Math 366 : Geometry
Midterm

This is a pledged midterm exam. It is due on Monday, March 11 at the start of class. There is no time limit. You are allowed to use the textbook, your notes, other books, and static internet resources like wikipedia; however, you are not allowed to talk to each other (or anyone else) or to use internet question/answer sites like `math.stackexchange.com` or `mathoverflow.net`.

1. Construct a convex set $U \subset \mathbb{R}^2$ and a continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(U)$ is not convex. Hint : You can take U to be a line segment.

   Solution : Set $U = \{(x, 0) \mid 0 \leq x \leq 1\}$. The set U is a line segment, and hence convex. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ via $f(x, y) = (x, y+x^2)$. Then $f(U) = \{(x, x^2) \mid 0 \leq x \leq 1\}$ is not convex; indeed, $f(U)$ contains the points $(0, 0)$ and $(1, 1)$ but does not contain the line segment between $(0, 0)$ and $(1, 1)$ (for instance, it does not contain $(1/2, 1/2)$).

2. Let $U_1 \subset U_2 \subset U_3 \subset \cdots$ be an increasing sequence of convex subsets of $\mathbb{R}^n$. Prove that $\cup_{i=1}^{\infty} U_i$ is convex.

   Solution : Consider $x, y \in \cup_{i=1}^{\infty} U_i$. We want to show that the line segment ℓ between x and y is contained in $\cup_{i=1}^{\infty} U_i$. Pick $n, m \geq 1$ such that $x \in U_n$ and $y \in U_m$. Set $p = \max(n, m)$. Then $x, y \in U_p$. The set U_p is convex, so $\ell \subset U_p \subset \cup_{i=1}^{\infty} U_i$, as desired.

3. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ be convex sets. Define $W = \{\vec{u} + \vec{v} \mid \vec{u} \in U \text{ and } \vec{v} \in V\}$. Prove that W is convex.

   Solution : Consider $x, y \in W$. We want to show that the line segment $\ell = \{tx + sy \mid s, t \geq 0,\ s + t = 1\}$ joining x and y is contained in W. Consider a point $tx + sy \in \ell$. Write $x = u_x + v_x$ and $y = u_y + v_y$ with $u_x, u_y \in U$ and $v_x, v_y \in V$. Since U and V are convex, we have $tu_x + su_y \in U$ and $tv_x + sv_y \in V$. We therefore have

   $$tx + sy = t(u_x + v_x) + s(u_y + v_y) = (tu_x + su_y) + (tv_x + sv_y) \in W,$$

   as desired.

4. Prove that the Sylvester-Gallai theorem does *not* hold for infinite sets. In other words, construct (with proof!) an infinite set $U \subset \mathbb{R}^2$ such that if a line ℓ passes through two distinct points $x, y \in U$, then ℓ passes through a third point of U.

   Solution : There are many possible solutions; one very cheap one is to take $U = \mathbb{R}^2$!

5. Let $P \subset \mathbb{R}^2$ be a convex polygon. Prove that there exists a line ℓ in $\mathbb{R}^2$ that divides P into two convex polygons whose perimeters are equal.

   Solution : Parametrize the boundary of P by the simple closed curve $\gamma : [0, 1] \to P$. For $0 \leq t \leq 1$, let $f(t)$ be the length of the path $\gamma([0, t])$. The function $f : [0, 1] \to \mathbb{R}$ is continuous. Letting h be the perimeter of P, we have $f(0) = 0$ and $f(1) = h$. The intermediate value theorem therefore implies that there exists some $t_0 \in [0, 1]$ such that $f(t_0) = h/2$. Observe that the length of $\gamma([t_0, 1])$ is also $h/2$. Let ℓ be the line through $f(0)$ and $f(t_0)$. The line ℓ divides P into two convex polygons, one of whose boundaries is $\gamma([0,t_0]) \cup (\ell \cap P)$ and the other of whose boundaries is $\gamma([t_0,1]) \cup (\ell \cap P)$. Both of these boundaries have length $h/2 + \text{length}(\ell \cap P)$, as desired.

Remark : One might be tempted to solve this problem by (for instance) letting ℓ_t be the vertical line $x = t$ and then defining $f(t)$ to be the perimeter of the portion of P to the left of ℓ_t. The problem is that $f(t)$ might not be continuous. For example, let P be the unit square $[0,1] \times [0,1]$. Then

$$f(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 + 2t & \text{if } 0 \leq t < 1, \\ 4 & \text{if } t \geq 1. \end{cases}$$

This is not continuous at $t = 0$ and $t = 1$. This could be corrected by first rotating P so that none of its line segments are vertical.

6. Recall that we proved in class that every finite tree T had a leaf, that is, a vertex v_0 such that v_0 is the endpoint of precisely one edge of T. Does this hold for infinite trees? Either prove it or construct a counterexample.

   Solution : It does not hold for infinite trees. For instance, let T be the tree whose vertices are the integers and where the edges join integers which differ by 1 (draw it; it looks like the real line).