31. (a) We show first that 5 points always determine a convex quadrilateral. If the convex hull of them is a quadrilateral we are done, so suppose that this convex hull is a triangle abc. This contains two further points d, e. Suppose that the line de intersects the edges ab and ac of the triangle. Then $bcde$ is a convex quadrilateral (Fig. 105).

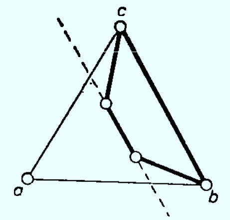

FIGURE 105

Now let us 2-color all quadruples of the given n points: let a quadruple be red, if it forms the vertices of convex quadrilateral and blue otherwise. If

(1) $$n \geq R_2^4(m, 5),$$

then we have either m points with all quadruples red or 5 points with all quadruples blue. The latter case is impossible by the above. Hence we have m points any 4 of which form a convex quadrilateral. Hence the m points form a convex m-gon.

(b) I. We show that (1) can be replaced by

$$n \geq \binom{2m-4}{m-2} + 1.$$

In fact, given any $\binom{2m-4}{m-2}+1$ points in general position, they always span an m-gon convex from below or from above by 14.30a; this is a convex m-gon.

II. We construct a set of 2^{m-2} points spanning no convex m-gon. Let T_i be a set of points in general position such that $|T_i| = \binom{n-2}{i}$ and T_i spans no $(i+2)$-gon convex from above a no $(n-i)$-gon convex from below. Let T_i have diameter less than 1 ($i=0,1,\dots,n-2$). Also we may suppose that any line formed by two points of T_i forms an angle less than 45° with e x-axis (by applying an affinity perpendicular to the x-axis, if necessary).

Let P be a large regular $(4n-4)$-gon with centre at the origin and let $v_0, \dots, v_{n-2}$ be its vertices in the segment of $\pm 45°$ around the positive half of

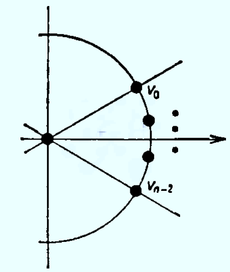

FIGURE 106

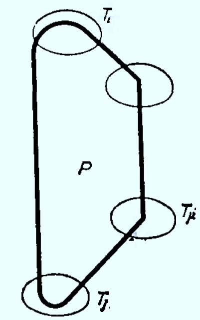

FIGURE 107

the x-axis (Fig. 106). Let us place T_i around v_i (i.e. let us translate one point of T_i into v_i). The resulting set T has

$$|T| = \sum_{i=0}^{n-2} |T_i| = \sum_{i=0}^{n-2} \binom{n-2}{i} = 2^{n-2}$$

elements. We claim that it contains no convex n-gon. Let P be any convex polygon spanned by T. If P is contained in a $T_i (0 \leq i \leq n-2)$, let a and b be the vertices of P with least and largest x-coordinate, respectively. The diagonal ab splits P into two polygons, convex from above and below, respectively. Hence by the definition of T_i, the number of vertices of these polygons is $\leq i+1$ and $\leq n-i-1$, respectively. Hence the number of vertices of P is $\leq (i+1)+(n-i-1)-2 = n-2$.

Now suppose that P intersects T_i and T_j, $i<j$. Let i be the least and j be the largest index for which this holds. Now P can intersect T_μ, $i < \mu < j$ in at most one point, and its pieces in T_i and T_j are convex from above and below, respectively (Fig. 107). Hence P has at most

$$(i+1) + (j-i-1) + (n-j-1) = n-1$$

points.

(c) $K(4) \geq 5$ trivially; $K(4) \leq 5$ by the first paragraph of the solution. Thus $K(4) = 5$.

$K(5) \geq 9$ by (b). So what we have to show is that any 9 points (in general position) contain a convex pentagon. If the convex hull of the 9 points is a pentagon we are done. So we may suppose that it is a quadrilateral or a triangle.

1° Suppose that the convex hull is a quadrilateral $xyzu$. Consider the 5 remaining points. If they form a convex pentagon we are done. Otherwise, there are 4 among them, which do not form a convex quadrilateral; let a point v be contained in a triangle abc (Fig. 108). Consider the angles avb, bvc, cva. One of these contains x and y. Then $xyavb$ is a convex pentagon.

2° Suppose the convex hull is a triangle abc. Again we have two subcases: