Math 366 : Geometry
Problem Set 10

1. Fix vector spaces W and V over $\mathbb{Q}$. Recall that $W \otimes_{\mathbb{Q}} V = A/B$, where A and B are as follows.

   - A is the vector space with basis $\{\vec{w} \otimes \vec{v} \mid \vec{w} \in W \text{ and } \vec{v} \in V\}$.
   - B is the subspace of A spanned by the following elements:

   $$\begin{aligned} &\{(\vec{w}+\vec{w}') \otimes \vec{v} - \vec{w} \otimes \vec{v} - \vec{w}' \otimes \vec{v} \mid \vec{w}, \vec{w}' \in W,\, \vec{v} \in V\} \\ \cup &\{\vec{w} \otimes (\vec{v}+\vec{v}') - \vec{w} \otimes \vec{v} - \vec{w} \otimes \vec{v}' \mid \vec{w} \in W,\, \vec{v}, \vec{v}' \in V\} \\ \cup &\{(c\vec{w}) \otimes \vec{v} - c(\vec{w} \otimes \vec{v}) \mid \vec{w} \in W,\, \vec{v} \in V,\, c \in \mathbb{Q}\} \\ \cup &\{\vec{w} \otimes (c\vec{v}) - c(\vec{w} \otimes \vec{v}) \mid \vec{w} \in W,\, \vec{v} \in V,\, c \in \mathbb{Q}\} \end{aligned}$$

   The problems are then as follows.

   (a) Consider linear maps $f : W \to \mathbb{Q}$ and $g : V \to \mathbb{Q}$. Define a linear map $H : A \to \mathbb{Q}$ via the formula $H(\vec{w} \otimes \vec{v}) = f(\vec{w})g(\vec{v})$ on a basis element $\vec{w} \otimes \vec{v}$ of A. Prove that $H(B) = 0$, and thus that H induces a linear map $h : W \otimes_{\mathbb{Q}} V \to \mathbb{Q}$.

   (b) Let $\vec{w}_1, \ldots, \vec{w}_n \in W$ be linearly independent. Also, let $\vec{v}_1, \ldots, \vec{v}_m \in V$ be linearly independent. Prove that the elements $\vec{w}_i \otimes \vec{v}_j \in W \otimes_{\mathbb{Q}} V$ are linearly independent. Hint : Assume that there is a linear dependence $\sum_{i,j} c_{ij} \vec{w}_i \otimes \vec{v}_j$. For any linear map $h : W \otimes_{\mathbb{Q}} V \to \mathbb{Q}$, we then have $\sum_{i,j} c_{ij} h(\vec{w}_i \otimes \vec{v}_j)$. For any fixed i_0 and j_0, use part a to construct a linear map h such that $h(\vec{w}_i \otimes \vec{v}_j)$ equals 1 if $i = i_0$ and $j = j_0$ and equals 0 otherwise. Use this to show that all the coefficients c_{ij} have to vanish.

2. Do either of the following two problems. The second is easier. I remark that you are *not* allowed to use the converse of Dehn's theorem that says that two polyhedra with equal volumes and Dehn invariants are scissors congruent.

   (a) Define a *prism* in $\mathbb{R}^3$ to be a polyhedron which has five sides, two of which are parallel (to help you visualize this, observe that any cross section parallel to the two parallel sides is a triangle). Prove that if $P, Q \subset \mathbb{R}^3$ are prisms whose volumes are equal, then P and Q are scissors congruent. Hint : Try to reduce this to the two-dimensional case.

   (b) Define a *cuboid* in $\mathbb{R}^3$ to be a polyhedron composed of three pairs of parallel sides which meet at right angles (for instance, a cube; a cuboid bears the same relation to a cube that a rectangle bears to a square). Prove that if $P, Q \subset \mathbb{R}^3$ are cuboids whose volumes are equal, then P and Q are scissors congruent.

3. Prove that a volume 1 octahedron is not scissors congruence to a cube of volume 1.