Math 366 : Geometry
Problem Set 6

1. Give a direct proof (without using any variant of the Jordan curve theorem) that a straight line $\ell \subset \mathbb{R}^2$ separates $\mathbb{R}^2$ into two pieces. Hint : Of course, those two pieces are the regions of the plane to the left and right of ℓ. Let U and V be those regions. First prove that U and V are path-connected (easy!) and then prove that there is no path in $\mathbb{R}^2 \setminus \ell$ connecting a point of U to a point of V. For this, you'll need the intermediate value theorem. It might be helpful to first rotate the plane so that ℓ is vertical.

2. Give a direct proof (without using any variant of the Jordan curve theorem) that the unit circle C around $(0,0)$ separates $\mathbb{R}^2$ into two pices. Hint : Of course, those regions are the disc $U = \{(x,y) \mid x^2 + y^2 < 1\}$ and the region $V = \{(x,y) \mid x^2 + y^2 > 1\}$. First prove that U and V are connected (easy!). Next, prove that there is no path in $\mathbb{R}^2 \setminus C$ connecting a point of U to a point of V. For this, you'll need the intermediate value theorem.

3. Let $x_1, \ldots, x_k \in \mathbb{R}^2$ be distinct points in general position ($k \geq 3$) and let $X = (x_1 x_2) \cup \cdots \cup (x_{k-1} x_k) \cup (x_k x_1)$ (here (pq) is the line segment connecting p and q). Prove that X separates $\mathbb{R}^2$ into at least 2 regions (remark: there is no assumption here that the line segments do not intersect). Hint : Figure out how to reduce this to the ordinary Jordan curve theorem.

4. Let $X \subset \mathbb{R}^2$ be a convex polygon and let $p \in \mathbb{R}^2$ be a point in the interior of X. Prove that there exists two points $a, b \in X$ such that p is the midpoint of the line segment (ab). Hint : Parameterize X by a path $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1)$. For all t, let r_t be the ray starting at $\gamma(t) \in X$ that passes through p. After the ray r_t passes through p, it intersects X at another point which we'll call $\delta(t)$. Our goal is to find some $t \in [0,1]$ such that $\operatorname{dist}(\gamma(t), p) = \operatorname{dist}(\delta(t), p)$. Show that some such t must exist using the intermediate value theorem.

5. Let ℓ_1 and ℓ_2 and ℓ_3 be distinct parallel lines in $\mathbb{R}^2$. Prove that there exists an equilateral triangle with vertices $x \in \ell_1$ and $y \in \ell_2$ and $z \in \ell_3$. Hint : First rotate the plane so that all three lines are vertical. Fix a point $x_0 \in \ell_1$. For $y \in \ell_2$, let $\phi(y) \in \mathbb{R}^2$ be the point such that $\{x_0, y, \phi(y)\}$ forms the vertices of an equilateral triangle and $\phi(y)$ lies to the right of the line from x_0 to y. Prove using the intermediate value theorem that there must exist some $y \in \ell_2$ such that $\phi(y) \in \ell_3$.