Math 366 : Geometry
Problem Set 2 Solutions

Problem 2 : Assume that $x_1, \ldots, x_n \in \mathbb{R}^2$ are distinct point such that for any $1 \leq i < j < k < \ell \leq n$, the four points $\{x_i, x_j, x_k, x_\ell\}$ form the vertices of a convex 4-gon. Prove that the points $\{x_1, \ldots, x_n\}$ form the vertices of a convex n-gon.
Solution : Assume that the x_i do not form the vertices of a convex n-gon. This implies that we can write $\{x_1, \ldots, x_n\}$ as the disjoint union of proper nonempty sets S_1 and S_2 such that S_1 forms the vertices of a convex polygon P and S_2 is contained in the convex hull of S_1. Pick some $x_i \in S_1$, and divide P into triangles each of which has x_i as one of its vertices (draw a picture to see what I mean!). Letting $x_j \in S_2$, we know that x_j has to be contained in one of those triangles, say with vertices x_i and $x_{i'}$ and $x_{i''}$. But then the four points $\{x_i, x_{i'}, x_{i''}, x_j\}$ do not form the vertices of a convex 4-gon, a contradiction.

Problem 5 : Consider $n \geq 4$ parallel line segments in $\mathbb{R}^2$. Assume that for every three of these line segments, there is a line in $\mathbb{R}^2$ meeting all three segments. Prove that there is a single line meeting all n of the line segments.
Solution : Let $S_1, \ldots, S_n$ be the line segments. By appropriately rotating the plane, we can assume that all of the S_i are vertical. For $1 \leq i \leq n$, let $c_i, d_i, e_i \in \mathbb{R}$ be such that S_i is the portion of the line vertical line $x = c_i$ satisfying $d_i \leq y \leq e_i$. Define

$$C(S_i) = \{(\alpha, \beta) \in \mathbb{R}^2 \mid \text{the line } y = \alpha x + \beta \text{ meets } S_i\} \subset \mathbb{R}^2.$$

I claim that $C(S_i)$ is convex. Indeed, consider (α, β) and (α', β') in $C(S_i)$ and $t, t' \geq 0$ with $t + t' = 1$. We want to prove that

$$t(\alpha, \beta) + t'(\alpha', \beta') = (t\alpha + t'\alpha', t\beta + t'\beta') \in C(S_i).$$

We know that there exists some $y_0, y_0' \in [d_i, e_i]$ such that

$$y_0 = \alpha x_i + \beta \quad \text{and} \quad y_0' = \alpha' x_i + \beta'.$$

Adding t time the first equality to t' times the second, we see that

$$ty_0 + t'y_0' = t(\alpha x_i + \beta) + t'(\alpha' x_i + \beta') = (t\alpha + t'\alpha')x_i + (t\beta + t'\beta').$$

Since the interval $[d_i, e_i]$ is convex, we have that $ty_0 + t'y_0' \in [d_i, e_i]$. We conclude that the line

$$y = (t\alpha + t'\alpha')x + (t\beta + t'\beta')$$

intersects the line segment S_i, i.e. that $(t\alpha + t'\alpha', t\beta + t'\beta') \in C(S_i)$, as claimed.
Now, the assumptions of the problem say that for any $1 \leq i_1 < i_2 < i_3 \leq n$, there exists a line meeting S_{i_1} and S_{i_2} and S_{i_3}. This is equivalent to saying that the convex sets $C(S_{i_1})$ and $C(S_{i_2})$ and $C(S_{i_3})$ must intersect. Helly's Theorem therefore says that all of the $C(S_i)$ must intersect, i.e. that there exists a single line meeting all of the S_i, as desired.