Math 464/564 : Algebra III
Final

This final exam is pledged. You may not talk about it with anyone. You are allowed to use your textbook and course notes, but are not allowed to use other books or to search the internet. You have unlimited time. This is due in my office by 5pm on May 2nd (earlier is fine!). Place it under the door if I am not there.

1. Let R be a commutative ring and let $P_1, P_2 \subset R$ be prime ideals such that $P_1 \nsubseteq P_2$ and $P_2 \nsubseteq P_1$. Prove that $P_1 \cap P_2$ is not prime.

2. Let $R = \mathbb{Q}[x, y]$ and let $I \subset R$ be the ideal (x, y). Prove that the R-modules I and R/I are neither flat nor projective.

3. Let R be a PID with field of fractions F. Let S be a ring with $R \subset S \subset F$.

   (a) Prove that all elements of S can be written as a/b with $a, b \in R$ and $1/b \in S$.

   (b) Prove that S is a PID.

   (c) Prove that if S is finitely generated as an R-module then $R = S$.

4. Consider the ideal $J = (x - t^2, y^2 - t^3) \subset \mathbb{Q}[x, y, t]$. Prove that $J \cap \mathbb{Q}[x, y]$ is generated by $x^3 - y^4$ as an ideal over $\mathbb{Q}[x, y]$.

5. Let a and b be distinct rational numbers and let m and n be positive integers. Set $R = \mathbb{Q}[x]$. Compute the following.

   (a) $\mathrm{Hom}_R(R/((x - a)^m), R/((x - b)^n))$.

   (b) $R/((x - a)^m) \otimes_R R/((x - b)^n))$.

   (c) $\dim_{\mathbb{Q}}(R/((x - a)^m) \otimes_{\mathbb{Q}} R/((x - b)^n))$, where the quotient rings are regarded as vector spaces over $\mathbb{Q}$.

   (d) $\mathrm{Tor}_1^R(R/((x - a)^m), R/((x - b)^n))$.