Math 444/539, Homework 2

1. Let M^n be a smooth n-manifold.
   (a) Prove that the tangent bundle TM^n is a smooth $2n$-dimensional manifold.
   (b) Construct (with proof!) a surjective submersion $\pi : TM^n \to M^n$.
2. Let $f : M_1^{n_1} \to M_2^{n_2}$ be a submersion.
   (a) Prove that if $U \subset M_1^{n_1}$ is open, then $f(U)$ is open.
   (b) If $M_1^{n_1}$ is compact and $M_2^{n_2}$ is connected, then prove that f is surjective.
3. Let M^n be a smooth n-manifold with boundary. Prove that the boundary ∂M^n is a smooth $(n-1)$-manifold (without boundary).
4. A *standard projection* of $\mathbb{R}^m$ onto an n-dimensional subspace is a linear map $\pi : \mathbb{R}^m \to \mathbb{R}^n$ that can be written in the form $\pi(x_1, \ldots, x_m) = (x_{i_1}, \ldots, x_{i_n})$ for some $1 \leq i_1 < \cdots < i_n \leq m$. Problem: For an embedding $f : M^n \to \mathbb{R}^m$ and a point $p \in M^n$, prove that there exists a chart $\phi : U \to V$ such that $p \in U$ and $\phi = \pi \circ (f|_U)$ for some standard projection $\pi : \mathbb{R}^m \to \mathbb{R}^n$. We remark that each chart in the system of charts we gave for S^n on the first day of class is of this form.
5. A polynomial $f(x_1, \ldots, x_k)$ is *homogeneous* of degree m if
   $$f(tx_1, \ldots, tx_k) = t^m f(x_1, \ldots, x_k) \qquad (t, x_1, \ldots, x_k \in \mathbb{R}).$$
   Fix some polynomial $f(x_1, \ldots, x_k)$ which is homogeneous of degree $m \geq 1$.
   (a) Prove *Euler's Identity*:
   $$mf = \sum_{i=1}^{k} x_i \frac{\partial f}{\partial x_i}.$$
   (b) Prove that all nonzero numbers $a \in \mathbb{R}$ are regular values of $f(x_1, \ldots, x_k)$, and hence that $f^{-1}(a)$ is a smooth submanifold of $\mathbb{R}^n$ of dimension $(n-1)$.
   (c) Prove that if $a, b > 0$, then the manifolds $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic, and similarly if $a, b < 0$.
6. Let A be an $n \times n$ real matrix whose entries are all nonnegative. Prove that A has a real nonnegative eigenvalue. Hint: This is trivial if A is singular (i.e. if 0 is an eigenvalue), so we can assume that A is nonsingular. Define a function $f : S^{n-1} \to S^{n-1}$ via the formula
   $$f(v) = \frac{Av}{\|Av\|};$$
   this makes sense since A is nonsingular. Prove that f preserves the set
   $$Q = \{(x_1, \ldots, x_n) \in S^{n-1} \mid x_i \geq 0 \text{ for all } i\}.$$
   Next, prove that Q is homeomorphic to an $(n-1)$-disc. Finally, apply the Brouwer Fixed Point Theorem.

7. A continuous function $f : X \to Y$ is *nullhomotopic* if there exists a continuous function $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ for all $x \in X$ and $x \mapsto F(x, 1)$ is a constant map. In other words, f can be "deformed" to a constant map.

   (a) Prove that if X is a topological space and $f : X \to \mathbb{R}^n$ is a continuous function, then f is nullhomotopic.

   (b) Prove that if X is a topological space and $f : X \to S^n$ is a continuous nonsurjective function, then f is nullhomotopic.

   (c) Prove that if M^k is a smooth k-manifold with $k < n$, then every smooth map $f : M^k \to S^n$ is nullhomotopic.