Math 366 : Geometry
Final

This is a pledged final exam. It is due on Wednesday, May 1st (the last day of the final exam period) at 5pm. Slide it under my office door. There is no time limit. You are allowed to use the textbook, your notes, other books, and static internet resources like wikipedia; however, you are not allowed to talk to each other (or anyone else) or to use internet question/answer sites like `math.stackexchange.com` or `mathoverflow.net`.

1. Consider a set C of n great circles on the 2-sphere S^2 no three of which pass through a common point. How many pieces do C divide S^2 into? Prove your answer. Hint: Draw a bunch of pictures to guess at a formula, and then confirm your guess with the aid of Euler's formula.

2. Let $P \subset \mathbb{R}^2$ be a convex polygon and let $v \in \mathbb{R}^2$ be a point in the interior of P. Prove that there are two points $a, b \in P$ such that v is the midpoint of the line segment from a to b. Hint: Move P so that v is at the origin. Let $Q = \{-x \mid x \in P\}$. Prove using areas that P and Q have to intersect. Why does this give us the desired pair of points?

3. Let P be a convex polygon in $\mathbb{R}^2$. Prove that there exists a convex polygon $Q \subset \mathbb{R}^2$ such that P is a closed billiard path in Q. Hint: Obviously there will have to be an edge of Q for each vertex of P.

4. Let $X \subset \mathbb{R}^2$ be a set of n points forming the vertices of a convex n-gon P. Let T be any triangulation with vertex set X. Prove that T contains a triangle two of whose sides lie on P. Hint: Let G be the following graph. The vertices are in bijection with the triangles in T. Two vertices are joined by an edge if their associated triangles share an edge. Prove that G is a tree, and then prove that the triangle associated to a leaf of G is the desired triangle.

5. Let $P, Q, R \subset \mathbb{R}^n$ be convex polytopes. Assume that P is scissors congruent to Q and that Q is scissors congruent to R. Prove that P is scissors congruent to R. Hint: The key fact you'll use is that the intersection of two convex polytopes is a convex polytope.