Math 444/539, Homework 4

In the first few problems, you will prove the *Jordan-Brouwer separation theorem*, which states that if M^n is a smooth compact connected n-dimensional submanifold of $\mathbb{R}^{n+1}$, then $\mathbb{R}^{n+1} \setminus M^n$ has two components. This generalizes the smooth version of the ordinary Jordan curve theorem.

1. This first problem is a warmup. Let M^n be a smooth compact connected n-dimensional submanifold of $\mathbb{R}^m$ with $m \geq n+2$. Prove that $\mathbb{R}^m \setminus M^n$ is connected.

2. Let M^n be a smooth compact connected n-dimensional submanifold of $\mathbb{R}^{n+1}$. Prove that $\mathbb{R}^{n+1} \setminus M^n$ has at most 2 components. Hint: you will probably use the tubular neighborhood theorem at some point in the argument. This hint might also apply to the previous problems, though there are other approaches for that one.

3. Let M^n be a smooth compact connected n-dimensional submanifold of $\mathbb{R}^{n+1}$. Set $U = \mathbb{R}^{n+1} \setminus M^n$. For $p \in U$, define a function $\pi_p : M^n \to S^n$ via the formula

$$\pi_p(q) = \frac{q-p}{\|q-p\|} \qquad (q \in M^n).$$

   Finally, define a function $\tau : U \to \mathbb{Z}/2$ by setting $\tau(q) = \deg_2(\pi_q)$. Problem: prove that τ is locally constant.

4. Prove that the function τ constructed in the previous question is surjective. Use this to deduce the Jordan-Brouwer separation theorem.

5. Let M^n be a smooth compact connected n-dimensional submanifold of $\mathbb{R}^{n+1}$. Prove that M^n has a *nonvanishing unit normal vector field*, that is, a nonvanishing function $f : M^n \to \mathbb{R}^{n+1}$ such that

$$f(p) \in N_{p,\mathbb{R}^{n+1}/M^n} = (T_pM^n)^\perp \subset \mathbb{R}^{n+1}$$

   and

$$\|f(p)\| = 1$$

   for all $p \in M^n$. Hint: You'll want to use the previous problem.

6. Let M^n be a smooth compact connected n-dimensional submanifold of $\mathbb{R}^{n+1}$. Prove that M^n is orientable. Hint: You'll want to use the previous problem. We remark that this shows for instance that a compact nonorientable 2-manifold (like $\mathbb{R}P^2$) cannot be embedded in $\mathbb{R}^3$. It does not apply to the Mobius band since the Mobius band either is noncompact or is a manifold with boundary (depending on how you define it).

7. Let Σ_g denote a genus g surface, that is, the compact oriented 2-manifold that looks like a "donut with g holes".
   (a) Prove that if $g \geq h$, then there exists a continuous map $f : \Sigma_g \to \Sigma_h$ with $\deg(f) = 1$.
   (b) For a fixed $k \in \mathbb{Z}$ and $h \geq 0$, find conditions on g that ensure that there exists a continuous map $f : \Sigma_g \to \Sigma_h$ with $\deg(f) = k$.

8. Define $X^n = \mathbb{R}^n / \sim$, where $\sim$ identifies $(x_1, x_2, x_3, \ldots, x_n)$ with $(x_1 + 1, -x_2, x_3, \ldots, x_n)$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$. For $n = 2$, this is the open Mobius strip. Problem: prove that X^n is not orientable. Hint: Assume that X^n is orientable and fix an orientation on it. Also, fix the usual orientation on $\mathbb{R}^n$ and let $\pi : \mathbb{R}^n \to X^n$ be the projection. Define a function $f : \mathbb{R} \to \mathbb{Z}/2$ as follows:

- For $x \in \mathbb{R}$, the map π is a local diffeomorphism at $(x, 0, \ldots, 0)$. Define $f(x) = 1$ if $D_{(x,0,\ldots,0)}\pi : T_{(x,0,\ldots,0)}\mathbb{R}^n \to T_{\pi(x,0,\ldots,0)}X^n$ preserves the orientations on these vector spaces and $f(x) = -1$ if it does not.

Prove that f is locally constant and that $f(x+1) = -f(x)$ for all x, and obtain a contradiction.