Math 464/564 : Algebra III
Problem Set 10

1. Let C_* and C'_* be chain complexes with differentials ∂_* and ∂'_*, respectively. Let $\phi_* : C_* \to C'_*$ be a chain map. Define $E_k = C_{k-1} \oplus C'_k$ and

$$\delta_k : E_k \to E_{k-1}$$
$$\delta_k(c, c') = (-\partial_{k-1}(c), \phi(c) + \partial_k(c')).$$

Prove that E_* is a chain complex with differential δ_*. Also, show that the natural inclusion $C'_* \hookrightarrow E_*$ is a chain map.

2. Consider a chain complex

$$0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to 0$$

of finite-dimensional vector spaces. Prove that

$$\sum_{i=0}^{n} (-1)^i \dim(C_i) = \sum_{i=0}^{n} \dim(H_i(C_*)).$$

3. Consider an exact sequence

$$0 \to A \to B \to P \to 0$$

of R-modules such that P is projective. Prove that the exact sequence splits.

4. Prove that if P is a finitely-generated projective module, then there is a finitely-generated Q such that $P \oplus Q$ is free.

5. Prove that all projective modules are flat.

6. Let $R = \mathbb{F}[x, y, z]$, and let M be the ideal $(xy, x+y, xz)$ of R (an R-module). Construct an explicit free resolution of M.