_1p_2, p_2 + 3), (2p_1x_1 - 9p_2x_2, p_2x_1 + p_1x_2, x_2))$$

for $p = (p_1, p_2) \in V_1$ and $\vec{x} = (x_1, x_2) \in T_pV_1 = \mathbb{R}^2$. □

To globalize the chain rule (Theorem 1.1), observe that if $V_3 \subset \mathbb{R}^\ell$ is an open set and $g : V_2 \to V_3$ is a smooth map, then we have derivative maps

$$Df : TV_1 \to TV_2$$

and

$$Dg : TV_2 \to TV_3$$

and

$$D(g \circ f) : TV_1 \to TV_3.$$

The chain rule can then be stated as follows.

Theorem 1.2 (Chain Rule II). *Let $V_1 \subset \mathbb{R}^n$ and $V_2 \subset \mathbb{R}^m$ and $V_3 \subset \mathbb{R}^\ell$ be open sets and let $f : V_1 \to V_2$ and $g : V_2 \to V_3$ be smooth maps. We then have*

$$D(g \circ f) = (Dg) \circ (Df).$$

CHAPTER 2

Smooth manifolds

This chapter defines smooth manifolds and gives some basic examples. We also discuss smooth partitions of unity.

2.1. The definition

We start with the definition of a manifold (not yet smooth).

DEFINITION. A *manifold* of dimension n is a paracompact Hausdorff space M^n such that for every $p \in M^n$ there exists an open set $U \subset M^n$ containing p and a homeomorphism $\phi : U \to V$, where $V \subset \mathbb{R}^n$ is open. The map $\phi : U \to V$ is a *chart* around p. We will often call V a *local coordinate system* around p and identify it via ϕ^{-1} with a subset of M^n. □

REMARK. We require M^n to be Hausdorff and paracompact to avoid various pathologies, some of which are discussed in the exercises. The existence of charts is the real fundamental defining property of a manifold. □

Our goal is to learn how to do calculus on manifolds. The idea is that notions like derivatives are *local*: they only depend on the behavior of functions in small neighborhoods of a point. We can thus use charts and local coordinate systems to identify small pieces of our manifold with open sets in $\mathbb{R}^n$ and thereby apply calculus in $\mathbb{R}^n$ to our manifolds. However, this does not quite work because different charts might give you completely unrelated notions of smooth functions, derivatives, etc. We therefore have to carefully choose our charts.

DEFINITION. Given two charts $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ on a manifold M^n, the *transition function* from U_1 to U_2 is the function $\tau_{21} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ defined via the formula $\tau_{21} = \phi_2 \circ (\phi_1|_{\phi_1(U_1 \cap U_2)})^{-1}$. Here observe that $\phi_1(U_1 \cap U_2)$ is an open subset of $V_1 \subset \mathbb{R}^n$ and $\phi_2(U_1 \cap U_2)$ is an open subset of $V_2 \subset \mathbb{R}^n$. □

DEFINITION. A *smooth atlas* for a manifold M^n is a set $\mathcal{A} = \{\phi_i : U_i \to V_i\}_{i \in I}$ of charts on M^n with the following properties.

- The U_i cover M^n, i.e. $M^n = \cup_{i \in I} U_i$.
- For all $i, j \in I$, the transition function from U_1 to U_2 is smooth. Of course, this only has content if $U_i \cap U_j \neq \emptyset$.

Two smooth atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are *compatible* if $\mathcal{A}_1 \cup \mathcal{A}_2$ is also a smooth atlas. This defines an equivalence relation on smooth atlases. A *smooth manifold* is a manifold equipped with an equivalence class of smooth atlases. □

REMARK. We will give examples of manifolds by describing an atlas for them. However, this atlas is not a fundamental property of the manifold, and when we subsequently make use of charts for the manifold we will allow ourselves to use charts from any equivalent atlas. The first place where this freedom will play an important role is when we define what it means for a function between two smooth manifolds to be smooth. □

2.2. Basic examples

Here are a number of examples.

Example. If $U \subset \mathbb{R}^n$ is an open set, then U is naturally a smooth manifold with the smooth atlas $\mathcal{A}$ consisting of a single chart $\phi : U \to V$ with $V = U$ and $\phi = \text{id}$. These can be complicated and wild; for instance, U might be the complement of a Cantor set embedded in $\mathbb{R}^n$. □

Example. An important special case of an open subset of Euclidean space is the general linear group $\text{GL}_n(\mathbb{R})$. The set $\text{Mat}(n, n)$ of $n \times n$ real matrices can be identified with $\mathbb{R}^{n^2}$ in the obvious way, and $\text{GL}_n(\mathbb{R})$ is the complement of the closed subset where the determinant vanishes. This is an example of a *Lie group*, that is, a smooth manifold which is also a group and for which the group operations are continuous (and, in fact, smooth). We will discuss these in much more detail in Chapter 9 □

Example. More generally, if M^n is a smooth manifold with smooth atlas $\mathcal{A} = \{\phi_i : U_i \to V_i\}_{i \in I}$ and $U \subset M^n$ is an open subset, then U is naturally a smooth manifold with smooth atlas $\{\phi_i|_{U \cap U_i} : U_i \cap U \to \phi_i(U \cap U_i)\}_{i \in I}$. □

Example. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$, i.e.

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

Then S^n can be endowed with the following smooth atlas. For $1 \leq i \leq n+1$, define

$$U_{x_i>0} = \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_i > 0\}$$

and

$$U_{x_i<0} = \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_i < 0\}.$$

Let $V \subset \mathbb{R}^n$ be the open unit disc. Define $\phi_{x_i>0} : U_{x_i>0} \to V$ via the formula

$$\phi_{x_i>0}(x_1, \ldots, x_{n+1}) = (x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}) \in V;$$

here $\widehat{x_i}$ indicates that this single coordinate should be omitted. Define $\phi_{x_i<0} : U_{x_i<0} \to V$ similarly. We claim that

$$\mathcal{A} = \{\phi_{x_i>0} : U_{x_i>0} \to V\}_{i=1}^{n+1} \cup \{\phi_{x_i<0} : U_{x_i<0} \to V\}_{i=1}^{n+1}$$

is a smooth atlas. Since the $U_{x_i>0}$ and $U_{x_i<0}$ clearly cover S^n, it is enough to check that the transition functions are smooth. As an illustration of this, we will verify that for $1 \leq i < j \leq n+1$ the transition function τ from $U_{x_i>0}$ to $U_{x_j>0}$ is smooth (all the other needed verifications are similar, and this will allow us to avoid introducing some terrible notation for the various special cases). Define $V_{ij} = \phi_i(U_{x_i>0} \cap U_{x_j>0})$ and $V_{ji} = \phi_j(U_{x_i>0} \cap U_{x_j>0})$, so V_{ij} consists of points $(y_1, \ldots, y_n) \in V$ such that $y_{j-1} > 0$ and V_{ji} consists of points $(y_1, \ldots, y_n) \in V$ such that $y_i > 0$. The transition function $\tau_{ji} : V_{ij} \to V_{ji}$ is then given by the formula

$$\begin{aligned}\tau_{ji}(y_1, \dots, y_n) &= \phi_{x_j>0}(\phi^{-1}_{x_i>0}(y_1, \dots, y_n))\\ &= \phi_{x_j>0}(y_1, \dots, y_{i-1}, \sqrt{1 - y_1^2 - \cdots - y_n^2}, y_i, \dots, y_n)\\ &= (y_1, \dots, y_{i-1}, \sqrt{1 - y_1^2 - \cdots - y_n^2}, y_i, \cdots, \widehat{y_{j-1}}, \dots, y_n).\end{aligned}$$

This is clearly a smooth function. □

EXAMPLE. Here is another smooth atlas for S^n. Let $U_1 = S^n \setminus \{(0, 0, 1)\}$ and $U_{-1} = S^n \setminus \{(0, 0, -1)\}$. Identifying $\mathbb{R}^n$ with the subspace of $\mathbb{R}^{n+1}$ consisting of points whose last coordinate is 0, define a function $\phi_1 : U_1 \to \mathbb{R}^n$ by letting $\phi_1(p)$ be the unique intersection point of the line joining $p \in U_1 \subset S^n \subset \mathbb{R}^{n+1}$ and $(0, 0, 1)$ with the plane $\mathbb{R}^n$. It is clear that ϕ_1 is a homeomorphism. Similarly, define $\phi_{-1} : U_{-1} \to \mathbb{R}^n$ by letting $\phi_{-1}(p)$ be the unique intersection point of the line joining $p \in U_{-1} \subset S^n \subset \mathbb{R}^{n+1}$ and $(0, 0, -1)$ with the plane $\mathbb{R}^n$. Again, ϕ_{-1} is a homeomorphism. In the exercises, you will show that the set $\{\phi_1 : U_1 \to \mathbb{R}^n, \phi_{-1} : U_{-1} \to \mathbb{R}^n\}$ is a smooth atlas for S^n which is equivalent to the smooth atlas for S^n given in the previous example. □

EXAMPLE. Define $\mathbb{RP}^n$ to be *real projective space*, i.e. the quotient $S^n/\sim$, where $\sim$ identifies antipodal points (that is, $x \sim -x$ for all $x \in S^n$). For $1 \leq i \leq n + 1$, define $U_i \subset \mathbb{RP}^n$ to be the image of $U_{x_i>0} \subset S^n$ under the quotient map $S^n \to \mathbb{RP}^n$. Since $U_{x_i>0}$ does not contain any antipodal points, the map $U_{x_i>0} \to U_i$ is a homeomorphism. Clearly the U_i cover $\mathbb{RP}^n$. Letting V be the unit disc in $\mathbb{R}^n$, we can define homeomorphisms $\phi_i : U_i \to V$ as the composition

$$U_i \cong U_{x_i>0} \xrightarrow{\phi_{x_i>0}} V.$$

The set $\mathcal{A} = \{\phi_i : U_i \to V\}_{i=1}^{n+1}$ then forms a smooth atlas for $\mathbb{RP}^n$; the fact that the transition maps for the sphere are smooth implies that the transition maps for $\mathcal{A}$ are. □

EXAMPLE. For $j = 1, 2$, let $M_j^{n_j}$ be a smooth n_j-dimensional manifold with smooth atlas $\{\phi_i^j : U_i^j \to V_i^j\}_{i\in I_j}$. Then $M_1^{n_1} \times M_2^{n_2}$ is a smooth $(n_1 + n_2)$-dimensional manifold with smooth atlas $\{\phi_i^1 \times \phi_{i'}^2 : U_i^1 \times U_{i'}^2 \to V_i^1 \times V_{i'}^2\}_{(i,i')\in I_1\times I_2}$. An important special case of a product is the *n-torus*, i.e. the product $S^1 \times \cdots \times S^1$ of n copies of S^1. □

For our final family of examples of smooth manifolds, we need the following definition.

DEFINITION. Let $X \subset \mathbb{R}^n$ be an arbitrary set and let $f : X \to \mathbb{R}^m$ be a function. We say that f is *smooth* if there exists an open set $U \subset \mathbb{R}^n$ with $X \subset U$ and a smooth function $g : U \to \mathbb{R}^n$ such that $g|_X = f$. If $Y \subset \mathbb{R}^m$ is the image of f, then we say that $f : X \to Y$ is a *diffeomorphism* if f is a homeomorphism and both $f : X \to Y$ and $f^{-1} : Y \to X$ are smooth. □

EXAMPLE. An *n-dimensional smooth submanifold of* $\mathbb{R}^m$ is a subset $M^n \subset \mathbb{R}^m$ such that for each point $p \in M^n$, there exists a chart $\phi : U \to V$ around p such that ϕ is a diffeomorphism. Here we emphasize that we are using the definition of diffeomorphism discussed in the previous definition. The collection of all such

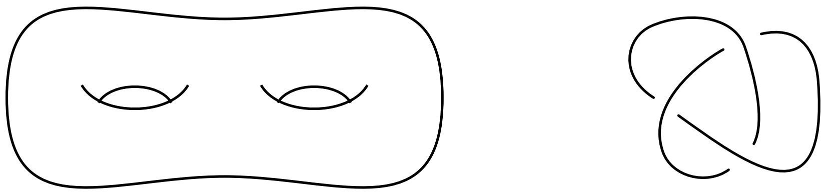

FIGURE 2.1. On the left is a genus 2 surface (a “donut with two holes”), which is a 2-dimensional smooth submanifold of $\mathbb{R}^3$. On the right is a trefoil knot, which is a 1-dimensional smooth submanifold of $\mathbb{R}^3$.

charts forms a smooth atlas on M^n; the fact that we require the charts to be diffeomorphisms makes the fact that the transition functions are smooth automatic. It is easy to draw many interesting examples of smooth submanifolds of $\mathbb{R}^3$; see, for example, the genus 2 surface and the knotted circle in Figure 2.1. □

REMARK. The charts in the first smooth atlas on S^n we gave above are diffeomorphisms, so we were really making use of the fact that S^n is an n-dimensional smooth submanifold of $\mathbb{R}^{n+1}$. □

REMARK. In fact, all smooth manifolds can be realized as smooth submanifolds of $\mathbb{R}^m$ for some $m \gg 0$ (in other words, all smooth manifolds can be “embedded” in $\mathbb{R}^m$). We will prove this for compact smooth manifolds in Theorem 5.1 below. □

2.3. Smooth functions

One of the reasons for introducing smooth atlases is to allow us to make the following definition; see the proof of the lemma that immediately follows.

DEFINITION. Let M^n be a smooth n-manifold and let $f : M^n \to \mathbb{R}$ be a function. We say that f is *smooth at a point* $p \in M^n$ if the following condition holds.

- Let $\phi : U \to V$ be a chart such that $p \in U$. Then the function $f \circ \phi^{-1} : V \to \mathbb{R}$ is smooth at $\phi(p)$; here recall that V is an open subset of $\mathbb{R}^n$, so smoothness for $f \circ \phi^{-1}$ means as in Chapter 1 that all of its mixed partial derivatives exist.

We say that f is *smooth* if it is smooth at all points $p \in M^n$. We will denote the set of all smooth functions on M^n by $C^\infty(M^n, \mathbb{R})$. □

LEMMA 2.1. *The notion of $f : M^n \to \mathbb{R}$ being smooth at a point $p \in M^n$ is well-defined, i.e. it does not depend on the choice of chart $\phi : U \to V$ such that $p \in U$.*

PROOF. Let $\phi_1 : U_1 \to V_1$ be another chart such that $p \in U_1$. We must prove that $f \circ \phi^{-1} : V \to \mathbb{R}$ is smooth at $\phi(p)$ if and only if $f \circ \phi_1^{-1} : V_1 \to \mathbb{R}$ is smooth at $\phi_1(p)$. Let $\tau : \phi(U \cap U_1) \to \phi_1(U \cap U_1)$ be the transition map between our two charts, so $\tau = \phi_1 \circ (\phi|_{U \cap U_1})^{-1}$. On $\phi(U \cap U_1)$, we have

$$f \circ \phi^{-1} = f \circ \phi_1^{-1} \circ \phi_1 \circ \phi^{-1} = f \circ \phi_1^{-1} \circ \tau.$$

Since τ is smooth, the function $f \circ \phi^{-1}$ is smooth at $\phi(p)$ if and only if the function $f \circ \phi_1^{-1}$ is smooth at $\phi_1(p)$, as desired. □

Definition. If $f : M^n \to \mathbb{R}$ is a smooth function on M^n and $\phi : U \to V$ is a chart on M^n, then the smooth function $f \circ \phi^{-1} : V \to \mathbb{R}$ will be called the *expression for f in the local coordinates V.* □

Remark. If M^n is a smooth submanifold of $\mathbb{R}^m$, then we now have two different definitions of what it means for a function $f : M^n \to \mathbb{R}$ to be smooth:

- The definition we just gave, and
- The definition given right before the definition of a smooth submanifold of $\mathbb{R}^m$, i.e. a function $f : M^n \to \mathbb{R}$ that can be extended to a smooth function $g : U \to \mathbb{R}$ for some open set $U \subset \mathbb{R}^m$ containing M^n.

In the exercises, you will prove that these two definitions are equivalent. By the way, this makes it easy to write down many examples of smooth functions. For example, the function $f : S^n \to \mathbb{R}$ defined via the formula

$$f(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} i x_i^{2i+1}$$

is smooth; here we are regarding S^n as a smooth submanifold of $\mathbb{R}^{n+1}$. □

Defining what it means for a map between arbitrary manifolds to be smooth is a little complicated. Consider the following example.

Example. Define a map $f : \mathbb{R} \to S^1$ via the formula $f(t) = (\cos(t), \sin(t)) \in S^1 \subset \mathbb{R}^2$. We clearly want f to be smooth. Recall that $\mathbb{R}$ is endowed with the smooth atlas with a single chart, namely the identity map $\mathbb{R} \to \mathbb{R}$. The image of this chart under f is not contained in any single chart for S^1, so we cannot define smoothness for f locally using this smooth atlas. □

The problem with the above example is that we really need to use “smaller” charts on $\mathbb{R}$. We now adapt the following convention to circumvent this.

Convention. If M^n is a smooth manifold with smooth atlas $\mathcal{A}$, then we will automatically enlarge $\mathcal{A}$ to the maximal atlas compatible with $\mathcal{A}$ (remember our equivalence relation on smooth atlases!). In particular, if $\phi : U \to V$ is a chart for M^n, then so is $\phi|_{U'} : U' \to \phi(U')$ for any open set $U' \subset U$. □

With this convention, we make the following definition.

Definition. Let $f : M_1^{n_1} \to M_2^{n_2}$ be a map between smooth manifolds. We say that f is *smooth at a point* $p \in M_1^{n_1}$ if there exist charts $\phi_1 : U_1 \to V_1$ for $M_1^{n_1}$ and $\phi_2 : U_2 \to V_2$ for $M_2^{n_2}$ with the following properties.

- $p \in U_1$.
- $f(U_1) \subset U_2$.
- The composition

  $$V_1 \xrightarrow{\phi_1^{-1}} U_1 \xrightarrow{f} U_2 \xrightarrow{\phi_2} V_2$$

  is smooth at $\phi_1(p)$; this makes sense since V_1 and V_2 are open subsets of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively.

We say that f is *smooth* if it is smooth at all points $p \in M_1^{n_1}$. We will denote the set of all smooth functions from $M_1^{n_1}$ to $M_2^{n_2}$ by $C^\infty(M_1^{n_1}, M_2^{n_2})$. A *diffeomorphism* is a smooth bijection whose inverse is also smooth. □

Just like for real-valued smooth functions, this does not depend on the choice of charts.

Definition. If $f : M_1^{n_1} \to M_2^{n_2}$ is a smooth function between smooth manifolds, $\phi_1 : U_1 \to V_1$ is a chart for $M_1^{n_1}$, and $\phi_2 : U_2 \to V_2$ is a chart for $M_2^{n_2}$ such that $f(U_1) \subset U_2$, then the smooth function $V_1 \to V_2$ obtained as the composition

$$V_1 \xrightarrow{\phi_1^{-1}} U_1 \xrightarrow{f} U_2 \xrightarrow{\phi_2} V_2$$

will be called the *expression for f in the local coordinates V_1 and V_2*. □

Example. It is immediate that the function $f : \mathbb{R} \to S^1$ discussed above defined via the formula $f(t) = (\cos(t), \sin(t)) \in S^1 \subset \mathbb{R}^2$ is smooth. □

Remark. Just as before, if M_1 and M_2 are smooth submanifolds of Euclidean space this definition agrees with the definition given just before the definition of smooth submanifolds. This allows us to write down many interesting examples of smooth maps. For example, regarding S^1 as a smooth submanifold of $\mathbb{R}^2$ we can define a smooth map $f : S^1 \to S^1$ via the formula $f(x_1, x_2) = (x_1^2 - x_2^2, 2x_1x_2)$. □

2.4. Manifolds with boundary

The following spaces are not manifolds.

Example. The set $\mathbb{D}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$ is not a manifold since the points of $S^{n-1} \subset \mathbb{D}^n$ do not have open neighborhoods in $\mathbb{D}^n$ homeomorphic to open subsets of $\mathbb{R}^n$. In particular, $[0, 1]$ is not a manifold. □

However, $\mathbb{D}^n$ is an example of a manifold with boundary, which we now define.

Notation. Define

$$\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$$

and

$$\partial\,\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\}.$$ □

Definition. A *smooth n-manifold with boundary* is a Hausdorff paracompact space M^n together with a smooth atlas $\{\phi_i : U_i \to V_i\}_{i \in I}$, which is defined exactly like for ordinary smooth manifolds except that now V_i is an open subset of $\mathbb{H}^n$. □

There is one subtle aspect of the above definition: since V_i is an open subset of $\mathbb{H}^n$, we need to be careful about what it means for the transition functions to be smooth. The correct definition of a smooth function on an arbitrary (not necessarily open) subset of $\mathbb{R}^n$ is as follows.

Definition. Let $X \subset \mathbb{R}^n$ be arbitrary and let $f : X \to \mathbb{R}^m$ be a function. We say that f is *smooth* if there exists an open set $U \subset \mathbb{R}^n$ such that $X \subset U$ as well as a smooth function $g : U \to \mathbb{R}^m$ such that $g|_X = f$. □

Smooth maps between manifolds with boundary are defined exactly like those between ordinary manifolds.

We now define the boundary of a smooth manifold with boundary.

Definition. Let M^n be a smooth manifold with boundary. The *boundary of M^n*, denoted ∂M^n, is the set of all points $x \in M^n$ such that there exists a chart $\phi : U \to V$ with $x \in U$ and $V \subset \mathbb{H}^n$ and $\phi(x) \in \partial\,\mathbb{H}^n$. The *interior of M^n*, denoted $\mathrm{Int}(M^n)$, is the set of all points $x \in M^n$ such that there exists a chart $\phi : U \to V$ with $x \in U$ and $V \subset \mathbb{H}^n$ and $\phi(x) \notin \partial\,\mathbb{H}^n$. □

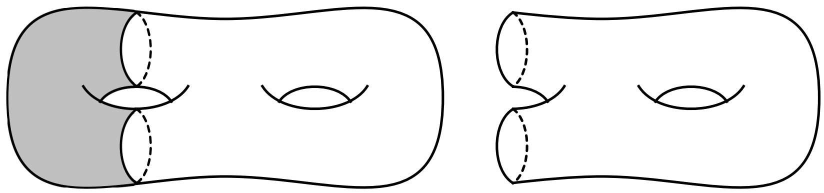

FIGURE 2.2. Removing the shaded submanifold of the genus 2 surface results in a surface with boundary whose boundary consists of the union of two circles.

Of course, with this definition it is not immediately obvious that ∂M^n is disjoint from $\text{Int}(M^n)$. However, the following lemma says that it is.

LEMMA 2.2. *Let M^n be a smooth manifold with boundary. Then $\partial M^n \cap \text{Int}(M^n) = \emptyset$.*

PROOF. To prove this, it is enough to prove that if $U \subset \mathbb{H}^n$ is an open set such that $U \cap \partial\,\mathbb{H}^n \neq \emptyset$, then there does not exist a diffeomorphism $f : U \to U'$, where $U' \subset \mathbb{H}^n$ satisfies $U' \cap \partial\,\mathbb{H}^n = \emptyset$. Assume that such a diffeomorphism $f : U \to U'$ exists. By definition, we can find an open set $V \subset \mathbb{R}^n$ such that $U = \mathbb{H}^n \cap V$ and a function $g : V \to \mathbb{R}^n$ such that $f = g|_U$. Let $p \in U \cap \partial\,\mathbb{H}^n$. Since f is a diffeomorphism, the derivative $D_p g = D_p f$ is an isomorphism. By Theorem 5.2 (the Implicit Function Theorem), the map g is a local diffeomorphism around p, i.e. there exists an open neighborhood V' of p such that $V' \subset V$ and such that g restricts to a diffeomorphism between V' and an open set W in $\mathbb{R}^n$. Since U' is open in $\mathbb{R}^n$ (after all, it does not intersect $\partial\,\mathbb{H}^n$), the set $U' \cap W$ is open in $\mathbb{R}^n$. But this implies that

$$g^{-1}(U' \cap W) = f^{-1}(U' \cap W) \subset \mathbb{H}^n$$

is an open subset of $\mathbb{R}^n$. Since $f^{-1}(U' \cap W)$ contains the point $p \in \partial\,\mathbb{H}^n$, this is impossible, as desired. □

We now discuss some examples.

EXAMPLE. Every smooth manifold is a smooth manifold with boundary. The point is that every open subset of $\mathbb{R}^n$ is diffeomorphic to an open subset of $\mathbb{H}^n$. The boundary of a smooth manifold is empty. □

EXAMPLE. The set $[0, 1]$ is a smooth 1-manifold with boundary and $\partial[0, 1]$ is $\{0, 1\}$. □

We will later prove (see Theorem 11.1) that all compact connected 1-manifolds with boundary are diffeomorphic to either S^1 or $[0, 1]$.

EXAMPLE. More generally, $\mathbb{D}^n$ is a smooth n-manifold with boundary and $\partial\mathbb{D}^n = S^{n-1}$. This is not hard to prove directly, but we will derive it from more general considerations in §5.6. □

EXAMPLE. Our final example will be intuitively plausible, but we will not be able to justify it until Chapter 5 (where it will appear in the exercises). Let M^n be a smooth n-manifold and let X^n be a smooth n-manifold with boundary that is a smooth submanifold of M^n (we have not yet defined what this means, but we hope that the idea is intuitively clear). Then $M^n \setminus \mathrm{Int}(X^n)$ is a smooth n-manifold with boundary and $\partial(M^n \setminus \mathrm{Int}(X^n)) = \partial X^n$. As an example, see Figure 2.2. This kind of example shows one important role played by manifolds with boundary: they appear during "cut-and-paste" operations on manifolds. □

2.5. Partitions of unity

We now introduce an important technical device. In calculus, we learned how to construct many interesting functions on open subsets of $\mathbb{R}^n$. To use these functions to prove theorems about manifolds, we need a tool for assembling local information into global information. This tool is called a *smooth partition of unity*, which we now define. Recall that if $f : M^n \to \mathbb{R}$ is a function, then the *support* of f, denoted $\mathrm{Supp}(f)$, is the closure of the set $\{x \in M^n \mid f(x) \neq 0\}$.

Definition. Let M^n be a smooth manifold with boundary and let $\{U_i\}_{i \in I}$ be an open cover of M^n. A *smooth partition of unity* subordinate to $\{U_i\}_{i=1}^k$ is a collection of smooth functions $\{f_i : M^n \to \mathbb{R}\}_{i \in I}$ satisfying the following properties.

- We have $0 \leq f_i(x) \leq 1$ for all $1 \leq i \leq k$ and $x \in M^n$.
- We have $\mathrm{Supp}(f_i) \subset U_i$ for all $1 \leq i \leq k$.
- For all $p \in M^n$, there exists an open neighborhood W of p such that the set $\{i \in I \mid W \cap \mathrm{Supp}(f_i) \neq \emptyset\}$ is finite.
- For all $p \in M^n$, we have $\sum_{i \in I} f_i(p) = 1$. This sum makes sense since the previous condition ensures that only finitely many terms in it are nonzero. □

Theorem 2.3 (Existence of partitions of unity). *Let M^n be a smooth manifold with boundary and let $\{U_i\}_{i \in I}$ be an open cover of M^n. Then there exists a smooth partition of unity subordinate to $\{U_i\}_{i \in I}$.*

For the proof of Theorem 2.3, we need the following lemma.

Lemma 2.4 (Bump functions, weak). *Let M^n be a smooth manifold with boundary, let $p \in M^n$ be a point, and let $U \subset M^n$ be a neighborhood of p. Then there exists a smooth function $f : M^n \to \mathbb{R}$ such that $0 \leq f(x) \leq 1$ for all $x \in M^n$, such that f equals 1 in some neighborhood of p, and such that $\mathrm{Supp}(f) \subset U$.*

Proof. We will construct f in a sequence of steps.

Step 1. *There exists a smooth function $g : \mathbb{R} \to \mathbb{R}$ such that $0 \leq g(x) \leq 1$ for all $x \in \mathbb{R}$, such that $g(x) = 1$ when $|x| \leq 1$, and such that $\mathrm{Supp}(g) \subset (-3, 3)$.*

Define $g_1 : \mathbb{R} \to \mathbb{R}$ via the formula

$$g_1(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-1/x} & \text{if } x > 0. \end{cases} \qquad (x \in \mathbb{R}).$$

The function g_1 is a smooth function such that $g_1(x) \geq 0$ for all $x \in \mathbb{R}$, such that $g_1(x) = 0$ when $x \leq 0$, and such that $g_1(x) > 0$ when $x > 0$. Next, define $g_2 : \mathbb{R} \to \mathbb{R}$ via the formula

$$g_2(x) = \frac{g_1(x)}{g_1(x) + g_1(1 - x)},$$

so g_2 is a smooth function such that $0 \leq g_2(x) \leq 1$ for all $x \in \mathbb{R}$, such that $g_2(x) = 0$ when $x \leq 0$, and such that $g_2(x) = 1$ when $x \geq 1$. Finally, define g via the formula

$$g(x) = g_1(2 + x) g_1(2 - x).$$

Clearly g satisfies the desired conditions.

STEP 2. *Let* $C_0 = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ *and* $U_0 = \{x \in \mathbb{R}^n \mid \|x\| < 2\}$. *Then there exists a smooth function* $h : \mathbb{R}^n \to \mathbb{R}$ *such that* $0 \leq h(x) \leq 1$ *for all* $x \in \mathbb{R}^n$, *such that* $h|_{C_0} = 1$, *and such that* $\mathrm{Supp}(h) \subset U_0$.

Let g be as in Step 1. Define h via the formula

$$h(x_1, \ldots, x_n) = g(x_1^2 + \cdots + x_n^2).$$

Clearly h satisfies the desired conditions.

STEP 3. *There exists a smooth function* f *as in the statement of the lemma.*

Let C_0 and U_0 and h be as in Step 2. We can then find an open set $U' \subset U$ such that $p \in U'$ and a diffeomorphism $\phi : U' \to V$, where V is either an open subset of $\mathbb{R}^n$ containing U_0 or an open subset of $\mathbb{H}^n$ containing $U_0 \cap \mathbb{H}^n$ and $\phi(p) = 0$. The function $f : M^n \to \mathbb{R}$ can then be defined via the formula

$$f(x) = \begin{cases} g(\phi(x)) & \text{if } x \in U', \\ 0 & \text{otherwise.} \end{cases} \qquad (x \in M^n).$$

Clearly f satisfies the conditions of the lemma. □

PROOF OF THEOREM 2.3. Since M^n is paracompact and locally compact, we can find open covers $\{U'_j\}_{j \in J}$ and $\{U''_j\}_{j \in J}$ of M^n with the following properties.

- The cover $\{U'_j\}_{j \in J}$ refines the cover $\{U_i\}_{i \in I}$, i.e. for all $j \in J$ there exists some $i_j \in I$ such that the closure of U'_j is contained in U_{i_j}.
- The cover $\{U'_j\}_{j \in J}$ is locally finite, i.e. for all $p \in M^n$ there exists some open neighborhood W of p such that $\{j \in J \mid W \cap U'_j \neq \emptyset\}$ is finite.
- The closure of U''_j is a compact subset of U'_j for all $j \in J$.

For each $p \in M^n$, choose j_p such that $p \in U''_{j_p}$ and use Lemma 2.4 to find a smooth function $g_p : M^n \to \mathbb{R}$ such that $0 \leq g_p(x) \leq 1$ for all $x \in M^n$, such that $\mathrm{Supp}(g_p) \subset U'_{j_p}$, and such that g_p equals 1 in some neighborhood V_p of p. Since the closure of U''_j in U'_j is compact for all $j \in J$, we can find a set $\{p_k\}_{k \in K}$ of points of M^n such that the set $\{V_{p_k} \mid k \in K,\, j_{p_k} = j\}$ is a finite cover of U''_j for all $j \in J$. For all $j \in J$, define $h_j : M^n \to \mathbb{R}$ to be the sum of all the g_{p_k} such that $j_{p_k} = j$ (a finite sum), so h_j is a smooth function such that $h_j(x) \geq 0$ for all $x \in M^n$, such that $h_j(x) > 0$ for all $x \in U''_j$, and such that $\mathrm{Supp}(h_j) \subset U'_j$. Finally, for all $i \in I$, define $f_i : M^n \to \mathbb{R}$ via the formula

$$f_i(x) = \frac{\sum_{i_j = i} h_j(x)}{\sum_{j \in J} h_j(x)} \qquad (x \in M^n).$$

These are not finite sums, but because the cover $\{U'_j\}_{j \in J}$ is locally finite and $\mathrm{Supp}(h_j) \subset U'_j$ for all $j \in J$, only finitely many terms in each are nonzero for any choice of $x \in M^n$ and the numerator and denominator are smooth functions. Also, the denominator is nonzero since $h_j(x) > 0$ for all $x \in U''_j$ and the set $\{U''_j\}_{j \in J}$ is a cover.

By construction, we have $\mathrm{Supp}(f_i) \subset U_i$. Moreover, for all $x \in M^n$ the fact that the cover $\{U'_j\}_{j \in J}$ is locally finite and $\mathrm{Supp}(h_j) \subset U'_j$ for all $j \in J$ implies that there exists some open neighborhood W of x such that the set $\{i \in I \mid W \cap \mathrm{Supp}(f_i) = \emptyset\}$

is finite. Finally, for all $x \in M^n$ we have

$$\sum_{i \in I} f_i(x) = \frac{\sum_{i \in I} \sum_{i_j = i} h_j(x)}{\sum_{j \in J} h_j(x)} = \frac{\sum_{j \in J} h_j(x)}{\sum_{j \in J} h_j(x)} = 1,$$

as desired. □

As a first illustration of how Theorem 2.3 can be used, we prove the following lemma.

Lemma 2.5 (Bump functions, strong). *Let M^n be a smooth manifold with boundary, let $C \subset M^n$ be a closed set, and let $U \subset M^n$ be an open set such that $C \subset U$. Then there exists a smooth function $f : M^n \to \mathbb{R}$ such that $0 \leq f(x) \leq 1$ for all $x \in M^n$, such that $f(x) = 1$ for all $x \in C$, and such that $\mathrm{Supp}(f) \subset U$.*

Proof. Set $U' = M^n \setminus C$. The set $\{U, U'\}$ is then an open cover of M^n. Using Theorem 2.3, we can find smooth functions $f : M^n \to \mathbb{R}$ and $g : M^n \to \mathbb{R}$ such that $0 \leq f(x), g(x) \leq 1$ for all $x \in M^n$, such that $\mathrm{Supp}(f) \subset U$ and $\mathrm{Supp}(g) \subset U'$, and such that $f + g = 1$. The function f then satisfies the conditions of the lemma. □

This has the following useful consequence. Just like for functions on Euclidean space, if C is an arbitrary subset of a smooth manifold M_1 and $f : C \to M_2$ is a function to another smooth manifold, then f is said to be smooth if there exists an open set $U \subset M_1$ containing C and a smooth function $g : U \to M_2$ such that $g|_C = f$.

Lemma 2.6 (Extending smooth functions). *Let M be a smooth manifold with boundary, let $C \subset M$ be a closed set, and let $U \subset M$ be an open set such that $C \subset U$. Let $f : C \to \mathbb{R}$ be a smooth function. Then there exists a smooth function $g : M \to \mathbb{R}$ such that $g|_C = f$ and such that $\mathrm{Supp}(g) \subset U$.*

Proof. By definition, there exists an open set $U' \subset M$ containing C and a smooth function $g_1 : U' \to \mathbb{R}$ such that $g_1|_C = f$. Shrinking U' if necessary, we can assume that $U' \subset U$. Use Lemma 2.5 to construct a smooth function $h : M \to \mathbb{R}$ such that $0 \leq h(x) \leq 1$ for all $x \in M$, such that $h(x) = 1$ for all $x \in C$, and such that $\mathrm{Supp}(h) \subset U'$. Define $g : M \to \mathbb{R}$ via the formula

$$g(x) = \begin{cases} h(x)g_1(x) & \text{if } x \in U', \\ 0 & \text{otherwise.} \end{cases} \qquad (x \in M).$$

Clearly g satisfies the conclusions of the lemma. □

2.6. Approximating continuous functions, I

As another illustration of how partitions of unity can be used, we will prove the following.

Theorem 2.7. *Let M^n be a smooth manifold with boundary and let $f : M^n \to \mathbb{R}^m$ be a continuous function. Then for all $\epsilon > 0$ there exists a smooth function $g : M^n \to \mathbb{R}^m$ such that $\|f(x) - g(x)\| < \epsilon$ for all $x \in M^n$.*

Remark. If M^n is not compact, then it is often useful to require that $\|f(x) - g(x)\| < \epsilon(x)$ for all $x \in M^n$, where $\epsilon : M^n \to \mathbb{R}$ is a fixed function such that $\epsilon(x) > 0$ for all $x \in M^n$. The proof is exactly the same. □

Remark. We will later use an important tool called the tubular neighborhood theorem to generalize Theorem 2.7 to show that continuous functions between arbitrary smooth manifolds can be approximated in an appropriate sense by smooth functions; see Theorem 6.5. □

For the proof of Theorem 2.7, we need the following lemma.

Lemma 2.8. *Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}^m$ be a continuous function such that* $\text{Supp}(f) \subset U$*. Then for all $\epsilon > 0$ there exists a smooth function $g : U \to \mathbb{R}^m$ such that* $\text{Supp}(g) \subset U$ *and such that $\|f(x) - g(x)\| < \epsilon$ for all $x \in M^n$.*

Proof. The Stone-Weierstrass theorem says that we can find a smooth function $g_1 : U \to \mathbb{R}^m$ such that $\|f(x) - g_1(x)\| < \epsilon$ for all $x \in U$ (in fact, it says that we can take g_1 to be a function whose coordinate functions are polynomials). Let $C = \text{Supp}(f)$, so C is a closed subset of U. Using Lemma 2.5, we can find a smooth function $\beta : U \to \mathbb{R}$ such that $0 \leq \beta(x) \leq 1$ for all $x \in U$, such that $\beta|_C = 1$, and such that $\text{Supp}(\beta) \subset U$. Define $g : U \to \mathbb{R}^m$ via the formula $g(x) = \beta(x) \cdot g_1(x)$. Since $\text{Supp}(\beta) \subset U$, we also have $\text{Supp}(g) \subset U$. Also, we clearly have $\|f(x) - g(x)\| < \epsilon$ for all $x \in C$. For $x \in U \setminus C$, we have $f(x) = 0$, so $\|g_1(x)\| < \epsilon$ and hence

$$\|f(x) - g(x)\| = \|\beta(x) \cdot g_1(x)\| \leq \|g_1(x)\| < \epsilon,$$

as desired. □

Proof of Theorem 2.7. In the exercises, you will construct a smooth atlas $\mathcal{A} = \{\phi_i : U_i \to V_i\}_{i \in I}$ for M^n and a large integer K such that for all $p \in M^n$, there exists a neighborhood W of p with $|\{i \in I \mid U_i \cap W \neq \emptyset\}| < K$. We remark that this is trivial if M^n is compact. Using Theorem 2.3, we can find a smooth partition of unity $\{\nu_i : U_i \to \mathbb{R}\}_{i \in I}$ subordinate to $\{U_i\}_{i \in I}$. Define $f_i : M^n \to \mathbb{R}^m$ via the formula $f_i(x) = \nu_i(x) \cdot f(x)$. We thus have

$$\sum_{i \in I} f_i(x) = (\sum_{i \in I} \nu_i(x)) \cdot f(x) = f(x) \qquad (x \in M^n).$$

These sums makes sense since only finitely many terms in them are nonzero for any fixed $x \in M^n$. Moreover, $\text{Supp}(f_i) \subset U_i$. Define $\widehat{f_i} : V_i \to \mathbb{R}^m$ to be the expression for f_i in the local coordinates V_i, so $\widehat{f_i} = f \circ \phi_i^{-1}$. Applying Lemma 2.8, we can find a smooth function $\widehat{g}_i : V_i \to \mathbb{R}^m$ such that $\text{Supp}(\widehat{g}_i) \subset V_i$ and such that $\|\widehat{f_i}(x) - \widehat{g}_i(x)\| < \epsilon/K$ for all $x \in V_i$. Define $g_i : M^n \to \mathbb{R}^m$ via the formula

$$g_i(x) = \begin{cases} \widehat{g}_i(\phi_i(x)) & \text{if } x \in U_i, \\ 0 & \text{otherwise} \end{cases} \qquad (x \in M^n).$$

Since $\text{Supp}(\widehat{g}_i) \subset V_i$, this is a smooth function on M^n satisfying $\text{Supp}(g_i) \subset U_i$. Moreover, $\|f_i(x) - g_i(x)\| < \epsilon/K$ for all $x \in M^n$. Define $g : M^n \to \mathbb{R}^m$ via the formula

$$g(x) = \sum_{i \in I} g_i(x) \qquad (x \in M^n);$$

this makes sense because $\text{Supp}(g_i) \subset U_i$, and hence only finitely many terms in this sum are nonzero for any fixed $x \in M^n$. The function g is a smooth function and

$$\|f(x) - g(x)\| = \|\sum_{i \in I}(f_i(x) - g_i(x))\| \leq \sum_{i \in I} \|f_i(x) - g_i(x)\| < K(\epsilon/K) = \epsilon,$$

as desired. ☐

The following "relative" version of Theorem 2.7 will also be useful.

THEOREM 2.9. *Let M^n be a smooth manifold with boundary and let $f : M^n \to \mathbb{R}^m$ be a continuous function. Assume that $f|_U$ is smooth for some open set U. Then for all $\epsilon > 0$ and all closed sets $C \subset M^n$ with $C \subset U$, there exists a smooth function $g : M^n \to \mathbb{R}^m$ such that $\|f(x) - g(x)\| < \epsilon$ for all $x \in M^n$ and such that $g|_C = f|_C$.*

PROOF. The proof is very similar to the proof of Theorem 2.7, so we only describe how it differs. The key is to choose the smooth atlas $\mathcal{A} = \{\phi_i : U_i \to V_i\}_{i \in I}$ for M^n at the beginning of the proof such that if $U_i \cap C \neq \emptyset$ for some $i \in I$, then $U_i \subset U$. For $i \in I$ with $U_i \subset U$, we can then take our "approximating functions" $\widehat{g}_i$ to simply equal $\widehat{f}_i$, and thus $g_i = f_i$. These choices ensure that the function $g : M^n \to \mathbb{R}^m$ constructed in the proof of Theorem 2.7 satisfies $g|_C = f|_C$, as desired. ☐

CHAPTER 3

The tangent bundle

In this chapter, we will construct the tangent bundle of a smooth manifold and describe how to differentiate smooth functions. We will then discuss vector fields and show how then can be integrated to flows. Finally, as an application we will prove that if M is a smooth manifold and $p, q \in M$ are points, then there exists a diffeomorphism $f : M \to M$ such that $f(p) = q$.

3.1. Tangent spaces

Let M^n be a smooth n-manifold and let $p \in M^n$. Our first goal is to construct an n-dimensional vector space $T_p M^n$ called the tangent space to M^n at p. If $\phi : U \to V$ is a chart around p, then vectors in $T_p M^n$ should be represented by elements of $T_{\phi(p)} V = \mathbb{R}^n$. To make a definition that does not depend on any particular choice of chart, we introduce the following equivalence relation.

Definition. Let M^n be a smooth n-manifold, let $p \in M^n$, and let $\{\phi_i : U_i \to V_i\}_{i \in I}$ be the set of charts around p. For $i, j \in I$, let $\tau_{ji} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ be the transition function from U_i to U_j. Finally, let $\mathcal{X}(M^n, p)$ be the set of pairs $(i, \vec{v})$, where $i \in I$ and $\vec{v} \in T_{\phi_i(p)} V_i$. Define $\sim$ to be the relation on $\mathcal{X}(M^n, p)$ where where $(i, \vec{v}) \sim (j, \vec{w})$ when $(D_{\phi_i(p)} \tau_{ji})(\vec{v}) = \vec{w}$. □

Lemma 3.1. *The relation $\sim$ defined in the previous definition is an equivalence relation on $\mathcal{X}(M^n, p)$.*

Proof. We must check reflexivity, symmetry, and transitivity.

For $(i, \vec{v}) \in \mathcal{X}(M^n, p)$, we have $(i, \vec{v}) \sim (i, \vec{v})$ since the relevant transition function $\tau_{ii} : \phi_i(U_i \cap U_i) \to \phi_i(U_i \cap U_i)$ is the identity.

If $(i, \vec{v}), (j, \vec{w}) \in \mathcal{X}(M^n, p)$ satisfy $(i, \vec{v}) \sim (j, \vec{w})$, then by definition we have $(D_{\phi_i(p)} \tau_{ji})(\vec{v}) = \vec{w}$. From its definition, we see that $\tau_{ij} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is the inverse of $\tau_{ji} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$. From Theorem 1.1 (the Chain Rule I), we have $(D_{\phi_j(p)} \tau_{ij}) \circ (D_{\phi_i(p)} \tau_{ji}) = \text{id}$, so $(D_{\phi_j(p)} \tau_{ij})(\vec{w}) = \vec{v}$ and hence $(j, \vec{w}) \sim (i, \vec{v})$.

If $(i, \vec{v}), (j, \vec{w}), (k, \vec{u}) \in \mathcal{X}(M^n, p)$ satisfy $(i, \vec{v}) \sim (j, \vec{w})$ and $(j, \vec{w}) \sim (k, \vec{u})$, then by definition we have $(D_{\phi_i(p)} \tau_{ji})(\vec{v}) = \vec{w}$ and $(D_{\phi_j(p)} \tau_{kj})(\vec{w}) = \vec{u}$. From its definition, we see that on $\phi_i(U_i \cap U_j \cap U_k)$ we have $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$. Again using Theorem 1.1 (the Chain Rule I), we see that $D_{\phi_i(p)} \tau_{ki} = (D_{\phi_j(p)} \tau_{kj}) \circ (D_{\phi_i(p)} \tau_{ji})$, so $(D_{\phi_i(p)} \tau_{ki})(\vec{v}) = \vec{u}$ and hence $(i, \vec{v}) \sim (k, \vec{u})$. □

This allows us to make the following definition.

Definition. Let M^n be a smooth manifold and let $p \in M^n$. Let $\{\phi_i : U_i \to V_i\}_{i \in I}$ be the set of charts around p. The *tangent space to M^n at p*, denoted $T_p M^n$, is the set of equivalence classes of elements of $\mathcal{X}(M^n, p)$ under the equivalence relation given by Lemma 3.1. □

Lemma 3.2. *Let M^n be a smooth manifold and let $p \in M^n$. Then the tangent space T_pM^n is an n-dimensional vector space.*

Proof. This follows from the fact that the derivatives used to define the equivalence relation are vector space isomorphisms, so the vector space structures on the various $T_{\phi_i(p)}V_i$ used to define T_pM^n descend to a vector space structure on T_pM^n. □

Convention. The notation $\mathcal{X}(M^n, p)$ that we used when defining T_pM^n will not be used again. In the future, instead of talking about elements of T_pM^n being equivalence classes of pairs $(i, \vec{v})$, we will simply say that a given element of T_pM^n is represented by some $\vec{v} \in T_{\phi_i(p)}V_i$. □

3.2. Derivatives I

Let $f : M_1^{n_1} \to M_2^{n_2}$ be a smooth map between smooth manifolds and let $p \in M_1^{n_1}$. We now show how to construct the derivative $D_pf : T_pM_1^{n_1} \to T_{f(p)}M_2^{n_2}$, which is a linear map between these vector spaces. Let $\phi_1 : U_1 \to V_1$ be a chart around p and let $\phi_2 : U_2 \to V_2$ be a chart around $\phi(p)$ such that $f(U_1) \subset U_2$. We thus have identifications $T_pM_1^{n_1} = T_{\phi_1(p)}V_1$ and $T_{f(p)}M_2^{n_2} = T_{\phi_2(f(p))}V_2$. We define $D_pf : T_pM_1^{n_1} \to T_{f(p)}M_2^{n_2}$ to be composition

$$T_pM_1^{n_1} \xrightarrow{=} T_{\phi_1(p)}V_1 \xrightarrow{D_{\phi_1(p)}(\phi_2 \circ f \circ \phi_1^{-1})} T_{\phi_2(f(p))}V_2 \xrightarrow{=} T_{f(p)}M_2^{n_2}.$$

Lemma 3.3. *This does not depend on the choice of charts.*

Proof. This is in the exercises; it provides good practice in the various identifications we have made. □

Theorem 1.1 (the Chain Rule I) immediately implies the following version of the chain rule.

Theorem 3.4 (Manifold Chain Rule I). *Let $f : M_1^{n_1} \to M_2^{n_2}$ and $g : M_2^{n_2} \to M_3^{n_3}$ be smooth maps between smooth manifolds. Then for all $p \in M_1^{n_1}$ we have*

$$D_p(g \circ f) = (D_{f(p)}g) \circ (D_pf).$$

3.3. The tangent bundle

Let M^n be a smooth manifold. The goal of this section is to construct the tangent bundle of M^n. Recall that if $V \subset \mathbb{R}^n$ is an open subset, then $TV = V \times \mathbb{R}^n$. This contains all the individual tangent spaces T_pV for $p \in V$, namely $T_pV = \{p\} \times \mathbb{R}^n \subset TV$. We wish to do a similar thing with the tangent spaces T_pM^n for $p \in M^n$. The result will be a $2n$-dimensional smooth manifold TM^n.

Just like for the tangent spaces, we will define TM^n using an equivalence relation.

Definition. Let M^n be a smooth n-manifold with smooth atlas $\{\phi_i : U_i \to V_i\}_{i \in I}$. For $i, j \in I$, let $\tau_{ji} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ be the transition function from U_i to U_j. Finally, let $\mathcal{Y}(M^n)$ be the set of triples $(i, p, \vec{v})$, where $i \in I$ and $p \in U_i$ and $\vec{v} \in T_{\phi_i(p)}V_i$. Define $\sim$ to be the relation on $\mathcal{Y}(M^n)$ where $(i, p, \vec{v})$ and $(j, q, \vec{w})$ satisfy $(i, p, \vec{v}) \sim (j, q, \vec{w})$ when $p = q$ and $(D_p\tau_{ji})(\vec{v}) = \vec{w}$. □

Lemma 3.5. *The relation $\sim$ defined in the previous definition is an equivalence relation on $\mathcal{Y}(M^n)$.*

PROOF. Immediate from Lemma 3.1. ☐

This allows us to make the following definition.

DEFINITION. Let M^n be a smooth manifold with smooth atlas $\{\phi_i : U_i \to V_i\}_{i \in I}$. The *tangent bundle of* M^n, denoted TM^n, is the set of equivalence classes of elements of $\mathcal{Y}(M^n)$ under the equivalence relation given by Lemma 3.5. ☐

We can identify $\mathcal{Y}(M^n)$ with the disjoint union of all the TV_i by identifying $(i, p, \vec{v})$ with $(\phi_i(p), \vec{v}) \in TV_i$. This endows $\mathcal{Y}(M^n)$ with a topology. We give TM^n the quotient topology, so by definition, a set $U \subset TM^n$ is open if its preimage under the projection

$$\mathcal{Y}(M^n) \xrightarrow{\text{mod } \sim} TM^n$$

is open. Under this projection, each TV_i maps injectively into TM^n; as temporary notation, let its image be $\overline{TV_i} \subset TM^n$. Since $TV_i = V_i \times \mathbb{R}^n$ is an open subset of $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ and there is an evident (and trivial) homeomorphism $\psi_i : \overline{TV_i} \to TV_i$, we deduce that TM^n is a manifold. Even better, the set $\{\psi_i : \overline{TV_i} \to TV_i\}_{i \in I}$ is a smooth atlas: the transition function from $\overline{TV_i}$ to $\overline{TV_j}$ equals the derivative $D\tau_{ji} : T\phi_i(U_i \cap U_j) \to T\phi_j(U_i \cap U_j)$ of the transition function $\tau_{ji} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$, which is clearly smooth. We have proved the following theorem.

THEOREM 3.6. *Let M^n be a smooth n-manifold. Then the tangent bundle TM^n of M^n is a smooth $2n$-dimensional manifold.*

CONVENTION. Just like for the tangent space, we will never again use the notation $\mathcal{Y}(M^n)$ or the formalism of triples $(i, p, \vec{v})$ when discussing TM^n. Instead, we will say that a given point of TM^n is represented by a given point of TV_i. ☐

REMARK. See §3.5 for a discussion of how to visualize the tangent bundle. ☐

3.4. Derivatives II

If $f : M_1 \to M_2$ is a smooth map between smooth manifolds, then we previously have defined linear maps $D_p f : T_p M_1 \to T_{f(p)} M_2$ for all $p \in M_1$. These piece together to define a map $Df : TM_1 \to TM_2$ that restricted to the subspace $T_p M_1$ of TM_1 equals $D_p f$. It is clear that this is a smooth map. Just like for Theorem 1.2 (Chain Rule II), Theorem 3.4 implies the following.

THEOREM 3.7 (Manifold Chain Rule II). *Let $f : M_1^{n_1} \to M_2^{n_2}$ and $g : M_2^{n_2} \to M_3^{n_3}$ be smooth maps between smooth manifolds. Then*

$$D(g \circ f) = (Dg) \circ (Df).$$

3.5. Visualizing the tangent bundle

Our construction of the tangent bundle was very abstract. In the case of smooth submanifolds of $\mathbb{R}^m$, there is a simpler construction which is a great aid to visualization. Consider a smooth submanifold $M^n \subset \mathbb{R}^m$. For $p \in M^n$, we can regard $T_p M^n$ as a subspace of $T_p \mathbb{R}^m = \mathbb{R}^m$ in the following way. By definition, there is a diffeomorphism $\phi : U \to V$, where $U \subset M^n$ is an open neighborhood of p and $V \subset \mathbb{R}^n$ is an open set. The inverse ϕ^{-1} can be regarded as a smooth map from V to $\mathbb{R}^m$, and thus it has a derivative

$$D_{\phi(p)} \phi^{-1} : T_{\phi(p)} V \to T_p \mathbb{R}^m = \mathbb{R}^m.$$

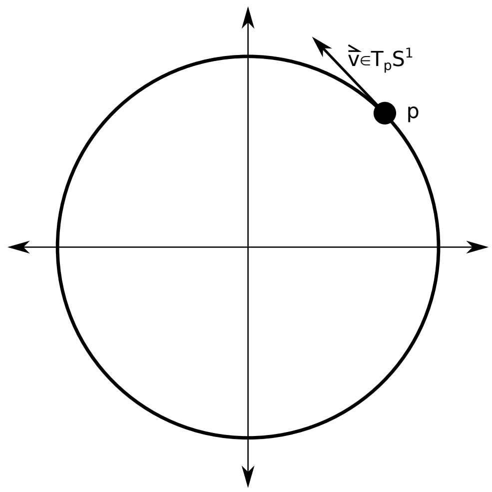

FIGURE 3.1. A vector $\vec{v} \in T_p S^1$ is orthogonal to the line from 0 to p.

The image of this derivative can be identified with the tangent space $T_p M^n$; it is easy to see that it does not depend on the choice of diffeomorphism $\phi : U \to V$.

Using this, we can regard the tangent bundle TM^n as the subspace

$$\{(p, \vec{v}) \in T\mathbb{R}^m \mid p \in M^n, \, \vec{v} \in T_p M^n \subset T_p \mathbb{R}^m\} \subset T\mathbb{R}^m = \mathbb{R}^m \times \mathbb{R}^m.$$

This results in the familar picture of tangent vectors to M^n as being arrows in $\mathbb{R}^m$ that "point in the direction of the tangent plane to M^n".

EXAMPLE. For $S^n \subset \mathbb{R}^{n+1}$, you will prove in the exercises that

$$TS^n = \{(p, \vec{v}) \in T\mathbb{R}^{n+1} \mid \|p\| = 1 \text{ and } \vec{v} \text{ is orthogonal to the line from 0 to } p\}.$$

See Figure 3.1. □

The derivative map can also be understood from this perspective. Let $M_1^{n_1} \subset \mathbb{R}^{m_1}$ and $M_2^{n_2} \subset \mathbb{R}^{m_2}$ be smooth submanifolds of Euclidean space and let $f : M_1^{n_1} \to M_2^{n_2}$ be a smooth map. By definition, this means that there exists an open set $U \subset \mathbb{R}^{m_1}$ and a smooth map $g : U \to \mathbb{R}^{m_2}$ such that $g|_{M_1^{n_1}} = f$. As discussed in Chapter 1 (our review of multivariable calculus), the map g induces a derivative map $Dg : TU \to T\mathbb{R}^{m_2}$; on $T_pU \subset TU$ for $p \in U$, this is just the linear derivative map $D_p g : T_p U \to T_{g(p)} \mathbb{R}^{m_2}$. The derivative $Df : TM_1^{n_1} \to TM_2^{n_2}$ is then just the restriction of Dg to $TM_1^{n_1} \subset TU$; this image of this restriction lies in $TM_2^{n_2} \subset T\mathbb{R}^{m_2}$.

Often the smooth map $f : M_1^{n_1} \to M_2^{n_2}$ is given by a formula which can be extended to an open set U (often all of $\mathbb{R}^{m_1}$, or at least $\mathbb{R}^{m_1}$ minus some isolated points where the formula has a singularity). Using this formula, it is easy to use the above recipe to work out the effect of Df.

3.6. Directional derivatives

Let M be a smooth manifold, let $p \in M$, and let $\vec{v} \in T_pM$. Our goal in this section is to construct a linear map $\nabla_{\vec{v}}$ from the set $C^\infty(M^n, \mathbb{R})$ of smooth real-valued functions on M^n to $\mathbb{R}$; for $f \in C^\infty(M^n, \mathbb{R})$, the value $\nabla_{\vec{v}}(f) \in \mathbb{R}$ will be called the *directional derivative* of f in the direction $\vec{v}$.

Consider a smooth function $f : M^n \to \mathbb{R}$. The derivative $D_p f$ is a linear map from T_pM^n to $T_{f(p)}\mathbb{R} = \mathbb{R}$. We define

$$\nabla_{\vec{v}}(f) = (D_p f)(\vec{v}) \in \mathbb{R}.$$

This can be easily related to the usual directional derivative from multivariable calculus. Namely, if $\phi : U \to V$ is a chart around p and $g : V \to \mathbb{R}$ is the expression for f in the local coordinates V (so $g = f \circ \phi^{-1}$), then we can regard $\vec{v}$ as an element of $T_{\phi(p)}V$ and $\nabla_{\vec{v}}(f)$ is easily seen to be the usual multivariable calculus directional derivative of g in the direction $\vec{v}$.

The operator $\nabla_{\vec{v}}$ has the following properties.

LEMMA 3.8. *Let M be a smooth manifold and let $p \in M$. The following hold.*

(1) For $\vec{v} \in T_pM$ and $f, g \in\in C^\infty(M^n, \mathbb{R})$, we have

$$\nabla_{\vec{v}}(f + g) = \nabla_{\vec{v}}(f) + \nabla_{\vec{v}}(g)$$

and

$$\nabla_{\vec{v}}(fg) = \nabla_{\vec{v}}(f) \cdot g(p) + f(p) \cdot \nabla_{\vec{v}}(g).$$

(2) For $\vec{v}, \vec{w} \in T_pM$ and $c, d \in \mathbb{R}$ and $f \in C^\infty(M^n, \mathbb{R})$, we have

$$\nabla_{c\vec{v}+d\vec{w}}(f) = c\nabla_{\vec{v}}(f) + d\nabla_{\vec{w}}(f).$$

PROOF. These properties are inherited from corresponding properties of directional derivatives of functions defined on open subsets of Euclidean space. □

REMARK. A linear map $\Psi : C^\infty(M^n, \mathbb{R}) \to \mathbb{R}$ such that

$$\Psi(fg) = \Psi(f) \cdot g(p) + f(p) \cdot \Psi(g) \qquad (f, g \in C^\infty(M^n, \mathbb{R}))$$

is called a *derivation of $C^\infty(M^n, \mathbb{R})$ at p*. In the exercises, you will prove that every derivation Ψ at p equals $\nabla_{\vec{v}}$ for some $\vec{v} \in T_pM$. Many sources define tangent vectors as derivations. □

3.7. Manifolds with boundary

Let M^n be a smooth n-manifold with boundary. The constructions of this chapter go through with little change to define the tangent space T_pM^n for $p \in M^n$ and the tangent bundle TM^n. The only potentially confusing point is that one has to define $TV = V \times \mathbb{R}^n$ for any open subset V of $\mathbb{H}^n$. The tangent space T_pM^n is thus an n-dimensional vector even when $p \in \partial M^n$; tangent vectors on ∂M^n are allowed to point "outwards".

3.8. Vector bundles

The tangent bundle TM of a smooth manifold M is an example of a vector bundle over M, whose definition is as follows. We will not use other vector bundles very often, but they will show up in a few places.

Definition. Let X be a topological space. A *k-dimensional vector bundle* over X is a topological space E together with a continuous map $\pi : E \to X$ such that the following hold for all all $x \in X$.

- The preimage $\pi^{-1}(x)$ is equipped with the structure of a k-dimensional vector space. We will denote this vector space by E_x.
- There exists an open neighborhood $U \subset X$ of x and a homeomorphism $\psi : U \times \mathbb{R}^k \to \pi^{-1}(U)$ such that for all $y \in U$, we have $\psi(\{y\} \times \mathbb{R}^k) = E_y$ and the composition
$$\mathbb{R}^k \xrightarrow{\cong} \{y\} \times \mathbb{R}^k \xrightarrow{\psi} E_y$$
is a vector space isomorphism.

The second condition is called *local triviality.* If X and E are smooth manifolds and both $\pi : E \to X$ and all the isomorphisms ψ appearing above are smooth, then E is a *smooth vector bundle.* □

Example. Let M^n be an n-dimensional smooth manifold. The projection $\pi : TM^n \to M^n$ taking T_pM^n to p makes TM^n into a smooth n-dimensional vector bundle over M^n. Indeed, the preimage $\pi^{-1}(M^n)$ is the n-dimensional vector space T_pM^n. Moreover, by definition for every chart $\phi : U \to V$ of M^n we have $\pi^{-1}(U) \cong TV \cong V \times \mathbb{R}^n$. □

Example. If X is a topological space, then $E = X \times \mathbb{R}^k$ is a k-dimensional vector bundle over X whose map $\pi : E \to X$ is simply the projection onto the first factor. This will be called the *trivial k-dimensional vector bundle* over X. If X is a smooth manifold, then this is a smooth vector bundle. □

The vector bundles that we will use will all be built out of the tangent bundle using linear-algebraic operations. Rather than prove a general theorem about such operations, we will give several examples.

Construction. Fix a topological space X, and for $i = 1, 2$, let $\pi_i : E_i \to X$ be a k_i-dimensional vector bundle over X. Define
$$E_1 \oplus E_2 = \{(e_1, e_2) \in E_1 \times E_2 \mid \pi(e_1) = \pi(e_2)\}$$
and let $\rho : E_1 \oplus E_2 \to X$ be the map taking (e_1, e_2) to $\pi(e_1)$. You will prove in the exercises that $\rho : E_1 \oplus E_2 \to X$ is a $(k_1 + k_2)$-dimensional vector bundle over X such that for $x \in X$ the vector space $(E_1 \oplus E_2)_x$ is the vector space $(E_1)_x \oplus (E_2)_x$. □

Construction. Let $\pi : E \to X$ be a k-dimensional vector bundle. Define
$$E^* = \{(x, \tau) \mid x \in X \text{ and } \tau : E_x \to \mathbb{R} \text{ is a linear map}\}$$
and let $\rho : E^* \to X$ take (x, τ) to x. You will prove in the exercises that $\rho : E^* \to X$ is a k-dimensional vector bundle over X such that for $x \in X$ the fiber $(E^*)_x$ is the dual vector space $(E_x)^*$. This is called the *dual bundle* to X. The dual bundle of the tangent bundle of a smooth manifold M is the *cotangent bundle* and is denoted T^*M. □

Construction. Let $\pi : E \to X$ be a k-dimensional vector bundle. Define
$$\wedge^i E = \{(x, \vec{v}) \mid x \in X \text{ and } \vec{v} \in \wedge^i E_x\}$$
and let $\rho : E^* \to X$ take $(x, \vec{v})$ to x. You will prove in the exercises that $\rho : \wedge^i E \to X$ is a $\binom{k}{i}$-dimensional vector bundle over X such that for $x \in X$ the fiber $(\wedge^i E)_x$ is the wedge product $\wedge^i E_x$. □

Remark. All of the above constructions take smooth vector bundles to smooth vector bundles. □

Maps between vector bundles are defined as follows.

Definition. A *vector bundle map* between vector bundles $\pi_1 : E_1 \to X_1$ and $\pi_2 : E_2 \to X_2$ is a pair of continuous maps $f : X_1 \to X_2$ and $g : E_1 \to E_2$ with the following two properties.

- We have $\pi_2 \circ g = f \circ \pi_1$, i.e. the diagram
$$\begin{array}{ccc} E_1 & \xrightarrow{g} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$
commutes.
- The previous condition implies that for $x \in X_1$, the map g restricts to a map from the vector space $(E_1)_x$ to the vector space $(E_2)_x$. We require that this map be linear.

If $X_1 = X_2 = X$ and $f = \text{id}$, then we say that this is a *vector bundle map over* X. We will often not write f and simply say that $g : E_1 \to E_2$ is a vector bundle map. If $g : E_1 \to E_2$ is a bijective map of vector bundles over X and g^{-1} is continuous, then we will call g an *isomorphism*. □

Example. If $f : M_1 \to M_2$ is a smooth map between smooth manifolds, then the derivative $Df : TM_1 \to TM_2$ is a vector bundle map. □

This allows us to define our final four vector bundle operations.

Construction. For $i = 1, 2$, let $\pi_i : E_i \to X$ be a k_i-dimensional vector bundle and let $g : E_1 \to E_2$ be a vector bundle map over X. Assume that the vector space map $(E_1)_x \to (E_2)_x$ induced by g is surjective for all $x \in X$. Define $\ker(g)$ to be the set of pairs

$$\{(x, \vec{v}) \mid x \in X \text{ and } \vec{v} \text{ lies in the kernel of the map } (E_1)_x \to (E_2)_x \text{ induced by } g\}$$

and let $\rho : \ker(g) \to X$ to be the map taking $(x, \vec{v})$ to x. You will prove in the exercises that $\rho : \ker(g) \to X$ is a $(k_1 - k_2)$-dimensional vector bundle over X such that for $x \in X$ the fiber $\ker(g)_x$ is the kernel of the map $(E_1)_x \to (E_2)_x$ induced by g. □

Construction. For $i = 1, 2$, let $\pi_i : E_i \to X$ be a k_i-dimensional vector bundle and let $g : E_1 \to E_2$ be a vector bundle map over X. Assume that the vector space map $(E_1)_x \to (E_2)_x$ induced by g is injective for all $x \in X$. Define $\text{coker}(g)$ to be the set of pairs

$$\{(x, \vec{v}) \mid x \in X \text{ and } \vec{v} \text{ lies in the quotient vector space } (E_2)_x / g((E_1)_x)\}$$

and let $\rho : \text{coker}(g) \to X$ to be the map taking $(x, \vec{v})$ to x. You will prove in the exercises that $\rho : \text{coker}(g) \to X$ is a $(k_2 - k_1)$-dimensional vector bundle over X such that for $x \in X$ the fiber $\text{coker}(g)_x$ is the quotient $(E_2)_x / g((E_1)_x)$. □

Construction. Let $\pi : E \to X$ be a k-dimensional vector bundle and let $f : Y \to X$ be a continuous map. Define

$$f^*(E) = \{(y, e) \mid y \in Y,\, e \in E,\, f(y) = \pi(e)\} \subset Y \times E.$$

The projection $Y \times E \to Y$ restricts to a map $f^*(\pi) : f^*(E) \to Y$. In the exercises, you will prove that $f^*(E)$ is a k-dimensional vector bundle with $f^*(E)_y = E_{f(y)}$ for all $y \in Y$. This fits into a map of vector bundles

$$\begin{array}{ccc} f^*(E) & \longrightarrow & E \\ {\scriptstyle f^*(\pi)}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\;f\;} & X \end{array}$$

where the top row is the restriction of the projection $Y \times E \to E$. We will call $f^*(E)$ the *pull-back* of E along f. □

Example. If X is a topological space, $X \times \mathbb{R}^k$ is the trivial k-dimensional vector bundle, and $f : Y \to X$ is any continuous map, then $f^*(X \times \mathbb{R}^k)$ is isomorphic as a vector bundle over Y to the trivial k-dimensional vector bundle $Y \times \mathbb{R}^k$. Indeed, by definition we have

$$f^*(X \times \mathbb{R}^k) = \{(y, (x, \vec{v})) \in Y \times (X \times \mathbb{R}^k) \mid y \in Y\};$$

the vector bundle isomorphism simply takes $(y, (x, \vec{v})) \in f^*(X \times \mathbb{R}^k)$ to $(y, \vec{v}) \in Y \times \mathbb{R}^k$. □

Construction. If X is a topological space, $\pi : E \to X$ is a vector bundle, and $Y \subset X$ is a subspace, then the *restriction* of E to Y, denoted $E|_Y$, is the pullback of E along the inclusion map $Y \hookrightarrow X$. □

Remark. Again, all of the above constructions take smooth vector bundles to smooth vector bundles. □

CHAPTER 4

Vector fields

In this chapter, we discuss some basic results about vector fields, including their integral curves and flows. As an application, we will prove that if M^n is a connected smooth manifold and $p, q \in M^n$, then there exists a diffeomorphism $f : M^n \to M^n$ such that $f(p) = q$.

4.1. Definition and basic examples

Let M^n be a smooth manifold with boundary. Intuitively, a smooth vector field on M^n is a smoothly varying choice of vector T_pM^n for each $p \in M^n$. More precisely, a smooth vector field on M^n is a smooth map $\nu : M^n \to TM^n$ such that $\nu(p) \in T_pM^n$ for all $p \in M^n$. Let $\mathfrak{X}(M^n)$ be the set of smooth vector fields on M^n. The vector space structures on each T_pM^n together endow $\mathfrak{X}(M^n)$ with the structure of a real vector space (infinite dimensional unless M^n is a compact 0-manifold).

If $\nu \in \mathfrak{X}(M^n)$ and $\phi : U \to V$ is a chart on M^n, then the *expression for ν in the local coordinates* V is the function $\eta : V \to \mathbb{R}^n$ such that $\eta(\phi(p)) \in T_{\phi(p)}V = \mathbb{R}^n$ represents $\nu(p)$ for all $p \in U$.

It is particularly easy to write down smooth vector fields on smooth submanifolds M^n of $\mathbb{R}^m$. Namely, recall that the embedding of M^n in $\mathbb{R}^m$ identifies each T_pM^n with an n-dimensional subspace of $T\mathbb{R}^m = \mathbb{R}^m$. A smooth vector field on M^n can thus be identified with a smooth map $\nu : M^n \to \mathbb{R}^m$ such that $\nu(p) \in T_pM^n \subset \mathbb{R}^m$ for each $p \in M^n$. We warn the reader that this is *different* from the expressions for ν in local coordinates defined above.

Example. Consider an odd-dimensional sphere $S^{2n-1} \subset \mathbb{R}^{2n}$. Recall that

$$TS^{2n-1} = \{(p, \vec{v}) \in T\mathbb{R}^{2n} \mid \|p\| = 1 \text{ and } \vec{v} \text{ is orthogonal to the line from 0 to } p\}.$$

We can then define a smooth vector field on S^{2n-1} via the formula

$$\nu(x_1, \ldots, x_{2n}) = (x_2, -x_1, x_4, -x_3, \ldots, x_{2n}, -x_{2n-1}) \in T_{(x_1,\ldots,x_{2n})}S^{2n-1} \subset \mathbb{R}^m$$

for each $(x_1, \ldots, x_{2n}) \in S^{2n-1}$. The smooth vector field ν has the property that $\nu(p) \neq 0$ for all $p \in S^{2n-1}$. We will later prove the "hairy ball theorem", which asserts that no such nonvanishing smooth vector field exists on an even-dimensional sphere. See Theorem 14.1. □

Example. Let M^n be a smooth submanifold of $\mathbb{R}^m$ and let $f : M^n \to \mathbb{R}$ be a smooth function. We can then define a smooth vector field $\text{grad}(f)$ on M^n in the following way. Consider $p \in M^n$. We can define a linear map $\eta_p : T_pM^n \to \mathbb{R}$ via the formula

$$\eta_p(\vec{v}) = \mathfrak{X}_{\vec{v}}(f).$$

This is linear because of the second conclusion of Lemma 3.8. Let $\omega(\cdot,\cdot)$ be the usual inner product on $\mathbb{R}^m$. There then exists a unique vector $\text{grad}(f)(p) \in T_pM^n$ such that

$$\eta_p(\vec{v}) = \omega(\text{grad}(f)(p), \vec{v}) \qquad (\vec{v} \in T_pM^n).$$

It is easy to see that this map $\text{grad}(f) : M^n \to TM^n$ is a smooth vector field. □

Remark. In the construction of $\text{grad}(f)$, we used the embedding of M^n into $\mathbb{R}^m$ to obtain an inner product on each T_pM^n. More generally, a Riemannian metric on M^n is a choice of a nondegenerate symmetric bilinear form on each T_pM^n that varies smoothly in an appropriate sense. Given a Riemannian metric on M^n, we can define a smooth vector field $\text{grad}(f)$ on M^n for any smooth function $F : M^n \to \mathbb{R}$ via the above procedure. □

Given a smooth vector field ν on M^n, we can define a map $\nabla_\nu : C^\infty(M^n,\mathbb{R}) \to C^\infty(M^n,\mathbb{R})$ by setting

$$\nabla_\nu(f)(p) = \nabla_{\nu(p)}(f) \qquad (f \in C^\infty(M^n,\mathbb{R}), p \in M^n).$$

This has the following properties.

Lemma 4.1. *Let M^n be a smooth manifold with boundary. The following then hold.*

(1) *For $\nu \in \mathfrak{X}(M^n)$ and $f, g \in C^\infty(M^n,\mathbb{R})$, we have*

$$\nabla_\nu(f+g) = \nabla_\nu(f) + \nabla_\nu(g)$$

and

$$\nabla_\nu(fg) = \nabla_\nu(f)\cdot g + f\cdot\nabla_\nu(g).$$

(2) *For $\nu_1, \nu_2 \in \mathfrak{X}(M^n)$ and $c, d \in \mathbb{R}$ and $f \in C^\infty(M^n,\mathbb{R})$, we have*

$$\nabla_{c\nu_1+d\nu_2}(f) = c\nabla_{\nu_1}(f) + d\nabla_{\nu_2}(f).$$

Proof. Immediate from Lemma 3.8. □

4.2. Extending vector fields

We now prove a vector field version of Lemma 2.6 (Extending smooth functions). First, some preliminaries. If M is a smooth manifold with boundary and $\nu \in \mathfrak{X}(M)$, then the *support* of ν, denoted $\text{Supp}(\nu)$, is the closure of the set of points $p \in M$ such that $\nu(p) \neq 0$. If $C \subset M$ is an arbitrary set, then the notion of a vector field on C can be defined in the obvious way. A vector field ν on C is said to be *smooth* if there exists an open subset $U \subset M$ containing C and a smooth vector field η on U such that $\eta|_C = \nu$.

Lemma 4.2 (Extending smooth vector fields). *Let M be a smooth manifold with boundary, let $C \subset M$ be a closed set, and let $U \subset M$ be an open set such that $C \subset U$. Let ν be a smooth vector field on C. Then there exists a smooth vector field η on M such that $\eta|_C = \nu$ and such that* $\text{Supp}(\eta) \subset U$.

Proof. By definition, there exists an open set $U' \subset M$ containing C and a smooth vector field function η_1 on U' such that $\eta_1|_C = \nu$. Shrinking U' if necessary, we can assume that $U' \subset U$. Use Lemma 2.5 to construct a smooth function

$h : M \to \mathbb{R}$ such that $0 \leq h(x) \leq 1$ for all $x \in M$, such that $h(x) = 1$ for all $x \in C$, and such that $\text{Supp}(h) \subset U'$. Define a vector field η on M via the formula

$$\eta(x) = \begin{cases} h(x)\eta_1(x) & \text{if } x \in U', \\ 0 & \text{otherwise.} \end{cases} \qquad (x \in M).$$

Clearly η satisfies the conclusions of the lemma. □

4.3. Integral curves of vector fields

Let M be a smooth manifold with boundary and let $\nu \in \mathfrak{X}(M)$. Informally, an integral curve of ν is a smoothly embedded curve that moves in the direction of ν. To make this precise, if $U \subset \mathbb{R}$ is a connected open set and $\gamma : U \to M$ is a smooth map, then for $t \in U$ we define $\gamma'(t) \in T_{\gamma(t)}M$ to be the image under the map $D_t\gamma : T_tU \to T_{\gamma(t)}M$ of the element $1 \in T_tU = \mathbb{R}^n$. The curve γ is an *integral curve of* ν if $U = \mathbb{R}$ and $\gamma'(t) = \nu(\gamma(t))$ for all $t \in \mathbb{R}$. Our main theorem then is as follows.

THEOREM 4.3 (Existence of integral curves). *Let M be a smooth manifold with boundary and let $\nu \in \mathfrak{X}(M)$. Assume that* $\text{Supp}(\nu)$ *is a compact subset of* $\text{Int}(M^n)$*. Then for all $p \in M$, there a unique integral curve γ of ν such that $\gamma(0) = p$.*

REMARK. The hypothesis that $\text{Supp}(\nu)$ is compact holds automatically if M is compact. □

REMARK. The theorem is not necessarily true if $\text{Supp}(\nu)$ is not compact. For instance, if $M = \mathbb{R}^n$, then an integral curve could diverge to infinity in finite time and thus not be defined for all points of $\mathbb{R}$. Similarly, the theorem is not necessarily true if $\text{Supp}(\nu)$ contains points of ∂M^n. The problem is if it contain such points, then an integral curve could cross the boundary and "leave the manifold" in finite time. □

The key technical input to the proof is the following lemma.

LEMMA 4.4. *Consider a chain of open sets $V'' \subset V' \subset V \subset \mathbb{R}^n$ such that the closure of V'' is a compact subset of V' and such that the closure of V' is a compact subset of V. Consider $\nu \in \mathfrak{X}(V)$. Then there is an $\epsilon > 0$ such that for all $p \in V''$, there exists a smooth map $\gamma : (-\epsilon, \epsilon) \to V$ such that $\gamma(0) = p$ and $\gamma'(t) = \nu(\gamma(t))$ for all $t \in (-\epsilon, \epsilon)$. The curve γ is unique in the following sense: if for some $\delta > 0$ there is another smooth map $\lambda : (-\delta, \delta) \to V$ with $\lambda(0) = p$ and $\lambda'(t) = \nu(\lambda(t))$ for all $t \in (-\delta, \delta)$, then $\gamma(t) = \lambda(t)$ for all $t \in (-\epsilon, \epsilon) \cap (-\delta, \delta)$.*

PROOF. This is simply a restatement into our language of the usual existence and uniqueness for solutions of systems of ordinary differential equations. □

This lemma provides the local result needed for the following.

LEMMA 4.5. *Let M be a smooth manifold with boundary and let $\nu \in \mathfrak{X}(M)$. Assume that* $\text{Supp}(\nu)$ *is a compact subset of* $\text{Int}(M)$*. There then exists some $\epsilon > 0$ such that for all $p \in M$, there exists a smooth map $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(t) = \nu(\gamma(t))$ for all $t \in (-\epsilon, \epsilon)$. The curve γ is unique in the following sense: if for some $\delta > 0$ there is another smooth map $\lambda : (-\delta, \delta) \to M$ with $\lambda(0) = p$ and $\lambda'(t) = \nu(\lambda(t))$ for all $t \in (-\delta, \delta)$, then $\gamma(t) = \lambda(t)$ for all $t \in (-\epsilon, \epsilon) \cap (-\delta, \delta)$.*

PROOF. Let $\{U_i\}_{i=1}^k$ and $\{U_i'\}_{i=1}^k$ and $\{U_i''\}_{i=1}^k$ be finite open covers of the compact set $\text{Supp}(\nu)$ such that the following hold for all $1 \leq i \leq k$.

- There exists a chart $\phi_i : U_i \to V_i$.
- The set U_i lies in $\text{Int}(M)$.
- The closure of U_i' is a compact subset of U_i.
- The closure of U_i'' is a compact subset of U_i'.

For $1 \leq i \leq k$, we can apply Lemma 4.4 to find some $\epsilon_i > 0$ such that for all $p \in U_i''$, there exists a smooth map $\gamma : (-\epsilon_i, \epsilon_i) \to U_i$ with $\gamma(0) = 0$ and $\gamma'(t) = \nu(\gamma(t))$ for all $t \in (-\epsilon_i, \epsilon_i)$. Let $\epsilon > 0$ be the minimum of the ϵ_i. Then the desired curve $\gamma : (-\epsilon, \epsilon) \to M$ exists and is unique for all $p \in \text{Supp}(\nu)$. But for $p \notin \text{Supp}(\nu)$ we have $\nu(p) = 0$, and thus the desired curve is the constant curve $\gamma : (\epsilon, \epsilon) \to M$ defined by $\gamma(t) = p$ for all t. □

PROOF OF THEOREM 4.3. Let $\epsilon > 0$ be the constant given by Lemma 4.5 and let $p \in M$. For $k \geq 1$, we will prove that there exists a unique smooth function $\gamma_k : (-k\epsilon/2, k\epsilon/2) \to M$ such that $\gamma_k(0) = p$ and $\gamma_k'(t) = \nu(\gamma_k(t))$ for all $t \in (-k\epsilon/2, k\epsilon/2)$. Before we do that, observe that the uniqueness of γ_k implies that $\gamma_{k+1}(t) = \gamma_k(t)$ for $t \in (-k\epsilon/2, k\epsilon/2)$, so the desired integral curve $\gamma : \mathbb{R} \to M$ can be defined by $\gamma(t) = \gamma_k(t)$, where k is chosen large enough such that $t \in (-k\epsilon/2, k\epsilon/2)$. The uniqueness of our integral curve follows from the uniqueness of the γ_k.

It remains to construct the γ_k. This construction will be inductive. First, we can use Lemma 4.5 to construct and prove unique the desired $\gamma_1 : (-\epsilon/2, \epsilon/2) \to M$ (in fact, we could ensure that γ_1 was defined on $(-\epsilon, \epsilon)$, but this will simplify our inductive procedure). Now assume that γ_k has been constructed and proven to be unique. Set $q_k = \gamma_k((k-1)\epsilon/2)$ and $r_k = \gamma_k(-(k-1)\epsilon/2)$. Another application of Lemma 4.5 implies that there exists smooth functions $\zeta_k : (-\epsilon, \epsilon) \to M$ and $\kappa_k : (-\epsilon, \epsilon) \to M$ such that

$$\zeta_k(0) = p_k \quad \text{and} \quad \kappa_k(0) = r_k$$

and such that

$$\zeta_k'(t) = \nu(\zeta_k(t)) \quad \text{and} \quad \kappa_k'(t) = \nu(\kappa_k(t))$$

for all $t \in (-\epsilon, \epsilon)$. The uniqueness statement in Lemma 4.5 implies that

$$\zeta_k(t) = \gamma_k((k-1)\epsilon/2 + t) \quad \text{and} \quad \kappa_k(t) = \gamma_k(-(k-1)\epsilon/2 + t)$$

for all $t \in (-\epsilon/2, \epsilon/2)$. The desired function $\gamma_{k+1} : (-(k+1)\epsilon/2, (k+1)\epsilon/2) \to M$ is then defined via the formula

$$\gamma_{k+1}(t) = \begin{cases} \kappa_k(t + (k-1)\epsilon/2) & \text{if } -(k+1)\epsilon/2 < t < -(k-1)\epsilon/2, \\ \gamma_k(t) & \text{if } -k\epsilon/2 < t < k\epsilon/2, \\ \zeta_k(t - (k-1)\epsilon/2) & \text{if } (k-1)\epsilon/2 < t < (k+11)\epsilon/2. \end{cases}$$

Its uniqueness follows from the uniqueness statement in Lemma 4.5. □

4.4. Flows

Let M be a smooth manifold with boundary and let $\nu \in \mathfrak{X}(M)$. In this section, we use the results of the previous section to prove an important theorem which says that in most cases ν determines a flow, that is, a family of diffeomorphisms of M that move points in the direction of ν. More precisely, a *flow on M in the direction* *of* ν consists of smooth maps $f_t : M \to M$ for each $t \in \mathbb{R}$ with the following properties.

- For all $t \in \mathbb{R}$, the map f_t is a diffeomorphism.
- Define $F : M \times \mathbb{R} \to M$ via the formula $F(p,t) = f_t(p)$. Then F is smooth.
- For all $t, s \in \mathbb{R}$, we have $f_{t+s} = f_t \circ f_s$. In particular, $f_0 = \mathrm{id}$.
- For all $p \in M$, define $\gamma_p : \mathbb{R} \to M$ via the formula $\gamma_p(t) = f_t(p)$. Then γ_p is an integral curve for ν starting at p.

Our main theorem is as follows.

THEOREM 4.6 (Existence of flows). *Let M be a smooth manifold with boundary and let $\nu \in \mathfrak{X}(M)$ be such that* $\mathrm{Supp}(\nu)$ *is a compact subset of* $\mathrm{Int}(M)$*. Then there exists a unique flow on M in the direction of ν.*

REMARK. Since $\mathrm{Supp}(\nu) \subset \mathrm{Int}(M)$, the flow in the direction of ν fixes ∂M pointwise. □

PROOF OF THEOREM 4.6. Theorem 4.3 implies that for all $p \in M$, there exists a unique integral curve $\gamma_p : \mathbb{R} \to M$ for ν starting at p. From the uniqueness of this integral curve, we see that

$$\gamma_p(s+t) = \gamma_{\gamma_p(s)}(t) \qquad (p \in M, s, t \in \mathbb{R}). \tag{1}$$

Define $F : M \times \mathbb{R} \to M$ via the formula $F(p,t) = \gamma_p(t)$. It follows from the smooth dependence on initial conditions of solutions to systems of ordinary differential equations that F is smooth. For $t \in \mathbb{R}$, define $f_t : M \to M$ via the formula $f_t(p) = F(p,t)$ for $p \in M$. The equation (1) implies that $f_{s+t} = f_s \circ f_t$ for all $s, t \in \mathbb{R}$. Since $f_0 = \mathrm{id}$ by construction, this implies that $f_{-t} \circ f_t = \mathrm{id}$ for all $t \in \mathbb{R}$, and hence each f_t is a diffeomorphism. The theorem follows. □

4.5. Moving points around by diffeomorphisms

As an application of the results in the previous section, we prove the following useful theorem.

THEOREM 4.7. *Let M be a connected smooth manifold with boundary and let $p, q \in \mathrm{Int}(M)$ be points. Then there exists a diffeomorphism $f : M \to M$ such that $f(p) = q$. In fact, f can be chosen as f_1 for some flow f_t on M.*

PROOF. Since M is connected, there exists a continuous function $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$. In fact, in the exercises you will show that we can choose γ such that it is a smooth homeomorphism onto its image. Set $C = \gamma([0,1])$. Let ν be the vector field on C defined via $\nu(p) = \gamma'(t)$, where $t \in [0,1]$ is such that $\gamma(t) = p$. Let $U \subset M$ be an open set containing C such that the closure of U is compact. Using Lemma 4.2 (Extending smooth vector fields), we can find a smooth vector field η on M such that $\eta|_C = \nu$ and such that $\mathrm{Supp}(\eta) \subset U$; in particular, $\mathrm{Supp}(\eta)$ is compact. Theorem 4.6 (Existence of flows) says that there is a flow $f_t : M \to M$ in the direction of η. By construction, the restriction to $[0,1]$ of the integral curve for η starting at p equals γ, so we deduce that $f_1(p) = q$, as desired. □

CHAPTER 5

The structure of smooth maps

In this section, we will discuss features of smooth maps, mostly focusing on local properties. Highlights include the fact that every manifold can be embedded in Euclidean space (see §5.2 and §5.9), the Brouwer fixed point theorem (see §5.11), and a topological proof of the fundamental theorem of algebra (see §5.10).

5.1. Embeddings

The first type of map we will discuss are embeddings, which are defined as follows.

DEFINITION. Let M_1 be a smooth manifold with boundary and M_2 be a smooth manifold. A smooth map $f : M_1 \to M_2$ is an *embedding* if f is a homeomorphism onto its image (i.e. a topological embedding) and the derivative map $D_p f : T_p M_1 \to T_{f(p)} M_2$ is injective for all $p \in M_1$. □

REMARK. The correct definition of an embedding $f : M_1 \to M_2$ when M_2 is a manifold with boundary is a little subtle, so we prefer to not give it. In general, manifolds with boundary are technical devices, so we do not dwell on them unless we are forced to. □

The canonical example is as follows.

EXAMPLE. If M^n is a smooth submanifold of $\mathbb{R}^m$, then the inclusion map $M^n \hookrightarrow \mathbb{R}^m$ is an embedding. □

More generally, we make the following definition.

DEFINITION. If $f : M_1 \to M_2$ is an embedding from a smooth manifold with boundary M_1 into a smooth manifold M_2, then we will call the image of f a *smooth submanifold* of M_2. □

We thus have two different definitions of smooth submanifolds of Euclidean space, one in terms of charts and the other as the image of an embedding. In the exercises, you will prove that these two definitions are equivalent and also show that smooth submanifolds of arbitrary manifolds can be characterized in terms of charts.

5.2. Embedding manifolds in Euclidean space I

We now prove that every smooth manifold with boundary can be realized as a smooth submanifold of Euclidean space.

THEOREM 5.1. *If M^n is a compact smooth manifold with boundary, then for some $m \gg 0$ there exists an embedding $f : M^n \to \mathbb{R}^m$.*

REMARK. This is also true for noncompact manifolds manifolds with boundary, though the proof is a little more complicated. Whitney proved a difficult theorem that says that we can take $m = 2n$. Later, we will prove a much easier theorem that says that we can take $m = 2n + 1$; see Theorem 5.9 below. □

PROOF OF THEOREM 5.1. Since M^n is compact, there exists a finite atlas

$$\mathcal{A} = \{\phi_i : U_i \to V_i\}_{i=1}^k.$$

Choose open subsets $W_i \subset U_i$ such that $\{W_i\}_{i=1}^k$ is still a cover of M^n and such that the closure of W_i in U_i is compact. Using Lemma 2.5, we can find a smooth function $\nu_i : M^n \to \mathbb{R}$ such that $(\nu_i)|_{W_i} = 1$ and $(\nu_i)|_{M^n \setminus U_i} = 0$. Next, define a function $\eta_i : M^n \to \mathbb{R}^n$ via the formula

$$\eta_i(p) = \begin{cases} \nu_i(p) \cdot \phi_i(p) & \text{if } p \in U_i, \\ 0 & \text{otherwise.} \end{cases}$$

Here we are regarding the image of ϕ_i as lying in $\mathbb{R}^n$ even though technically it lies in $\mathbb{H}^n$. Clearly η_i is a smooth function. Finally, define $f : M^n \to \mathbb{R}^{k(n+1)}$ via the formula

$$f(p) = (\nu_1(p), \eta_1(p), \ldots, \nu_k(p), \eta_k(p)).$$

The function f is then a smooth map. In the exercises you will prove that f is an embedding. □

5.3. Local diffeomorphisms

The next property of smooth maps we will study is as follows.

DEFINITION. Let $f : M_1 \to M_2$ be a smooth map between smooth manifolds with boundary and let $p \in M_1$. The map f is a *local diffeomorphism at* p if there exists an open neighborhood U_1 of p such that $U_2 := f(U_1)$ is an open subset of M_2 and $f|_{U_1} : U_1 \to U_2$ is a diffeomorphism. The map f is a *local diffeomorphism* if it is a local diffeomorphisms at all points. □

REMARK. This implies that M_1 and M_2 have the same dimension. □

EXAMPLE. Let $f : \mathbb{R} \to S^1$ be the smooth map defined via the formula $f(t) = (\cos(t), \sin(t)) \in S^1 \subset \mathbb{R}^2$. Then f is a local diffeomorphism. Since f is not injective, f is not itself a diffeomorphism. □

EXAMPLE. Recall that $\mathbb{RP}^n$ is the quotient space of S^n via the equivalence relation $\sim$ that identifies antipodal points $x \in S^n$ and $-x \in S^n$. The projection map $f : S^n \to \mathbb{RP}^n$ is a smooth map which is a local diffeomorphism. □

The following is an easy criterion for recognizing a local diffeomorphism. As we will see, it is essentially a restatement of the implicit function theorem.

THEOREM 5.2 (Implicit Function Theorem). *Let $f : M_1 \to M_2$ be a smooth map between smooth manifolds with boundary and let $p \in \text{Int}(M_1)$. Then f is a local diffeomorphism at $p \in M_1$ if and only if the linear map $D_p f : T_p M_1 \to T_{f(p)} M_2$ is an isomorphism.*

PROOF. Assume first that f is a local diffeomorphism at $p \in M_1$ and let $U_1 \subset \text{Int}(M_1)$ be an open neighborhood of p such that $U_2 := f(U_1)$ is open and $f|_{U_1} : U_1 \to U_2$ is a diffeomorphism. Replacing U_1 with a smaller open subset if necessary, we can find charts $\phi_1 : U_1 \to V_1$ for M_1 and $\phi_2 : U_2 \to V_2$ for M_2. Let $F : V_1 \to V_2$ be the expression for f in these local coordinates, i.e. the composition

$$V_1 \xrightarrow{\phi_1^{-1}} U_1 \xrightarrow{f} U_2 \xrightarrow{\phi_2} V_2.$$

Setting $q = \phi_1(p)$, we have identifications $T_q V_1 \cong T_p M_1$ and $T_{F(q)} V_2 = T_{f(q)} M_2$, and it is enough to prove that $D_q F : T_q V_1 \to T_{F(q)} V_2$ is an isomorphism. Since F is a diffeomorphism, it has an inverse $G : V_2 \to V_1$. Applying Theorem 1.1 (Chain Rule I) to $\mathrm{id}_{V_1} = G \circ F$, we see that

$$\mathrm{id} = D_q \mathrm{id}_{V_1} = (D_{F(q)} G) \circ (D_p F).$$

Similarly, we have

$$\mathrm{id} = D_{F(q)} \mathrm{id}_{V_2} = (D_p F) \circ (D_{F(q)} G).$$

We conclude that $D_p F$ is an isomorphism, as desired.

Now assume conversely that the linear map $D_p f : T_p M_1 \to T_{f(p)} M_2$ is an isomorphism. Choose charts $\phi_1 : U_1 \to V_1$ for M_1 and $\phi_2 : U_2 \to V_2$ for M_2 such that $p \in U_1$ and $f(U_1) \subset U_2$ and $U_1 \subset \mathrm{Int}(M_1)$ and $U_2 \subset \mathrm{Int}(M_2)$. Let $F : V_1 \to V_2$ be the expression for f in these local coordinates, i.e. the composition

$$V_1 \xrightarrow{\phi_1^{-1}} U_1 \xrightarrow{f} U_2 \xrightarrow{\phi_2} V_2.$$

Setting $q = \phi_1(p)$, our assumptions imply that $D_q F : T_q V_1 \to T_{F(q)} V_2$ is an isomorphism. Since V_1 and V_2 are open subsets of Euclidean space, we can now apply the ordinary inverse function theorem to deduce that F is a local diffeomorphism at q. This implies that f is a local diffeomorphism at p, as desired. □

5.4. Immersions

We now turn to the following property.

DEFINITION. Let $f : M_1 \to M_2$ be a smooth map between smooth manifolds with boundary and let $p \in M_1$. The map f is an *immersion at* p if the derivative $D_p f : T_p M_1 \to T_{f(p)} M_2$ is an injective linear map. The map f is an *immersion* if it is an immersion at all points. □

REMARK. This implies that the dimension of M_2 is at least the dimension of M_1. □

EXAMPLE. If $f : M_1 \to M_2$ is a local diffeomorphism at p, then f is an immersion at p. □

EXAMPLE. If $f : M \to \mathbb{R}^m$ is an embedding of a smooth manifold, then f is an immersion. □

EXAMPLE. Consider the smooth map $f : \mathbb{R} \to \mathbb{R}^2$ whose image is as in Figure 5.1. Then f is an immersion but is not an embedding. □

EXAMPLE. If M_1 and M_2 are smooth manifolds and $x \in M_2$, then the map $f : M_1 \to M_1 \times M_2$ defined via the formula $f(p) = (p, x)$ is an immersion. □

The following theorem says that all immersions look locally like the final example above.

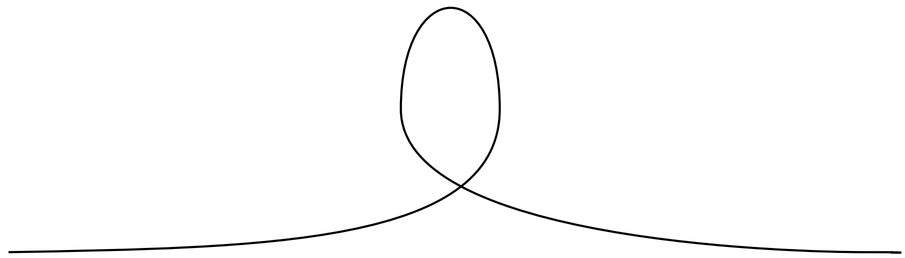

FIGURE 5.1. An immersion $f : \mathbb{R} \to \mathbb{R}^2$ that is not an embedding.

THEOREM 5.3 (Local Immersion Theorem). *Let $f : M_1^{n_1} \to M_2^{n_2}$ be a smooth map between smooth manifolds with boundary that is an immersion at $p \in \mathrm{Int}(M_1^{n_1})$. There then exists an open neighborhood $U_1 \subset M_1^{n_1}$ of p and an open subset $U_2 \subset M_2^{n_2}$ satisfying $f(U_1) \subset U_2$ such that the following hold. There exists an open subset $W \subset \mathbb{R}^{n_2-n_1}$, a point $w \in W$, and a diffeomorphism $\psi : U_2 \to U_1 \times W$ such that the composition*

$$U_1 \xrightarrow{f} U_2 \xrightarrow{\psi} U_1 \times W$$

takes $u \in U_1$ to $(u, w) \in U_1 \times W$.

PROOF. Choose charts $\phi_1 : U_1 \to V_1$ for M_1 and $\phi_2 : U_2 \to V_2$ for M_2 such that $p \in U_1$ and $f(U_1) \subset U_2$ and $U_1 \subset \mathrm{Int}(M_1)$ and $U_2 \subset \mathrm{Int}(M_2)$. Let $F : V_1 \to V_2$ be the expression for f in these local coordinates, i.e. the composition

$$V_1 \xrightarrow{\phi_1^{-1}} U_1 \xrightarrow{f} U_2 \xrightarrow{\phi_2} V_2.$$

Set $q = \phi_1(p)$. The map F is an immersion at q, and it is enough to prove the theorem for this immersion.

By assumption, the map $D_q F : T_q V_1 \to T_{F(q)} V_2$ is an injection. Let

$$X \subset T_{F(q)} V_2 = \mathbb{R}^{n_2}$$

be a vector subspace such that

$$T_{F(q)} V_2 = \mathrm{Im}(D_q F) \oplus X.$$

We thus have $X \cong \mathbb{R}^{n_2-n_1}$. Define $G : V_1 \times X \to \mathbb{R}^{n_2}$ via the formula

$$G(p, x) = F(q) + x.$$

We have $T_{(q,0)}(V_1 \times X) = (T_q V_1) \oplus X$ and by construction the derivative $D_{(q,0)} G : T_{(q,0)}(V_1 \times X) \to T_{F(q)} V_2$ is an isomorphism. Theorem 5.2 (the Implicit Function Theorem) thus implies that G is a local diffeomorphism at $(q, 0)$. This implies that we can find open subsets $V_1' \times W \subset V_1 \times X$ and $V_2' \subset V_2$ such that $(q, 0) \in V_1' \times W$ and $G(V_1' \times W) = V_2'$ and such that G restricts to a diffeomorphism between $V_1' \times W$ and V_2'. The composition

$$V_1' \xrightarrow{F} V_2' \xrightarrow{G^{-1}} V_1' \times W$$

then takes $v \in V_1'$ to $(v, 0) \in V_1' \times W$, as desired. $\square$

5.5. Submersions

We now turn to the following.

DEFINITION. Let $f : M_1 \to M_2$ be a smooth map between smooth manifolds with boundary and let $p \in M_1$. The map f is a *submersion at* p if the derivative $D_p f : T_p M_1 \to T_{f(p)} M_2$ is a surjective linear map. The map f is a *submersion* if it is a submersion at all points. □

REMARK. This implies that the dimension of M_1 is at least the dimension of M_2. □

EXAMPLE. If $f : M_1 \to M_2$ is a local diffeomorphism at p, then f is a submersion at p. □

EXAMPLE. If M_1 and M_2 are smooth manifolds, then the map $f : M_1 \times M_2 \to M_1$ defined via the formula $f(p_1, p_2) = p_2$ is a submersion. □

The following theorem says that all submersions look locally like the final example above.

THEOREM 5.4 (Local Submersion Theorem). *Let $f : M_1^{n_1} \to M_2^{n_2}$ be a smooth map between smooth manifolds with boundary that is a submersion at $p \in \text{Int}(M_1^{n_1})$. There then exists an open neighborhood $U_1 \subset M_1^{n_1}$ of p and an open subset $U_2 \subset M_2^{n_2}$ satisfying $f(U_1) \subset U_2$ such that the following hold. There exists an open subset $W \subset \mathbb{R}^{n_1 - n_2}$ and a diffeomorphism $\psi : U_2 \times W \to U_1$ such that the composition*

$$U_2 \times W \xrightarrow{\psi} U_1 \xrightarrow{f} U_2$$

takes $(u, w) \in U_2 \times W$ to $u \in U_2$.

PROOF. Choose charts $\phi_1 : U_1 \to V_1$ for M_1 and $\phi_2 : U_2 \to V_2$ for M_2 such that $p \in U_1$ and $f(U_1) \subset U_2$ and $U_1 \subset \text{Int}(M_1)$ and $U_2 \subset \text{Int}(M_2)$. Let $F : V_1 \to V_2$ be the expression for f in these local coordinates, i.e. the composition

$$V_1 \xrightarrow{\phi_1^{-1}} U_1 \xrightarrow{f} U_2 \xrightarrow{\phi_2} V_2.$$

Set $q = \phi_1(p)$. The map F is a submersion at q, and it is enough to prove the theorem for this submersion.

By assumption, the map $D_q F : T_q V_1 \to T_{F(q)} V_2$ is a surjection. Let $X = \ker(D_q F)$, so $X \cong \mathbb{R}^{n_1 - n_2}$. Identifying $T_q V_1$ with $\mathbb{R}^{n_1}$, let $\pi : \mathbb{R}^{n_1} \to X$ be a linear map such that $\pi|_X = \text{id}$. Define $G : V_1 \to V_2 \times X$ via the formula

$$G(v) = (F(v), \pi(v)).$$

We have $T_{(F(q),\pi(q))}(V_2 \times X) = (T_{F(q)} V_2) \times X$ and by construction the derivative $D_q G : T_q V_1 \to T_{(F(q),\pi(q))}(V_2 \times X)$ is an isomorphism. Theorem 5.2 (the Implicit Function Theorem) thus implies that G is a local diffeomorphism at q. This implies that we can find open subset $V_1' \subset V_1$ and $V_2' \times W \subset V_2 \times X$ such that $q \in V_1'$ and $G(V_1') \subset V_2' \times W$ and such that G restricts to a diffeomorphism between V_1' and $V_2' \times W$. The composition

$$V_2' \times W \xrightarrow{G^{-1}} V_1' \longrightarrow V_2'$$

then takes $(v, w) \in V_2' \times W$ to $v \in V_2'$, as desired. □

5.6. Regular values

We now discuss regular values, which are defined as follows.

DEFINITION. Let $f : M_1 \to M_2$ be a smooth map between smooth manifolds with boundary and let $q \in M_2$. Then $q \in M_2$ is a *regular value* if f is a submersion at each point of $f^{-1}(q)$. □

Before we discuss some examples, we prove the following theorem.

THEOREM 5.5. *Let $f : M_1^{n_1} \to M_2^{n_2}$ be a smooth map between smooth manifolds (without boundary) and let $q \in M_2^{n_2}$ be a regular value such that $f^{-1}(q)$ is nonempty. Then $f^{-1}(q)$ is a smooth $(n_1 - n_2)$-dimensional smooth submanifold of $M_1^{n_1}$.*

PROOF. Consider $p \in f^{-1}(q)$. Theorem 5.4 (the Submersion Theorem) implies that there exists an open neighborhood $U_1 \subset M_1^{n_1}$ of p and an open subset $U_2 \subset M_2^{n_2}$ satisfying $f(U_1) \subset U_2$ such that the following hold. There exists an open subset $W \subset \mathbb{R}^{n_1-n_2}$ and a diffeomorphism $\psi : U_2 \times W \to U_1$ such that the composition

$$U_2 \times W \xrightarrow{\psi} U_1 \xrightarrow{f} U_2$$

takes $(u, w) \in U_2 \times W$ to $u \in U_2$. This implies that ψ^{-1} restricts to a diffeomorphism between $f^{-1}(q) \cap U_1$ and $\{q\} \times W$, i.e. that the point $p \in f^{-1}(q)$ has a neighborhood diffeomorphic to the open subset W of $\mathbb{R}^{n_1-n_2}$, as desired. □

We now discuss a large number of illustrations of Theorem 5.5.

EXAMPLE. Let $f : M_1^{n_1} \to M_2^{n_2}$ be a smooth map such that $n_1 < n_2$. For instance, f might be an embedding of an n-manifold into $\mathbb{R}^m$ for some $m > n$. Then f is clearly not a submersion anywhere, so the only regular values of f are the points *not* in the image of f. For such a point q, we have $f^{-1}(q) = \emptyset$, which is what Theorem 5.5 predicts. Theorem 5.8 (Sard's Theorem) implies that such regular values must exist. This implies in particular that there does not exist a smooth surjective map $f : S^1 \to \mathbb{R}^n$ with $n \geq 2$. This is in contrast to the fact that there exist continuous space-filling curves. □

EXAMPLE. As in Figure 5.2, consider the 2-torus T embedded in $\mathbb{R}^3$ and let $f : T \to \mathbb{R}$ be the "height function", i.e. the function defined by the formula $f(x, y, z) = z$ for all $(x, y, z) \in T$. The only non-regular values of f are then $\{0, 2, 4, 6\}$. For a regular value $x \in \mathbb{R} \setminus \{0, 2, 4, 6\}$, the subset $f^{-1}(x) \subset T$ is a 1-manifold. There are several cases:

- If $x < 0$ or $x > 6$, then $f^{-1}(x) = \emptyset$.
- If $0 < x < 2$ or $4 < x < 6$, then $f^{-1}(x)$ consists of a single circle.
- If $2 < x < 4$, then $f^{-1}(x)$ consists of the disjoint union of two circles.

For $x \in \{0, 2, 4, 6\}$, the set $f^{-1}(x)$ is not a 1-manifold. For $x \in \{0, 6\}$, the set $f^{-1}(x)$ consists of a single point (a 0-manifold). For $x \in \{2, 4\}$, the set $f^{-1}(x)$ is not even a manifold (it is a "figure 8"). □

EXAMPLE. Consider the map $f : \mathbb{R}^{n+1} \to \mathbb{R}$ defined via the formula

$$f(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2.$$

The derivative of this at $p = (p_1, \ldots, p_{n+1})$ is the linear map $D_p f : T_p\mathbb{R}^{n+1} \to T_{f(p)}\mathbb{R}^1$ represented by the $1 \times (n + 1)$-matrix

$$\begin{pmatrix} 2p_1 & 2p_2 & \cdots & 2p_{n+1} \end{pmatrix}.$$

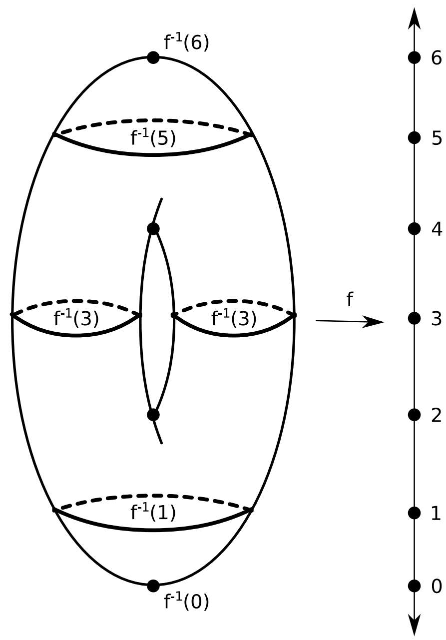

FIGURE 5.2. The torus T in $\mathbb{R}^3$ together with the height function $f : T \to \mathbb{R}$.

This is surjective as long as it is nonzero. We conclude that f is a submersion at every point except for $0 \in \mathbb{R}^{n+1}$, and thus that every nonzero point of $\mathbb{R}$ is a regular value. Since $f^{-1}(1) = S^n$, applying Theorem 5.5 furnishes us with another proof that S^n is a smooth n-manifold. □

Many smooth manifolds can be constructed like S^n was above. The following example is a very important case of this.

EXAMPLE. We can identify the set Mat_n of $n \times n$ real matrices with $\mathbb{R}^{n^2}$, and thus endow it with the structure of a smooth manifold. The map $f : \mathrm{Mat}_n \to \mathbb{R}$ defined via $f(A) = \det(A)$ is clearly a smooth map. We claim that f is a submersion at all points $A \in \mathrm{Mat}_n$ such that $f(A) \neq 0$. Indeed, fixing such an A we define a smooth map $g : \mathbb{R} \to \mathrm{Mat}_n$ via the formula $g(t) = tA$. We have

$$f(g(t)) = \det(tA) = t^n \det(A).$$

The ordinary calculus derivative of the map $f \circ g : \mathbb{R} \to \mathbb{R}$ is thus nonzero at $t = 1$, which implies that the derivative map $D_1(f \circ g) : T_1\mathbb{R} \to T_{\det(A)}\mathbb{R}$ is a surjective linear map (it is just multiplication by our nonzero ordinary calculus derivative!). Theorem 3.4 (the Manifold Chain Rule I) implies that

$$D_1(f \circ g) = (D_A f) \circ (D_1 g).$$

Since $D_1(f \circ g)$ is surjective, we conclude that $D_A f$ is surjective, i.e. that f is a submersion at A, as claimed. The upshot is that all nonzero numbers are regular values of $f : \mathrm{Mat}_n \to \mathbb{R}$. In particular, Theorem 5.5 implies that

$$\mathrm{SL}_n(\mathbb{R}) = f^{-1}(1)$$

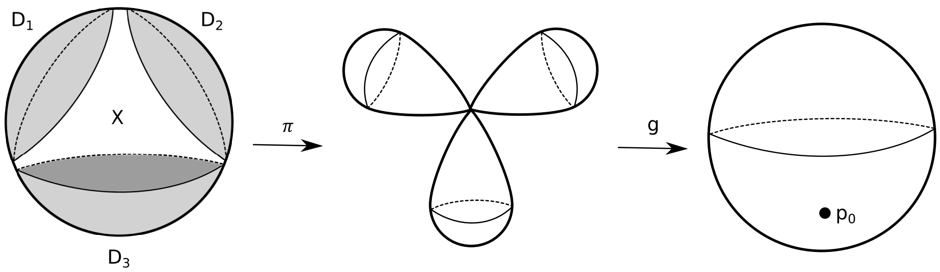

FIGURE 5.3. The function $f : S^2 \to S^2$ equals $g \circ \pi$. It takes X to p_0 and each open disc D_i diffeomorphically to $S^2 \setminus \{p_0\}$.

is a smooth manifold of dimension $n^2 - 1$. Just like $\mathrm{GL}_n(\mathbb{R})$, this is an example of a Lie group (a group which is also a smooth manifold and for which the group operations are smooth). We will discuss Lie groups in more detail in Chapter 9. □

EXAMPLE. As in Figure 5.3, let D_1 and D_2 and D_3 be three disjoint open round discs in S^2 and let $X = S^2 \setminus (D_1 \cup D_2 \cup D_3)$. We construct a function $f : S^2 \to S^2$ as follows.

- Let $S^2 \vee S^2 \vee S^2$ be the result of gluing three copies of S^2 together at a single point (which we will call the "wedge point"). The space $S^2 \vee S^2 \vee S^2$ is not a manifold because the wedge point does not have a neighborhood homeomorphic to an open set in Euclidean space. There is a map $\pi : S^2 \to S^2 \vee S^2 \vee S^2$ obtained by collapsing the subset X to a single point; the map π takes X to the wedge point and each open disc D_i homeomorphically to the result of removing the wedge point from one of the S^2's.
- Fix some basepoint $p_0 \in S^2$. There is a map $g : S^2 \vee S^2 \vee S^2 \to S^2$ that takes each copy of S^2 homeomorphically onto S^2 and takes the wedge point to p_0.
- We define $f = g \circ \pi$.

If one is careful in the above construction, we can ensure that f is a smooth map. The regular values of f are $S^2 \setminus \{p_0\}$. For $x \in S^2 \setminus \{p_0\}$, the set $f^{-1}(x)$ consists of three point, one in each disc D_i. As we expect, this is a 0-manifold. The set $f^{-1}(p_0)$ is X; this is not even a manifold. □

5.7. Regular values and manifolds with boundary

We now discuss two variants of Theorem 5.5 for manifolds with boundary. The first is as follows.

THEOREM 5.6. *Let M^n be a smooth n-manifold (without boundary) and let $f : M^n \to \mathbb{R}$ be a smooth map.*

- *If $a \in \mathbb{R}$ is a regular value of f, then $f^{-1}((\infty, a])$ is a smooth n-manifold with boundary and $\partial f^{-1}((\infty, a])) = f^{-1}(a)$. Similarly, $f^{-1}([a, \infty))$ is a smooth n-manifold with boundary and $\partial f^{-1}([a, \infty)) = f^{-1}(a)$.*
- *If $a, b \in \mathbb{R}$ are regular values with $a < b$, then $f^{-1}([a, b])$ is a smooth n-manifold with boundary and $\partial f^{-1}([a, b]) = f^{-1}(a) \cup f^{-1}(b)$.*

PROOF. This can be proved using Theorem 5.4 (the Local Submersion Theorem) just like Theorem 5.5. We omit the proof, though we point out that if $U \subset \mathbb{R}$

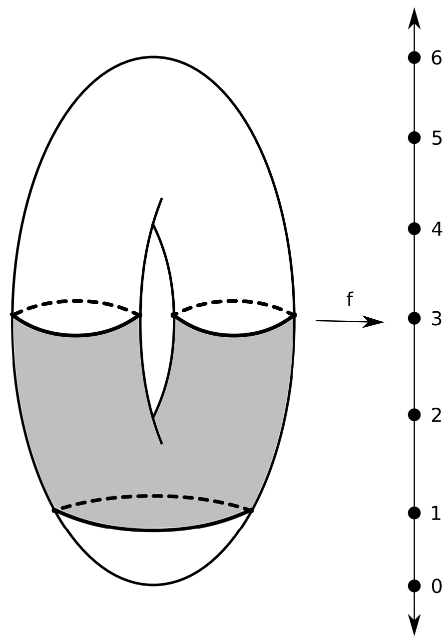

FIGURE 5.4. The torus T in $\mathbb{R}^3$ together with the height function $f : T \to \mathbb{R}$.

is an open set, then $f^{-1}(U)$ is an open subset of M^n, and thus a smooth n-manifold. The only place where Theorem 5.4 needs to be used therefore is on the boundary (i.e. the pullbacks of the regular values in question). $\square$

EXAMPLE. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined via the formula $f(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$. Then as we proved before, every nonzero point of $\mathbb{R}$ is a regular value, so we can apply Theorem 5.6 to deduce that $\mathbb{D}^n = f^{-1}((-\infty, 1])$ is a smooth n-manifold with boundary and $\partial \mathbb{D}^n = f^{-1}(1) = S^{n-1}$. $\square$

EXAMPLE. As in Figure 5.4, consider the 2-torus T embedded in $\mathbb{R}^3$ and let $f : T \to \mathbb{R}$ be the "height function", i.e. the function defined by the formula $f(x, y, z) = z$ for all $(x, y, z) \in T$. The only non-regular values of f are then $\{0, 2, 4, 6\}$. As is illustrated in Figure 5.4, we can apply Theorem 5.6 to deduce that $f^{-1}([1, 3])$ is a smooth 2-manifold with boundary and that $\partial f^{-1}([1, 3]) = f^{-1}(1) \cup f^{-1}(3)$, a union of three circles. $\square$

The other variant of Theorem 5.5 we need is as follows.

THEOREM 5.7. *Let $M_1^{n_1}$ be a smooth manifold with boundary and let $M_2^{n_2}$ be a smooth manifold (with empty boundary). Let $f : M_1^{n_1} \to M_2^{n_2}$ be a smooth function and let $p \in M_2^{n_2}$ be a point which is a regular value for both f and $f|_{\partial M_1^{n_1}}$. Then $f^{-1}(p) \subset M_1^{n_1}$ is a smooth $(n_1 - n_2)$-dimensional manifold with boundary satisfying*

$$\partial f^{-1}(p) = (f|_{\partial M_1^{n_1}})^{-1}(p).$$

PROOF. PROVE IT!!! $\square$

5.8. Sard's theorem

The following important theorem says that every smooth map has many regular values.

THEOREM 5.8 (Sard's theorem). *Let $f : M_1 \to M_2$ be a smooth map between smooth manifolds with boundary. Assume that M_1 is compact. Then the set of regular values of f is open and dense in M_2.*

PROOF. PROVE IT!!! □

5.9. Embedding manifolds in Euclidean space II

We now strengthen Theorem 5.1.

THEOREM 5.9. *If M^n is a compact smooth n-manifold with boundary, then there exist an embedding $f : M^n \to \mathbb{R}^{2n+1}$.*

PROOF. PROVE IT!!! □

5.10. Application: the fundamental theorem of algebra

We now show how to apply the ideas we have introduced to prove the fundamental theorem of algebra, which can be stated as follows.

THEOREM 5.10 (Fundamental theorem of algebra). *Let $f(z)$ be a nonconstant polynomial whose coefficients are complex numbers. Then there exists some $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$.*

For the proof of Theorem 5.10, we will need the following.

LEMMA 5.11. *Let M^n be a compact connected manifold whose dimension n is at least 2 and let $f : M^n \to M^n$ be a smooth map which is a submersion except at possibly finitely many points. Then for all regular values $p_1 \in M^n$ and $p_2 \in M^n$, we have $|f^{-1}(p_1)| = |f^{-1}(p_2)|$.*

PROOF. Let R be the set of regular values of f. Define $\psi : R \to \mathbb{Z} \cup \{\infty\}$ via the formula

$$\psi(p) = |f^{-1}(p)| \qquad (p \in R).$$

We must prove that ψ is constant. Our assumptions imply that all but finitely many points of M^n are regular values for f. Since $n \geq 2$, this implies that R is open and connected, so it is enough to prove that the function ψ is locally constant. In other words, fixing some $p \in R$ we must prove that there exists some neighborhood of p in R such that ψ restricted to that neighborhood is constant.

By Theorem 5.5, the set $f^{-1}(p)$ is a 0-dimensional submanifold of M^n. Since M^n is compact, this implies that $f^{-1}(p)$ is a finite set; enumerate it as $\{q_1, \ldots, q_k\}$. The function f is a submersion at each q_i, so by Theorem 5.2 (the Implicit Function Theorem) the function f is a local diffeomorphism at q_i, i.e. there exists neighborhoods U_i of q_i and W_i of p such that f restricts to a diffeomorphism from U_i to W_i. Shrinking the U_i if necessary, we can assume that they are all disjoint. Set

$$W = W_1 \cap W_2 \cap \cdots \cap W_k$$

and

$$U_i' = U_1 \cap f^{-1}(W_i).$$

By construction, W is a neighborhood of p and $f|_{U_i'}$ is a diffeomorphism between U_i' and W. What we would really like would be for $f^{-1}(W)$ to equal $U_1' \cup \cdots \cup U_k'$; however, this might not hold. To fix this, let C be the closure of the set

$$f^{-1}(W) \setminus (U_1' \cup \cdots \cup U_k').$$

The set C is a closed (hence compact) subset of M^n that does not contain any q_i. The image $f(C)$ is thus a compact subset of M^n that does not contain p, so we can find an open neighborhood W' of p such that $W' \subset W$ and $W' \cap f(C) = \emptyset$. Set $U_i'' = f^{-1}(W') \cap U_i'$. The restriction $f|_{U_i''}$ is thus a diffeomorphism from U_i'' to W' and $f^{-1}(W') = U_1'' \cup \cdots \cup U_k''$.

We are now done: for $p' \in W'$, the preimage $f^{-1}(p')$ contains exactly one point from each U_i'' and nothing else. In other words, $\psi|_{W'} = k$, as desired. □

Proof of Theorem 5.10. We will use the second smooth atlas for S^2 that was discussed in §2.2, which we now recall. Let $U_1 = S^2 \setminus \{(0,0,1)\}$ and $U_{-1} = S^2 \setminus \{(0,0,-1)\}$. Identifying $\mathbb{R}^2$ with the subspace of $\mathbb{R}^3$ consisting of points whose last coordinate is 0, define a function $\phi_1 : U_1 \to \mathbb{R}^2$ by letting $\phi_1(p)$ be the unique intersection point of the line joining $p \in U_1 \subset S^2 \subset \mathbb{R}^3$ and $(0,0,1)$ with the plane $\mathbb{R}^2$. It is clear that ϕ_1 is a homeomorphism. Similarly, define $\phi_{-1} : U_{-1} \to \mathbb{R}^2$ by letting $\phi_{-1}(p)$ be the unique intersection point of the line joining $p \in U_{-1} \subset S^2 \subset \mathbb{R}^3$ and $(0,0,-1)$ with the plane $\mathbb{R}^2$. Again, ϕ_{-1} is a homeomorphism. Then the set $\{\phi_1 : U_1 \to \mathbb{R}^n, \phi_{-1} : U_{-1} \to \mathbb{R}^n\}$ is a smooth atlas for S^2.

Identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way, so we can plug points of $\mathbb{R}^2$ into the polynomial f. Also, for simplicity set $\infty = (0,0,1) \in S^2$. Define a function $F : S^2 \to S^2$ via the formula

$$F(x) = \begin{cases} \infty & \text{if } x = \infty, \\ \phi_1^{-1}(f(\phi_1(x))) & \text{if } x \in S^2 \setminus \{\infty\}. \end{cases} \qquad (x \in S^2).$$

It is easy to see that F is a smooth map and that F takes $S^2 \setminus \{\infty\}$ to itself (the omitted calculation showing that F is smooth at ∞ uses the fact that f is nonconstant; if f were constant, then F might not even be continuous at ∞). To prove the fundamental theorem of algebra, it is enough to prove that F is surjective. An easy calculation shows that the only points of S^2 where F is not a submersion are

- $\phi_1^{-1}(z_0)$, where z_0 is a root of the derivative $f'(z)$, and
- possibly ∞.

Since $f'(z)$ is a polynomial that is not identically 0, it has finitely many roots. We deduce that F is a submersion except at finitely many points. Letting $R \subset S^2$ be the set of regular values of F, Lemma 5.11 therefore implies that the function $\Psi : R \to \mathbb{Z}$ defined via the formula $\Psi^{-1}(p) = |F^{-1}(p)|$ is constant. Clearly Ψ is not identically 0, so this implies that the image of F contains all points of R. Since F definitely contains all points of $S^2 \setminus R$, we deduce that F is surjective, as desired. □

5.11. Application: the Brouwer fixed point theorem

We now apply the ideas we have introduced to prove the following theorem. Let $\mathbb{D}^n$ denote the closed unit disc $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$, so $\partial\,\mathbb{D}^n = S^n \subset \mathbb{D}^n$.

THEOREM 5.12 (Brouwer Fixed Point Theorem). *Let $f : \mathbb{D}^n \to \mathbb{D}^n$ be a continuous function. Then there exists some point $x \in \mathbb{D}^n$ such that $f(x) = x$.*

REMARK. To illustrate what is going on here, consider the case $n = 1$, so $\mathbb{D}^n = [-1, 1] \subset \mathbb{R}$. The theorem asserts that if $f : [-1, 1] \to [-1, 1]$ is a continuous function, then there exists some point $x \in [-1, 1]$ such that $f(x) = x$. Another way of saying this is that the theorem is asserting that the function $g : [-1, 1] \to \mathbb{R}$ defined via the formula $g(x) = f(x) - x$ has a zero. Since $g(-1) = f(-1) + 1 \geq 0$ and $g(1) = f(1) - 1 \leq 0$, this is an immediate consequence of the intermediate value theorem. □

PROOF OF THEOREM 5.12. We will do this in several steps.

STEP 1. *There does not exist a smooth function $f : \mathbb{D}^n \to \partial\,\mathbb{D}^n$ such that $f|_{\partial\,\mathbb{D}^n} = id$.*

Assume that such a function exists. Using Theorem 5.8 (Sard's Theorem), there exists a regular value $p \in \partial\,\mathbb{D}^n$ for f. Since $f|_{\partial\,\mathbb{D}^n} = \text{id}$, the point p is also a regular value for $f|_{\partial\,\mathbb{D}^n}$. We can therefore apply Theorem 5.7 to deduce that $f^{-1}(p)$ is a smooth 1-manifold with boundary embedded in $\mathbb{D}^n$ such that

$$\partial f^{-1}(p) = (f|_{\partial\,\mathbb{D}^n})^{-1}(p) = \{p\}.$$

Recall that every connected compact 1-manifold with boundary is diffeomorphic to either S^1 or $[0, 1]$ (this will be proven in Theorem 11.1 below). But this implies that that any compact 1-manifold with boundary (connected or not) has a boundary consisting of an even number of points, so this is a contradiction.

STEP 2. *Let $f : \mathbb{D}^n \to \mathbb{D}^n$ be a smooth function. Then there exists some point $x \in \mathbb{D}^n$ such that $f(x) = x$.*

Assume that $f(x) \neq x$ for all $x \in \mathbb{D}^n$. Define a function $g : \mathbb{D}^n \to \partial\,\mathbb{D}^n$ as follows. For $x \in \mathbb{D}^n$, let ℓ_x be the ray starting at x and passing through $f(x)$. This is well-defined since $f(x) \neq x$. The ray ℓ_x intersects $\partial\,\mathbb{D}^n$ at a unique point; let $g(x)$ be this point of intersection. Writing out equations for ℓ_x, it is clear that the function g is smooth. Moreover, by definition we have $g|_{\partial\,\mathbb{D}^n} = \text{id}$. This contradicts Step 1.

STEP 3. *Let $f : \mathbb{D}^n \to \mathbb{D}^n$ be a continuous function. Then there exists some point $x \in \mathbb{D}^n$ such that $f(x) = x$.*

We will reduce this to Step 2 by approximating f by a smooth function. Assume that $f(x) \neq x$ for all $x \in \mathbb{D}^n$. Set

$$\epsilon = \inf\{\|f(x) - x\| \mid x \in \mathbb{D}^n\}.$$

Since $\mathbb{D}^n$ is compact, this infimum is realized and $\epsilon > 0$. Use Lemma 2.8 to find a smooth function $g : \mathbb{D}^n \to \mathbb{R}^n$ such that $\|g(x) - f(x)\| < \epsilon/3$ for all $x \in \mathbb{D}^n$. The image of g need not lie in $\mathbb{D}^n$; however, we have $\|g(x)\| \leq \|f(x)\| + \epsilon/3 \leq 1 + \epsilon/3$ for all $x \in \mathbb{D}^n$. Defining $h : \mathbb{D}^n \to \mathbb{R}^n$ via the formula

$$h(x) = \frac{g(x)}{1 + \epsilon/3},$$

we deduce that $h(\mathbb{D}^n) \subset \mathbb{D}^n$. To get a contradiction to Step 2, we will prove that h has no fixed points. For all $x \in \mathbb{D}^n$, it follows from the definitions that

$\|h(x) - g(x)\| \leq \epsilon/3$. The triangle inequality then implies that

$$\|h(x) - x\| \geq \|f(x) - x\| - \|f(x) - g(x)\| - \|g(x) - h(x)\| \geq \epsilon - \epsilon/3 - \epsilon/3 = \epsilon/3.$$

Since $\epsilon > 0$, this implies that $h(x) \neq x$. This contradicts Step 2, and we are done. $\square$

CHAPTER 6

Tubular neighborhoods

In this chapter, we discuss tubular neighborhoods together with some of their applications. For simplicity, we will restrict ourselves to tubular neighborhoods of submanifolds of Euclidean space.

6.1. Normal bundles

We first must discuss normal bundles. To define these, we will use the vector bundle operations that we discussed in §3.8.

DEFINITION. Let $M_1^{n_1}$ and $M_2^{n_2}$ be smooth manifolds and let $f : M_1^{n_1} \to M_2^{n_2}$ be an embedding. There is a map $g : TM_1^{n_1} \to f^*(TM_2^{n_1})$ of vector bundles over $M_1^{n_1}$ which since f is an embedding restricts to an injection $(TM_1^{n_1})_p \to f^*(TM_2^{n_1})_p$ for each $p \in M_1^{n_1}$. The *normal bundle* to f, denoted N_f, is the cokernel of g. This is an $(n_2 - n_1)$-dimensional vector bundle over $M_1^{n_1}$. □

The following special case of this will be particularly important.

DEFINITION. Let M^n be a smooth submanifold of $\mathbb{R}^m$. We will denote by $N_{\mathbb{R}^m/M^n}$ the normal bundle to the inclusion map $M^n \hookrightarrow \mathbb{R}^m$. □

The normal bundle to a smooth submanifold of $\mathbb{R}^m$ can be expressed in a particularly simple form. Recall that if M^n is a smooth submanifold of $\mathbb{R}^m$, then T_pM^n is canonically identified with an n-dimensional subspace of $T_p\mathbb{R}^m = \mathbb{R}^m$ for all $p \in M^n$.

DEFINITION. If M^n is a smooth submanifold of $\mathbb{R}^m$ and $p \in M^n$, then denote by $N_{p,\mathbb{R}^m/M^n}$ the orthogonal complement to T_pM^n in $T_p\mathbb{R}^m = \mathbb{R}^m$. □

LEMMA 6.1. *Let M^n be a smooth submanifold of $\mathbb{R}^m$. Then*

$$N_{\mathbb{R}^m/M^n} \cong \{(p, \vec{v}) \in T\mathbb{R}^m \mid p \in M_n \text{ and } \vec{v} \in N_{p,\mathbb{R}^m/M^n}\}.$$

PROOF. This is an exercise. □

EXAMPLE. For $S^n \subset \mathbb{R}^{n+1}$ and $p \in S^n$, recall that T_pS^n consists of all vectors in $T_p\mathbb{R}^{n+1} = \mathbb{R}^{n+1}$ that are orthgonal to the line from 0 to p. This implies that

$$N_{\mathbb{R}^{n+1}/S^n} = \{(p, tp) \mid p \in S^n, t \in \mathbb{R}\} \cong S^n \times \mathbb{R}.$$ □

Since $N_{\mathbb{R}^m/M^n}$ is an $(m-n)$-dimensional vector bundle over M^n, it is a smooth m-dimensional manifold. The following lemma identifies its tangent space. In it, we make use of the fact that

$$T(T\mathbb{R}^n) = T(\mathbb{R}^n \times \mathbb{R}^n) = (\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n).$$

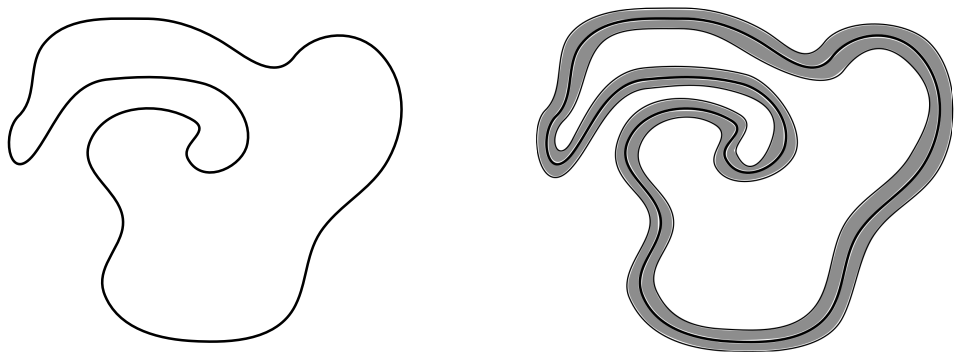

FIGURE 6.1. A tubular neighborhood of a loop S^1 embedded in $\mathbb{R}^2$

LEMMA 6.2. *Let M^n be a smooth submanifold of $\mathbb{R}^m$ and let $(p, \vec{v}) \in N_{\mathbb{R}^m/M^n} \subset T\mathbb{R}^m = \mathbb{R}^m \times \mathbb{R}^m$. Then the tangent space $T_{(p,\vec{v})}N_{\mathbb{R}^m/M^n}$ consists of all points $(\vec{w}, \vec{w}') \in T_{(p,\vec{v})}(\mathbb{R}^m \times \mathbb{R}^m) = \mathbb{R}^m \times \mathbb{R}^m$ such that $\vec{w} \in T_pM^n \subset \mathbb{R}^m$ and $\vec{w}' \in N_{p,\mathbb{R}^m/M^n} \subset \mathbb{R}^m$.*

PROOF. This is another exercise. □

6.2. The tubular neighborhood theorem

We now come to the tubular neighborhood theorem. This requires a definition. In it, we make use of the identification described in Lemma 6.1

DEFINITION. Let M^n be a smooth submanifold of $\mathbb{R}^m$. For $\epsilon > 0$, define $N^{\epsilon}_{\mathbb{R}^m/M^n}$ to be the subspace of $N_{\mathbb{R}^m/M^n}$ consisting of points $(p, \vec{v})$ such that $p \in M^n$ and $\vec{v} \in N_{p,\mathbb{R}^m/M^n} \subset \mathbb{R}^m$ and $\|\vec{v}\| < \epsilon$. □

THEOREM 6.3 (Tubular Neighborhood Theorem). *Let M^n be a smooth compact submanifold of $\mathbb{R}^m$. Define $\mathfrak{n} : N_{\mathbb{R}^m/M^n} \to \mathbb{R}^m$ to be the map taking $(p, \vec{v})$ to $p + \vec{v}$. Then there exists some ϵ such that the restriction of $\mathfrak{n}$ to $N^{\epsilon}_{\mathbb{R}^m/M^n}$ is an embedding.*

The image of the restriction of $\mathfrak{n} : N_{\mathbb{R}^m/M^n} \to \mathbb{R}^m$ to $N^{\epsilon}_{\mathbb{R}^m/M^n}$ will be called an ϵ-tubular neighborhood of M^n. To illustrate why it is called this, see Figure 6.1.

PROOF OF THEOREM 6.3. For $\epsilon > 0$ and a set $U \subset M^n$, let $N^{\epsilon}_{\mathbb{R}^m/M^n}(U)$ denote the subspace of $N^{\epsilon}_{\mathbb{R}^m/M^n}$ consisting of points $(p, \vec{v})$ such that $p \in U$. The proof will have two steps.

STEP 1. *There exists an open cover $\{U_i\}_{i=1}^k$ of M^n and an $\epsilon > 0$ such that $\mathfrak{n}$ restricts to an embedding from $N^{\epsilon}_{\mathbb{R}^m/M^n}(U_i)$ into $\mathbb{R}^m$ for all $1 \leq i \leq k$.*

Since M^n is compact, it is enough to prove that the map $\mathfrak{n}$ is a local diffeomorphism at each point $(p, 0) \in N_{\mathbb{R}^m/M^n}$. By Theorem 5.2 (the Implicit Function Theorem), this is equivalent to showing that $D_{(p,0)}\,\mathfrak{n} : T_{(p,0)}N_{\mathbb{R}^m/M^n} \to T_{(p,0)}\mathbb{R}^m = \mathbb{R}^m$ is an isomorphism. Lemma 6.2 says that $T_{(p,0)}N_{\mathbb{R}^m/M^n}$ consists of all point $(\vec{w}, \vec{w}') \in T_{(p,0)}(\mathbb{R}^m \times \mathbb{R}^m) = \mathbb{R}^m \times \mathbb{R}^m$ such that $\vec{w} \in T_pM^n \subset \mathbb{R}^m$ and $\vec{w}' \in N_{p,\mathbb{R}^m/M^n} \subset \mathbb{R}^m$. Under this identification, the derivative $D_{(p,0)}\,\mathfrak{n}$ takes $(\vec{w}, \vec{w}')$ to $\vec{w} + \vec{w}'$. Since $\mathbb{R}^m$ is the orthogonal direct sum of T_pM^n and $N_{p,\mathbb{R}^m/M^n}$, it follows that $D_{(p,0)}\,\mathfrak{n}$ is an isomorphism, as desired.

STEP 2. *There exists some $0 < \epsilon' < \epsilon$ such that the restriction of $\mathfrak{n}$ to $N^{\epsilon'}_{\mathbb{R}^m/M^n}$ is an embedding.*

Set $\epsilon'' = \epsilon/2$ and define $N^{\leq \epsilon'}_{\mathbb{R}^m/M^n}$ to be the subspace of $N_{\mathbb{R}^m/M^n}$ consisting of points $(p, \vec{v})$ such that $p \in M^n$ and $\vec{v} \in N_{p,\mathbb{R}^m/M^n} \subset \mathbb{R}^m$ and $\|\vec{v}\| \leq \epsilon'$. Define $N^{\leq \epsilon'}_{\mathbb{R}^m/M^n}(U_i)$ in the obvious way. Since M^n is compact, it follows that $N^{\leq \epsilon'}_{\mathbb{R}^m/M^n}$ is compact. Let $X = N^{\leq \epsilon'}_{\mathbb{R}^m/M^n} \times N^{\leq \epsilon'}_{\mathbb{R}^m/M^n}$ and $Y = \{(x, y) \in X \mid \mathfrak{n}(x) = \mathfrak{n}(y)\}$ and $\Delta = \{(x, y) \in X \mid x = y\}$, so both Y and Δ are closed (hence compact) and $\Delta \subset Y$. Define $Z = Y \setminus \Delta$. We claim that Z is compact. Indeed, define

$$W = \bigcup_{i=1}^{k} N^{\leq \epsilon'}_{\mathbb{R}^m/M^n}(U_i) \times N^{\leq \epsilon'}_{\mathbb{R}^m/M^n}(U_i) \subset X.$$

The set W is open and by Step 1 we have $W \cap Y = \Delta$. This implies that $Z = Y \setminus W$, so Z is compact, as claimed.

Define

$$A = \{((p, \vec{v}), (p', \vec{v}')) \in X \mid \vec{v} = \vec{v}' = 0\} \cong M^n \times M^n,$$

so A is compact. Since $\mathfrak{n}$ takes $\{(p, \vec{v}) \in N^{\leq \epsilon'}_{\mathbb{R}^m/M^n} \mid \vec{v} = 0\}$ diffeomorphically onto M^n, it follows that $Z \cap A = \emptyset$. Since both Z and A are compact, this implies that we can choose $0 < \epsilon' < \epsilon''$ such that for all $((p, \vec{v}), (p', \vec{v}')) \in Z$ we have $\|\vec{v}\| \geq \epsilon'$ and $\|\vec{v}'\| \geq \epsilon'$. Unwinding the definitions, this implies that the restriction of $\mathfrak{n}$ to $N^{\epsilon'}_{\mathbb{R}^m/M^n}$ is injective. Since Step 1 implies that the restriction of $\mathfrak{n}$ to $N^{\epsilon'}_{\mathbb{R}^m/M^n}$ is a local diffeomorphism, we conclude that the restriction of $\mathfrak{n}$ to $N^{\epsilon'}_{\mathbb{R}^m/M^n}$ is an embedding, as desired. □

The following corollary to Theorem 6.3 will be frequently used.

COROLLARY 6.4. *Let M^n be a smooth compact submanifold of $\mathbb{R}^m$. Then for all $\epsilon > 0$, there exists some open set $U_\epsilon \subset \mathbb{R}^m$ containing M^n and a smooth function $\pi : U_\epsilon \to M^n$ with the following properties.*

- *$\pi(p) = p$ for all $p \in M^n$.*
- *$\|\pi(p) - p\| < \epsilon$ for all $p \in U_\epsilon$.*

PROOF. Decreasing ϵ if necessary, we can apply Theorem 6.3 to construct an ϵ-tubular neighborhood U_ϵ of M^n. We have $U_\epsilon \cong N^{\epsilon}_{\mathbb{R}^m/M^n}$; under this identification, the desired function $\pi : U_\epsilon \to M^n$ is function that takes $(p, \vec{v}) \in N^{\epsilon}_{\mathbb{R}^m/M^n}$ to p. This clearly satisfies the claimed properties. □

6.3. Approximating continuous functions by smooth ones, II

Recall that in Theorem 2.7 we proved that if M is a smooth manifold and $f : M \to \mathbb{R}^m$ is a continuous function, the f can be approximated arbitrarily well by smooth functions. As a first application of the tubular neighborhood, we now show how to approximate continuous functions between arbitrary compact manifolds by smooth ones.

THEOREM 6.5. *Let M_1 be a smooth manifold with boundary, let M_2 be a smooth compact manifold, and let $f : M_1 \to M_2$ be a continuous function. Let $d_{M_2}(\cdot, \cdot)$ be a metric (in the sense of metric spaces) on M_2 that induces the topology on*

M_2. Then for all $\epsilon > 0$ there exists a smooth function $g : M_1 \to M_2$ such that $d_{M_2}(f(x), g(x)) < \epsilon$ for all $x \in M_1$.

PROOF. Using Theorem 5.1, we can embed M_2 into $\mathbb{R}^m$ for some $m \gg 0$. The subspace topology on M_2 induced by $\mathbb{R}^m$ is the same as the topology induced by the metric $d_{M_2}(\cdot, \cdot)$, so we can find some $\epsilon' > 0$ such that if $\|y_1 - y_2\| < \epsilon'$ for some $y_1, y_2 \in M_2$, then $d_{M_2}(y_1, y_2) < \epsilon$. Shrinking ϵ' if necessary, we can apply Corollary 6.4 to construct an open neighborhood $U_{\epsilon'/2}$ of M_2 together with a function $\pi : U_{\epsilon'/2} \to M_2$ such that $\pi(p) = p$ for all $p \in M_2$ and $\|\pi(p) - p\| < \epsilon'/2$ for all $p \in U_{\epsilon'/2}$. Applying Theorem 2.7 to our continuous function $f : M_1 \to M_2 \subset \mathbb{R}^m$, we can find a smooth function $g_1 : M_1 \to \mathbb{R}^m$ such that $\|f(x) - g_1(x)\| < \epsilon'/2$ for all $x \in M_1$. The image of g_1 will no longer lie in the subspace M_2 of $\mathbb{R}^m$, but it will lie in $U_{\epsilon'/2}$. Define $g = \pi \circ g_1$, so g is a smooth function from M_1 to M_2. For $x \in M_1$, we then have

$$\|f(x) - g(x)\| \leq \|f(x) - g_1(x)\| + \|g_1(x) + \pi(g_1(x))\| < \epsilon'/2 + \epsilon'/2 = \epsilon',$$

and hence $d_{M_2}(f(x), g(x)) < \epsilon$, as desired. □

The following "relative" version of Theorem 6.5 will also be useful.

THEOREM 6.6. *Let M_1 be a smooth manifold with boundary, let M_2 be a smooth compact manifold, and let $f : M_1 \to M_2$ be a continuous function. Also, let $U \subset M_1$ be an open set such that $f|_U$ is smooth and let $C \subset M_1$ be a closed set with $C \subset U$. Let $d_{M_2}(\cdot, \cdot)$ be a metric (in the sense of metric spaces) on M_2 that induces the topology on M_2. Then for all $\epsilon > 0$, there exists a smooth function $g : M_1 \to M_2$ such that $d_{M_2}(f(x), g(x)) < \epsilon$ for all $x \in M_1$ and such that $g|_C = f|_C$.*

PROOF. Simply replace the invocation of Theorem 2.7 in the proof of Theorem 6.5 with Theorem 2.9. □

CHAPTER 7

The degree of a map

7.1. Homotopies and smooth homotopies

We begin with the following topological relationship between functions.

Definition. Let $f_0, f_1 : X \to Y$ be continuous functions between topological spaces. We say that f_0 and f_1 are *homotopic* if there exists a continuous function $F : X \times I \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for all $x \in X$. The function F will be called a *homotopy*. □

In other words, the function f_0 can be "deformed" to the function f_1. Here is one easy example of this.

Example. Let X be a topological space and let $f_0, f_1 : X \to \mathbb{R}^n$ be continuous functions. Then f_0 and f_1 are homotopic via the homotopy $F : X \times I \to \mathbb{R}^n$ defined by the formula

$$F(x, t) = (1 - t)f_0(x) + tf_1(x) \qquad (x \in X).$$

In other words, F moves the point $f_0(x)$ to the point $f_1(x)$ along the straight line connecting these two points. □

Lemma 7.1. *For topological spaces X and Y, the relation of homotopy between continuous functions from X to Y is an equivalence relation.*

Proof. Trivial. □

If M_1 and M_2 are smooth manifolds, then we have two seemingly different equivalence classes of functions from M_1 to M_2.

- Continuous functions up to homotopy.
- Smooth functions up to smooth homotopy, that is, homotopies $F : M_1 \times I \to M_2$ that are themselves smooth.

The two main theorems in this section show that these are really the same thing. The first is as follows.

Theorem 7.2. *Let $f : M_1 \to M_2$ be a continuous function between smooth compact manifolds. Then f is homotopic to a smooth function $g : M_1 \to M_2$.*

Proof. Using Theorem 5.1, we can embed M_2 into $\mathbb{R}^m$ for some $m \gg 0$. Using Corollary 6.4, we can find an open neighborhood U of M_2 in $\mathbb{R}^m$ and a smooth function $\pi : U \to M_2$ such that $\pi(p) = p$ for all $p \in M_2$ (the constant ϵ in that corollary does not matter for this proof). We can now pick $\delta > 0$ small enough such that if $p_1, p_2 \in M_2$ satisfy $\|p_1 - p_2\| < \delta$, then the straight line segment from p_1 to p_2 lies in U. Now use Theorem 6.5 to construct a smooth function $g : M_1 \to M_2$ such that $\|f(p) - g(p)\| < \delta$ for all $p \in M_1$. We claim that f is homotopic to g. Indeed, the function $F : M_1 \times I \to M_2$ defined via the formula

$$F(p, t) = \pi((1 - t)f(p) + tg(p)) \qquad (p \in M_1, t \in I)$$

is a homotopy from f to g. This is well-defined since $(1-t)f(p)+tg(p) \in U$ (the domain of π) for all $t \in I$ and $p \in I$, which is a consequence of the fact that $\|f(p)-g(p)\| < \delta$. □

The second is as follows.

THEOREM 7.3. *Let $f_0 : M_1 \to M_2$ and $f_1 : M_1 \to M_2$ be homotopic smooth functions between smooth compact manifolds. Then there exists a smooth homotopy between f_0 and f_1.*

PROOF. Let $F : M_1 \times I \to M_2$ be a continuous homotopy between f_0 and f_1. Modifying F, we can assume that $F(p,t) = f_0(p)$ for $0 \leq t \leq 1/3$ and $F(p,t) = f_1(p)$ for $2/3 \leq t \leq 1$. Thus F is smooth on the open set $M \times ([0,1/3) \cup (2/3,1])$. Applying Theorem 6.6, we can find a smooth function $G : M_1 \times I \to M_2$ such that $G(p,0) = F(p,0) = f_0(p)$ and $G(p,1) = F(p,1) = f_1(p)$ for all $p \in M_1$ (we could also ensure that $\|G(p,t) - F(p,t)\| < \epsilon$ for all $(p,t) \in M_1 \times I$, but this is not necessary). The function G is the desired smooth homotopy. □

7.2. Homotopies and regular values

This technical section will discuss the relationship between homotopy classes of functions and regular values. Our two results augment Theorems 7.2 and 7.3 from the previous section.

LEMMA 7.4. *Let $f : M_1 \to M_2$ be a continuous function between smooth compact manifolds and let $p \in M_2$. Then f is homotopic to a smooth function $g : M_1 \to M_2$ such that p is a regular value of g.*

PROOF. By Theorem 7.2, we can assume that f is itself smooth. Using Theorem 5.8 (Sard's Theorem), we can find a regular value $q \in M_2$ of f. Theorem 4.7 implies that we can find a flow $h_t : M_2 \to M_2$ of a vector field such that $h_1(q) = p$. Define $g = h_1 \circ f$. Since h_1 is a diffeomorphism, the point p is a regular value of g. Moreover, f is homotopic to g via the homotopy $F : M_1 \times I \to M_2$ defined via the formula

$$F(x,t) = h_t(f(x)) \qquad (x \in M_1, t \in I);$$

here we are using the fact that $h_0 = \mathrm{id}$, which follows from the definition of a flow. □

LEMMA 7.5. *Let $f_0 : M_1 \to M_2$ and $f_1 : M_1 \to M_2$ be homotopic smooth functions between smooth compact manifolds and let $p \in M_2$ be a point which is a regular value of both f_0 and f_1. Then there exists a smooth homotopy $F : M_1 \times I \to M_2$ such that p is a regular value of both f_0 and f_1.*

PROOF. PROVE IT!!! □

7.3. The degree modulo 2

We now come to the first and most primitive notion of the degree of a smooth map between compact manifolds of the same dimension.

DEFINITION. Let M_1^n and M_2^n be smooth compact manifolds of the same dimension and let $f : M_1^n \to M_2^n$ be a continuous function. The *mod-2 degree* of f, denoted $\deg_2(f)$, is the element of $\mathbb{Z}/2$ defined as follows.

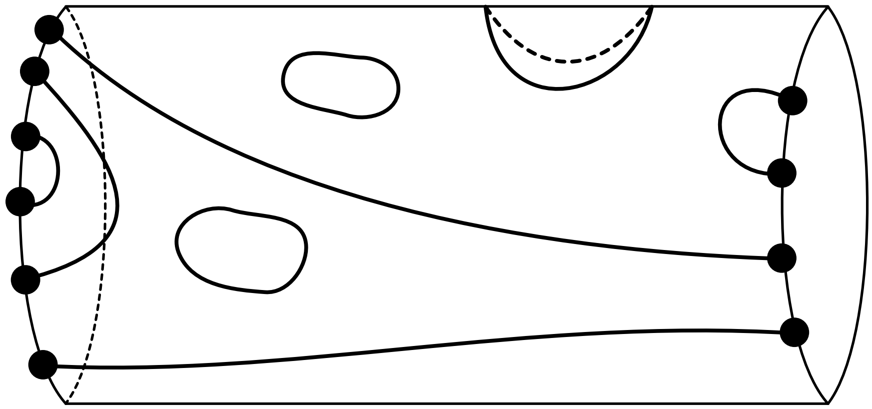

FIGURE 7.1. A schematic drawing of $M_1^n \times I$. The left side is $M_1^n \times \{0\}$, the right side is $M_1^n \times \{1\}$, the dots on the two boundary components are the points of $g_0^{-1}(p)$ and $g_1^{-1}(p)$, and the 1-submanifold is $\Lambda = F^{-1}(p)$

- Let $g : M_1^n \to M_2^n$ be a smooth function which is homotopic to f, which exists by Theorem 7.2. Let $p \in M_2^n$ be a regular value of g, which exists by Theorem 5.8 (Sard's theorem). Then $\deg_2(f)$ is the reduction modulo 2 of the number of points in $g^{-1}(p)$, which is a 0-dimensional submanifold of the compact manifold M_1^n, i.e. a finite collection of points. □

Of course, as defined the number $\deg_2(f)$ depends on the choice of g and p, but the following theorem says that this dependence is illusionary.

THEOREM 7.6. *The mod-2 degree is well-defined.*

PROOF. We will prove this in three steps. The first step is the most important and contains the geometric heart of the proof. As notation, if $g : M_1^n \to M_2^n$ is a smooth function that is homotopic to f and $p \in M_2^n$ is a regular value of g, then let $\deg_2(f, g, p) \in \mathbb{Z}/2$ denote the number of points modulo 2 of $g^{-1}(p)$. We want to show that $\deg_2(f, g, p)$ does not depend on g or p, which will be the content of the third step.

STEP 1. *Fix some $p \in M_2^n$. Let $g_0 : M_1^n \to M_2^n$ and $g_1 : M_1^n \to M_2^n$ be smooth functions that are homotopic to f and which have p as a regular value. Then* $\deg_2(f, g_0, p) = \deg_2(f, g_1, p)$.

Using Lemma 7.5, we can find a smooth homotopy $F : M_1^n \times I \to M_2^n$ from g_0 to g_1 such that p is a regular value of F. Define $\Lambda = F^{-1}(p)$. Using Theorem 5.7, we see that Λ is a smooth 1-dimensional submanifold of $M_1^n \times I$ such that

$$\partial\Lambda = (g_0^{-1}(p) \times \{0\}) \cup (g_1^{-1}(p) \times \{1\});$$

see Figure 7.1. Theorem 11.1 implies that each component of Λ is diffeomorphic to either a circle S^1 or an interval $[0, 1]$. Those that are intervals can be divided into three types (see Figure 7.1):

(1) intervals connecting points of $g_0^{-1}(p) \times \{0\}$ to other points of $g_0^{-1}(p) \times \{0\}$, and

(2) intervals connecting points of $g_1^{-1}(p) \times \{1\}$ to other points of $g_1^{-1}(p) \times \{1\}$, and
(3) intervals connecting points of $g_0^{-1}(p) \times \{0\}$ to points of $g_1^{-1}(p) \times \{1\}$.

Let k_0 (resp. k_1) be the number of points of $g_0^{-1}(p) \times \{0\}$ (resp. $g_1^{-1}(p) \times \{1\}$) that occur as endpoints of intervals of the first (resp. second) type. Both k_0 and k_1 are even. Next, let ℓ_0 (resp. ℓ_1) be the number of points of $g_0^{-1}(p) \times \{0\}$ (resp. $g_1^{-1}(p) \times \{1\}$) that occur as endpoints of intervals of the third type. We clearly have $\ell_0 = \ell_1$. We have

$$|g_0^{-1}(p)| = k_0 + \ell_0 \quad \text{and} \quad |g_1^{-1}(p)| = k_1 + \ell_1.$$

Since the k_i are even and the ℓ_i are equal, we deduce that reductions modulo 2 of $|g_0^{-1}(p)|$ and $|g_1^{-1}(p)|$ are the same, as desired.

Step 2. *Let $g : M_1^n \to M_2^n$ be a smooth function that is homotopic to f and let $p, q \in M_2$ be regular values of g. Then $\deg_2(f, g, p) = \deg_2(f, g, q)$.*

Using Theorem 4.7, we can find a flow $h_t : M_2 \to M_2$ of a smooth vector field such that $h_t(p) = q$. Define $g_1 = h_1 \circ g$. Since h_1 is a diffeomorphism of M_2, the point q is a regular value of g_1. In fact, we have $g^{-1}(p) = g_1^{-1}(q)$. Moreover, g_1 is homotopic to g (and hence f) via the homotopy $F : M_1 \times I \to M_2$ defined via $F(x,t) = h_{1-t}(g(x))$ for $(x,t) \in M_1 \times I$. This ends at g since $h_0 = \text{id}$. We thus have that

$$\deg_2(f, g, p) = \deg_2(f, g_1, q) = \deg_2(f, g, q),$$

where the first equality follows from the fact that $g^{-1}(p) = g_1^{-1}(q)$ and the second from Step 1.

Step 3. *Let $g_0 : M_1^n \to M_2^n$ and $g_1 : M_1^n \to M_2^n$ be smooth functions that are homotopic to f. Let $p_0 \in M_2^n$ be a regular value of g_0 and let $p_1 \in M_2^n$ be a regular value of g_1. Then $\deg_2(f, g_0, p_0) = \deg_2(f, g_1, p_1)$.*

By Theorem 5.8 (Sard's theorem), the regular values of g_0 are open and dense in M_2^n, and similarly for g_1. We can thus find some point $q \in M_2^n$ that is a regular value of both g_0 and g_1. Applying Steps 1 and 2, we see that

$$\deg_2(f, g_0, p_0) = \deg_2(f, g_0, q) = \deg_2(f, g_1, q) = \deg_2(f, g_1, p_1),$$

as desired. □

7.4. Simple applications of the mod-2 degree

One simple application of the mod-2 degree is as follows.

Theorem 7.7. *Let M^n be a smooth compact manifold and let $f : M^n \to M^n$ be a diffeomorphism. Then f is not homotopic to a constant map.*

Proof. Since every point of M^n is a regular value of f and has a single preimage, we see that $\deg_2(f) = 1$. However, if $g : M^n \to M^n$ is a constant map, then the regular values of g are exactly those points not in its image, so $\deg_2(g) = 0$. Since $\deg_2(f) \neq \deg_2(g)$, we see that f and g are not constant. □

This gives an alternate proof of the following result, which we recall is the key topological fact used to prove the Brouwer fixed point theorem (Theorem 5.12).

Lemma 7.8. *There does not exist a smooth map $f : \mathbb{D}^n \to S^n$ such that $f(p) = p$ for all $p \in S^n$.*

Proof. Assume that such an f exists. The function $F : S^n \times I \to S^n$ defined via the formula

$$F(p,t) = f((1-t)p) \qquad (p \in S^n, t \in I)$$

is a homotopy from the identity map on S^n to the constant map with image $f(0)$, which constradicts Theorem 7.7. $\square$

7.5. Orientations on vector spaces

Our next goal will be to refine the mod-2 degree to a degree that takes integer values. To do this, we will have to introduce orientations on our manifolds. We begin in this section by discussing orientations on vector spaces.

Definition. Let V be an n-dimensional real vector space with $n \geq 1$. An *orientation* on V is an equivalence class of ordered basis $(\vec{v}_1, \ldots, \vec{v}_n)$ for V under the following equivalence relation:

- If $b = (\vec{v}_1, \ldots, \vec{v}_n)$ and $b' = (\vec{v}'_1, \ldots, \vec{v}'_n)$ are ordered bases for V, then $b \sim b'$ if $\det(a_{ij}) > 0$, where (a_{ij}) is the $n \times n$ change of basis matrix defined via the identities

$$\vec{v}'_i = \sum_{j=1}^{n} a_{ij} \vec{v}_j \qquad (1 \leq i \leq n).$$

If V is equipped with a fixed orientation, then we will call V an *oriented vector space* and any ordered basis representing that orientation an *oriented basis* for V. $\square$

The first basic property of orientations is as follows.

Lemma 7.9. *Let V be an n-dimensional real vector space with $n \geq 1$. Then V has exactly two orientations.*

Proof. Let $b = (\vec{v}_1, \ldots, \vec{v}_n)$ and $b' = (\vec{v}'_1, \ldots, \vec{v}'_n)$ be two ordered bases for V. Since multiplying a column of a matrix by -1 has the effect of multiplying its determinant by -1, it follows that b represents the same orientation as either b' of $(-\vec{v}'_1, \vec{v}'_2, \ldots, \vec{v}'_n)$. $\square$

This lemma implies that the following definition makes sense.

Definition. Let V be an n-dimensional real vector space and let b be an orientation of V. Then $-b$ will denote the other orientation. $\square$

The following lemma will be very useful.

Lemma 7.10. *Let $f : V \to W$ be a surjective map between finite-dimensional real vector spaces of positive dimension. Set $U = \ker(f)$ and assume that U has positive dimension. Assume that two out of the three vector spaces U and V and W are equipped with an orientation. There is then a unique way to choose an orientation for the third such that the following holds.*

- *Let $\vec{w}_1, \ldots, \vec{w}_k \in V$ be vectors such that $(f(\vec{w}_1), \ldots, f(\vec{w}_k))$ is an oriented basis for W. Also, let $(\vec{u}_1, \ldots, \vec{u}_\ell)$ be an oriented basis for U. Then $(\vec{w}_1, \ldots, \vec{w}_k, \vec{u}_1, \ldots, \vec{v}_\ell)$ is an oriented basis for V.*

Proof. Homework! I'll insert this proof after the homework is due. $\square$

7.6. Orientation on manifolds

We now define orientations on smooth manifolds. Informally, an orientation on a smooth manifold is a choice of orientation on each tangent space that "vary smoothly". Since there are two orientations on a vector space, the set of possible orientations on a particular tangent space is a discrete set and the notion of "vary smoothly" should really mean "is locally constant". We formalize this in the following definition.

DEFINITION. An *orientation* on a smooth manifold with boundary M^n is a choice of orientation b_p of $T_p M^n$ for all $p \in M^n$ that satisfy the following continuity condition:

- Let $\phi : U \to V$ be any chart. For any $p \in U$, let $\overline{\mathrm{b}}_p$ be the orientation on $T_{\phi(p)} V = \mathbb{R}^n$ induced by b_p under the canonical identification of $T_p M^n$ with $T_{\phi(p)} V$. Then we require that $\overline{\mathrm{b}}_p = \overline{\mathrm{b}}_{p'}$ for all $p, p' \in U$.

A manifold with boundary M^n is *orientable* it there exists an orientation on it. An *oriented manifold with boundary* is a smooth manifold with boundary that is equipped with an orientation. □

The easiest example is as follows.

EXAMPLE. Let V be an open subset of $\mathbb{R}^n$. Equip each tangent space $T_p V = \mathbb{R}^n$ with the orientation corresponding to the standard basis of $\mathbb{R}^n$. This gives an orientation on V. □

To give more examples, we need the following lemma.

LEMMA 7.11. *Let $f : M_1 \to M_2$ be a smooth map between a smooth manifold with boundary M_1 and a smooth oriented manifold M_2 and let $p \in M_2$ be a regular value of both f and $f|_{\partial M_1}$. Then $f^{-1}(p)$ is orientable.*

PROOF. Set $X = f^{-1}(p)$. Recall that Theorem 5.7 says that X is a smooth manifold with boundary and that $\partial X = (f|_{\partial M_1})^{-1}(p)$. For each $q \in X$, we have that

$$T_q X = \ker(D_q f : T_q M_1 \to T_p M_2).$$

Since p is a regular value, the map $D_q f$ is surjective. Using Lemma 7.10, our given orientations on $T_p M_2$ and $T_q M_1$ induce a canonical orientation on $T_q X$. It is easy to see that these orientations on the tangent spaces of X vary smoothly, so this gives us an orientation on X. □

EXAMPLE. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be the function $f(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} x_i^2$. We know that 1 is a regular value of f and that $S^n = f^{-1}(1)$. Since both $\mathbb{R}^{n+1}$ and $\mathbb{R}$ are orientable, we deduce from Lemma 7.11 that S^n is orientable. □

Our next goal is to show how to orient the boundary of an orientable manifold with boundary. This requires the following definition.

DEFINITION. Let M^n be a smooth manifold with boundary, let $p \in \partial M^n$, and let $\vec{v} \in T_p M^n$. Choose a chart $\phi : U \to V$ with $p \in U$ and $V \subset \mathbb{H}^n$.

- We say that $\vec{v}$ is *tangent to the boundary* if the last coordinate of the element of $T_{\phi(p)} V = \mathbb{R}^n$ corresponding to $\vec{v}$ is zero. Equivalently, $\vec{v}$ lies in $T_p(\partial M^n) \subset T_p M^n$.
- We say that $\vec{v}$ is *inward facing* if the last coordinate of the element of $T_{\phi(p)} V = \mathbb{R}^n$ corresponding to $\vec{v}$ is positive.

- We say that $\vec{v}$ is *outward facing* if the last coordinate of the element of $T_{\phi(p)}V = \mathbb{R}^n$ corresponding to $\vec{v}$ is negative.

It is easy to see that these notions are well-defined. □

LEMMA 7.12. *Let M^n be a smooth oriented manifold with boundary. Then there exists a unique orientation on ∂M^n with the following property:*

- *Let $(\vec{v}_1, \ldots, \vec{v}_{n-1})$ be an oriented basis for $T_p(\partial M^n) \subset T_pM^n$ and let $\vec{v}_n \in T_pM^n$ be an inward facing vector. Then $(\vec{v}_1, \ldots, \vec{v}_n)$ is an oriented basis for T_pM^n.*

PROOF. That this condition picks out a unique orientation on each $T_p(\partial M^n)$ follows from Lemma 7.10. It is easy to see that it varys smoothly, and thus gives an orientation on ∂M^n. □

We will call the orientation given by Lemma 7.12 the *inward facing orientation* on ∂M^n. In a similar way, we can define the *outward facing orientation* on ∂M^n.

EXAMPLE. Since $\mathbb{D}^n \subset \mathbb{R}^n$ is orientable, Lemma 7.12 gives another proof that $S^n = \partial \mathbb{D}^n$ is orientable. □

7.7. The integral degree

We now define the degree of a continuous function between compact oriented n-manifolds. This requires the following preliminary definition.

DEFINITION. Let $g : M_1^n \to M_2^n$ be a smooth map between orientable n-manifolds and let $q \in M_1^n$ be such that g is a submersion at q. The derivative map $T_qg : T_qM_1^n \to T_{f(q)}M_2^n$ is thus an isomorphism. Define $\epsilon_{g,q}$ to be $+1$ if T_qg takes the given orientation on $T_qM_1^n$ to the given orientation on $T_{f(q)}M_2^n$ and to be -1 if it does not. □

We then define the degree as follows.

DEFINITION. Let M_1^n and M_2^n be smooth compact oriented manifolds of the same dimension and let $f : M_1^n \to M_2^n$ be a continuous function. The *degree* of f, denoted $\deg(f)$, is the element of $\mathbb{Z}$ defined as follows.

- Let $g : M_1^n \to M_2^n$ be a smooth function which is homotopic to f, which exists by Theorem 7.2. Let $p \in M_2^n$ be a regular value of g, which exists by Theorem 5.8 (Sard's theorem). Then

$$\deg(f) = \sum_{q \in f^{-1}(p)} \epsilon_{g,q},$$

which is a finite sum since $g^{-1}(p)$ is a 0-dimensional submanifold of the compact manifold M_1^n, i.e. a finite collection of points. □

Just like for the mod-2 degree, we must prove that this is well-defined.

THEOREM 7.13. *The degree is well-defined.*

PROOF. The proof of this is very similar to the proof of Theorem 7.6, which proves the analogous fact for the mod-2 degree. As notation, if $g : M_1^n \to M_2^n$ is a smooth function that is homotopic to f and $p \in M_2^n$ is a regular value of g, then let

$$\deg(f, g, p) = \sum_{q \in f^{-1}(p)} \epsilon_{g,q}.$$

We want to show that $\deg(f, g, p)$ does not depend on g or p

Again, there will be three steps.

STEP 1. *Fix some $p \in M_2^n$. Let $g_0 : M_1^n \to M_2^n$ and $g_1 : M_1^n \to M_2^n$ be smooth functions that are homotopic to f and which have p as a regular value. Then $\deg(f, g_0, p) = \deg(f, g_1, p)$.*

Using Lemma 7.5, we can find a smooth homotopy $F : M_1^n \times I \to M_2^n$ from g_0 to g_1 such that p is a regular value of F. Define $\Lambda = F^{-1}(p)$. Using Theorem 5.7, we see that Λ is a smooth 1-dimensional submanifold of $M_1^n \times I$ such that

$$\partial\Lambda = (g_0^{-1}(p) \times \{0\}) \cup (g_1^{-1}(p) \times \{1\});$$

see Figure 7.1.

At this point, we have to be careful with orientations. Observe first that we can orient $M_1^n \times I$ in such a way that the inward-facing orientation on $M_1^n \times \{0\} \subset \partial(M_1^n \times I)$ is the given orientation on M_1^n. However, with this choice of orientation the given orientation on M_1^n is the outward-facing orientation on $M_1^n \times \{1\} \subset \partial(M_1^n \times I)$. Now, Lemma 7.11 says that our given orientations on M_1^n and M_2^n induce a canonical orientation on Λ. An orientation on a 1-dimensional manifold is just a choice of direction for each component. Chasing through the proof of Lemma 7.11, we see that our orientation on Λ satisfies the following property:

- Consider a point $q \in \Lambda$. We then have that

  $$T_q\Lambda = \ker(D_qF : T_q(M_1^n \times I) \to T_pM_2^n).$$

  Let $\vec{v} \in T_q\Lambda$ be a vector such that $(\vec{v})$ is an oriented basis for $T_q\Lambda$. Also, let $\vec{w}_1, \ldots, \vec{w}_n \in T_q(M_1^n \times I)$ be elements such that $(D_qF(\vec{w}_1), \ldots, D_qF(\vec{w}_n))$ is an oriented basis for $T_pM_2^n$. Then $(\vec{w}_1, \ldots, \vec{w}_n, \vec{v})$ is an oriented basis for $T_q(M_1^n \times I)$.

Theorem 11.1 implies that each component of Λ is diffeomorphic to either a circle S^1 or an interval $[0, 1]$. Those that are intervals can be divided into three types (see Figure 7.1). In our descriptions of these types, we use our given orientation on Λ to speak of an interval having an initial and a terminal point.

(1) Intervals having an initial point $q \in g_0^{-1}(p) \times \{0\}$ and a terminal point $q' \in g_0^{-1}(p) \times \{0\}$. We claim that $\epsilon_{g_0,q} = 1$ and $\epsilon_{g_0,q'} = -1$. Indeed, let $\vec{v} \in T_q\Lambda$ be a vector such that $(\vec{v})$ is an oriented basis for $T_q\Lambda$. We have $q \in \partial(M_1^n \times I)$ and $\vec{v}$ is an inward-facing vector. Choose an ordered basis $(\vec{w}_1, \ldots, \vec{w}_n)$ for $T_q(\partial M_1^n \times I) \subset T_q(M_1^n \times I)$ such that $(D_qF(\vec{w}_1), \ldots, D_qF(\vec{w}_n))$ is an oriented basis for $T_pM_2^n$. Then by the above discussion of the orientation on Λ we have that $(\vec{w}_1, \ldots, \vec{w}_n)$ is an oriented basis for $T_q(M_2^n \times \{0\})$, which implies that $\epsilon_{g_0,q} = 1$. In a similar way, we see that $\epsilon_{g_0,q'} = -1$; the reason for the change in sign is that if $\vec{v}' \in T_{q'}\Lambda$ is such that $(\vec{v}')$ is an oriented basis for $T_{q'}\Lambda$, then $\vec{v}'$ is an outward-facing vector.

(2) Intervals having an initial point $q \in g_1^{-1}(p) \times \{1\}$ and a terminal point $q' \in g_1^{-1}(p) \times \{1\}$. Just like in the first case, we have $\epsilon_{g_1,q} = -1$ and $-\epsilon_{g_1,q'} = 1$.

(3) Intervals having an initial point $q \in g_0^{-1}(p) \times \{0\}$ and a terminal point $q' \in g_1^{-1}(p) \times \{1\}$. In this case, an argument similar to that above shows that $\epsilon_{g_0,q} = \epsilon_{g_1,q'}$. The key point here is that the relevant orientation on $M_1^n \times \{0\}$ is the inward-facing orientation but the relevant orientation on $M_1^n \times \{1\}$ is the outward-facing orientation.

Adding all the points up, we see that the positive and negative signs of the points appearing at the endpoints of the intervals of the first and second types all cancel, while the intervals of the third type match up points with identical signs. This implies that $\deg(f, g_0, p) = \deg(f, g_1, p)$, as desired.

STEP 2. *Let $g : M_1^n \to M_2^n$ be a smooth function that is homotopic to f and let $p, q \in M_2$ be regular values of g. Then* $\deg(f, g, p) = \deg(f, g, q)$.

The proof is identical to that of the analogous step of the proof of Theorem 7.6, so we omit it.

STEP 3. *Let $g_0 : M_1^n \to M_2^n$ and $g_1 : M_1^n \to M_2^n$ be smooth functions that are homotopic to f. Let $p_0 \in M_2^n$ be a regular value of g_0 and let $p_1 \in M_2^n$ be a regular value of g_1. Then* $\deg_2(f, g_0, p_0) = \deg_2(f, g_1, p_1)$.

Again, the proof is identical to that of the analogous step of the proof of Theorem 7.6, so we omit it. □

CHAPTER 8

Foliations and Frobenius's theorem

CHAPTER 9

Lie groups

CHAPTER 10

Transversality

CHAPTER 11

Morse theory

THEOREM 11.1 (Classification of 1-manifolds). *Every compact connected* 1-*manifold with boundary is diffeomorphic to either* S^1 *or* $[0, 1]$.

CHAPTER 12

Orientations and integral degrees

CHAPTER 13

Winding numbers and the Hopf invariant

CHAPTER 14

The Poincare-Hopf theorem

THEOREM 14.1 (Hairy ball theorem). *There does not exist a nonvanishing vector field on an even-dimensional sphere* S^{2n}.